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DEGREE SUM SPECTRA AND DEGREE SUM ENERGY OF CERTAIN FAMILIES OF GRAPHS

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Abstract. For any simple graph G, the degree sum matrix is defined as a matrix in which each entry represents the sum of the degrees of a pair of vertices. The degree sum energy is the absolute sum of the eigenvalues of the degree sum matrix of G. In this paper, we determine the degree sum spectra and the degree sum energy of certain classes of graphs and their complements.

Keywords: Degree sum spectra, degree sum energy, sunlet graph, windmill graph, complement of a graph.

AMS Subject Classification: 05C50, 05C31, 05C76.

1. Introduction

Let G be a simple graph of order n (number of vertices) and size m (number of edges). The degree of a vertex v_k is the number of edges incident to v_k. It is denoted as $d(v_k)$ or, simply by d_k. The *adjacency matrix* of a graph G is defined as $A(G) = [a_{jk}]$, in which $a_{jk} = 1$, if vertex v_j is adjacent to vertex v_k, otherwise $a_{jk} = 0$. The characteristic polynomial of $A(G)$ is given by, $P_{A(G)}(\lambda) = det(\lambda I - A(G))$, where I is the identity matrix of order n. The roots of the equation $P_{A(G)}(\lambda) = 0$ are called the eigenvalues of G and are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The spectrum $Spec(G)$ of a graph G is the collection of eigenvalues of G. The ordinary energy of a graph G, introduced by I. Gutman [4] in 1978, is defined as the absolute sum of eigenvalues of G and is denoted by $\mathcal{E}_A(G)$

The concept of *degree sum matrix* $DS(G)$ of a graph G is introduced by H. S. Ramane et al. [7], is defined as $DS(G) = [d_{jk}]$, where

$$d_{jk} = \begin{cases} d_j + d_k & \text{if } j \neq k, \\ 0 & \text{otherwise .} \end{cases}$$

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The characteristic polynomial of the degree sum matrix is given by,

$$P_{DS(G)}(\alpha) = det(\alpha I - DS(G)) = \alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + \cdots + c_n.$$

The *degree sum energy* of a graph G, introduced by H. S. Ramane et al. [7], is defined as the absolute sum of degree sum eigenvalues of G and is denoted by $\mathcal{E}_{DS}(G)$. In the literature, several authors have established the bounds for the degree sum eigenvalue and degree sum energy of a graph [7, 9], the degree sum polynomial of graph valued functions on regular graphs [5] and computed the degree sum energy for various graph families [6]. These results naturally motivate further exploration of the degree sum spectra and the degree sum energy for specific families of graphs.

The results mentioned below are useful for the computation of the degree sum spectra of graphs.

Lemma 1.1. [2] If a and b are real numbers, then

$$\begin{vmatrix} a & b & b & b & \cdots & b \\ b & a & b & b & \cdots & b \\ b & b & a & b & \cdots & b \\ b & b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & b & \cdots & a \end{vmatrix} = (a–b)^{n–1}(a + (n–1)b).$$

Lemma 1.2. [8] If a, b, c, d and α are real numbers, then the determinant of the form

$$\begin{vmatrix} (\alpha + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\alpha + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$(\alpha + a)^{n_1-1}(\alpha + b)^{n_2-1}((\alpha - (n_1 - 1)a)(\alpha - (n_2 - 1)b) - n_1n_2cd)$, where J is the matrix with all its entries equal to 1.

For undefined terminology and the results related to graph spectra, we follow [1].

Definition 1.1. [3] The sunlet graph $S_q; q \geq 3$ is a graph with $2q$ vertices, constructed by attaching q pendant edges to the vertices of the cycle C_q. An illustration of the sunlet graph is provided in Figure 1.

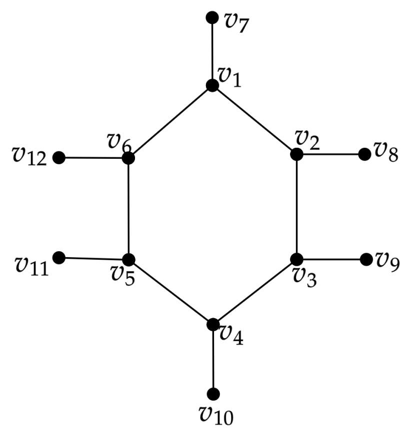

FIGURE 1. Sunlet graph S_6

Definition 1.2. [3] The pentagonal snake graph $PS_q; q \geq 2$ is a graph with $4q-3$ vertices, constructed by replacing each edge of the path P_q with a pentagonal cycle C_5. An illustration of the pentagonal snake graph is provided in Figure 2.

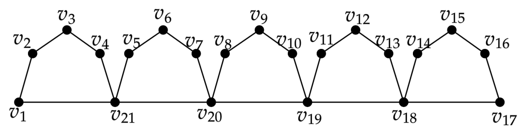

FIGURE 2. Pentagonal snake graph PS_6

Definition 1.3. [3] The book graph $B_q; q \geq 1$ is a graph with $2q+2$ vertices, constructed by joining q copies of the cycle C_4, all sharing one common edge. An illustration of the book graph is provided in Figure 3.

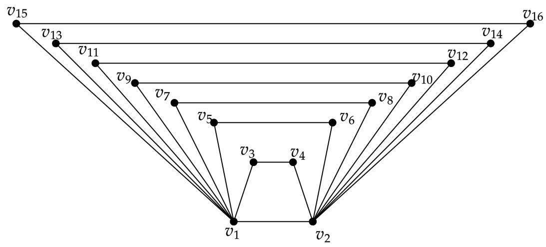

FIGURE 3. Book graph B_7

Definition 1.4. [3] The windmill graph $W_q^3; q \geq 2$ is a graph with $3q-2$ vertices, constructed by joining three copies of the complete graph K_q at a single common vertex. An illustration of the windmill graph is provided in Figure 4.

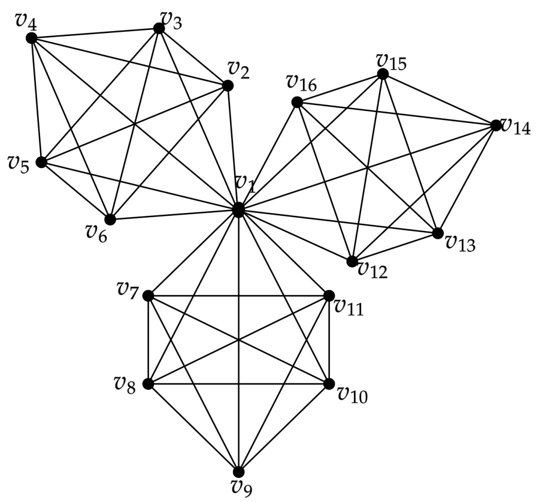

FIGURE 4. Windmill graph W_6^3

Definition 1.5. [3] The friendship graph $F_q; q \geq 1$ is a graph with $2q+1$ vertices, constructed by joining q copies of the cycle C_3 at a single common vertex. An illustration of the friendship graph is provided in Figure 5.

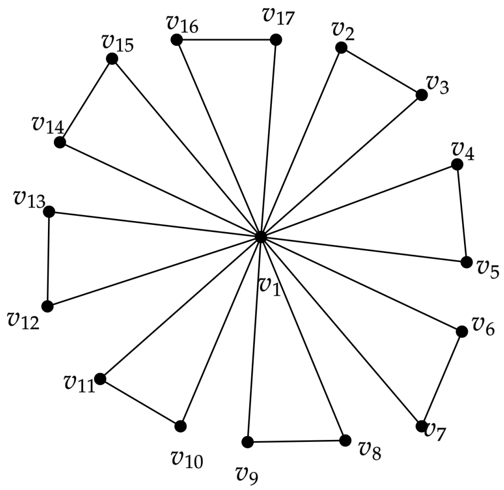

FIGURE 5. Friendship graph F_8

2. DEGREE SUM SPECTRA AND DEGREE SUM ENERGY OF GRAPHS

In the subsequent discussion, we compute the degree sum spectrum and the degree sum energy of some class of graphs.

Theorem 2.1. *The degree sum spectrum of the sunlet graph S_q is as follows:*

(1) $-6; q-1$ *times*
(2) $-2; q-1$ *times and*
(3) $4(q-1) \pm 2\sqrt{5q^2 - 2q + 1}$.

The degree sum energy of S_q is:

$$\mathcal{E}_{DS}(S_q) = 8(q-1) + 4\sqrt{5q^2 - 2q + 1}.$$

Proof. The degree sum matrix of S_q is

$$DS(S_q) = \begin{pmatrix} 0 & 6 & 6\cdots 6 & 4 & 4 & 4\cdots 4 \\ 6 & 0 & 6\cdots 6 & 4 & 4 & 4\cdots 4 \\ 6 & 6 & 0\cdots 6 & 4 & 4 & 4\cdots 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 6 & 6 & 6\cdots 0 & 4 & 4 & 4\cdots 4 \\ 4 & 4 & 4\cdots 4 & 0 & 2 & 2\cdots 2 \\ 4 & 4 & 4\cdots 4 & 2 & 0 & 2\cdots 2 \\ 4 & 4 & 4\cdots 4 & 2 & 2 & 0\cdots 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4 & 4 & 4\cdots 4 & 2 & 2 & \cdots 0 \end{pmatrix}.$$

This matrix can be written as,

$$DS(S_q) = \begin{pmatrix} 6J_q - 6I_q & 4J_{q\times q} \\ 4J_{q\times q} & 2J_q - 2I_q \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(S_q)}(\alpha) &= \mid \alpha I - DS(S_q) \mid \\ &= \begin{vmatrix} (\alpha+6)I_q - 6J_q & -4J_{q\times q} \\ -4J_{q\times q} & (\alpha+2)I_q - 2J_q \end{vmatrix}. \end{aligned}$$

Now, by Lemma 1.2, we obtain

$$\begin{aligned} P_{DS(S_q)}(\alpha) &= (\alpha+6)^{q-1}(\alpha+2)^{q-1}((\alpha-6(q-1))(\alpha-2(q-1))-16q^2) \\ &= (\alpha+6)^{q-1}(\alpha+2)^{q-1}(\alpha^2-8(q-1)\alpha-4(q^2+6q-3)), \end{aligned}$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of S_q. □

Theorem 2.2. *The degree sum spectrum of the complement of the sunlet graph* $\overline{S_q}$ *is as follows:*

(1) $-4(q-2); q-1$ *times*
(2) $-4(q-1); q-1$ *times and*
(3) $2(q-1)(2q-3) \pm \sqrt{4(q-1)^2(2q-3)^2 - 16(q-1)^3(q-2) + q^2(4q-6)^2}$.

The degree sum energy of $\overline{S_q}$ *is:*

$$\mathcal{E}_{DS}(\overline{S_q}) = 4(q-1)(2q-3) + 2\sqrt{4(q-1)^2(2q-3)^2 - 16(q-1)^3(q-2) + q^2(4q-6)^2}.$$

Proof. The degree sum matrix of $\overline{S_q}$ is

$$DS(\overline{S_q}) = \begin{pmatrix} 0 & 4(q-2) & 4(q-2) & \cdots & 4(q-2) & 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 \\ 4(q-2) & 0 & 4(q-2) & \cdots & 4(q-2) & 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 \\ 4(q-2) & 4(q-2) & 0 & \cdots & 4(q-2) & 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4(q-2) & 4(q-2) & 4(q-2) & \cdots & 0 & 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 \\ 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 & 0 & 4(q-1) & 4(q-1) & \cdots & 4(q-1) \\ 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 & 4(q-1) & 0 & 4(q-1) & \cdots & 4(q-1) \\ 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 & 4(q-1) & 4(q-1) & 0 & \cdots & 4(q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4q-6 & 4q-6 & 4q-6 & \cdots & 4q-6 & 4(q-1) & 4(q-1) & 4(q-1) & \cdots & 0 \end{pmatrix}.$$

This matrix can be written as,

$$DS(\overline{S_q}) = \begin{pmatrix} 4(q-2)J_q - 4(q-2)I_q & (4q-6)J_{q\times q} \\ (4q-6)J_{q\times q} & 4(q-1)J_q - 4(q-1)I_q \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(\overline{S_q})}(\alpha) &= \mid \alpha I - DS(\overline{S_q}) \mid \\ &= \begin{vmatrix} (\alpha+4(q-2))I_q - 4(q-2)J_q & (4q-6)J_{q\times q} \\ (4q-6)J_{q\times q} & (\alpha+4(q-1))I_q - 4(q-1)J_q \end{vmatrix}. \end{aligned}$$

Now, by Lemma 1.2, we obtain

$$\begin{aligned} P_{DS(\overline{S_q})}(\alpha) &= (\alpha+4(q-2))^{q-1}(\alpha+4(q-1))^{q-1}((\alpha-4(q-1)(q-2)) \\ &\quad (\alpha-4(q-1)^2) - q^2(4q-6)^2) \\ &= (\alpha+4(q-2))^{q-1}(\alpha+4(q-1))^{q-1}(\alpha^2-4(q-1)(2q-3)\alpha \\ &\quad +16(q-1)^3(q-2) - q^2(4q-6)^2), \end{aligned}$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of $\overline{S_q}$. □

Theorem 2.3. *The degree sum spectrum of the book graph B_q is as follows:*

(1) $-2(q+1)$,
(2) $-4; 2q-1$ *times and*
(3) $5q-1 \pm \sqrt{(5q-1)^2 - 4(2(2q-1)(q+1) + q(q+3)^2)}$.

The degree sum energy of B_q is:

$$\mathcal{E}_{DS}(B_q) = 2(5q-1) + 2\sqrt{(5q-1)^2 - 4(2(2q-1)(q+1) + q(q+3)^2)}.$$

Proof. The degree sum matrix of B_q is

$$DS(B_q) = \begin{pmatrix} 0 & 2(q+1) & q+3 & q+3 & q+3 & \cdots & q+3 \\ 2(q+1) & 0 & q+3 & q+3 & q+3 & \cdots & q+3 \\ q+3 & q+3 & 0 & 4 & 4 & \cdots & 4 \\ q+3 & q+3 & 4 & 0 & 4 & \cdots & 4 \\ q+3 & q+3 & 4 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q+3 & q+3 & 4 & 4 & 4 & \cdots & 0 \end{pmatrix}.$$

This matrix can be written as,

$$DS(B_q) = \begin{pmatrix} 2(q+1)J_2 - 2(q+1)I_2 & (q+3)J_{2\times 2q} \\ (q+3)J_{2q\times 2} & 4J_{2q} - 4I_{2q} \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(B_q)}(\alpha) &= \mid \alpha I - DS(B_q) \mid \\ &= \begin{vmatrix} (\alpha + 2(q+1))I_2 - 2(q+1)J_2 & -(q+3)J_{2\times 2q} \\ -(q+3)J_{2q\times 2} & (\alpha+4)I_{2q} - 4J_{2q} \end{vmatrix}. \end{aligned}$$

Now, by Lemma 1.2, we obtain,

$$\begin{aligned} P_{DS(B_q)}(\alpha) &= (\alpha + 2(q+1))^{2-1}(\alpha+4)^{2q-1}((\alpha - 2(q+1))(\alpha - 4(2q-1)) - 4q(q+3)^2). \\ &= (\alpha + 2(q+1))(\alpha+4)^{2q-1}(\alpha^2 - (2(q+1) + 4(2q-1))\alpha + 8(2q-1)(q+1) \\ &\quad -4q(q+3)^2), \end{aligned}$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of B_q. □

Theorem 2.4. *The degree sum spectrum of the complement of the book graph $\overline{B_q}$ is as follows:*

(1) $-2q$,
(2) $-2(2q-1); 2q-1$ *times and*
(3) $4q^2 - 3q + 1 \pm \sqrt{(4q^2 - 3q + 1)^2 - 4q((2q-1)^2 - (3q-1)^2)}$.

The degree sum energy of $\overline{B_q}$ is:

$$\mathcal{E}_{DS}(\overline{B_q}) = 2(4q^2 - 3q + 1) + 2\sqrt{(4q^2 - 3q + 1)^2 - 4q((2q-1)^2 - (3q-1)^2)}.$$

Proof. The degree sum matrix of $\overline{B_q}$ is

$$DS(\overline{B_q}) = \begin{pmatrix} 0 & 2q & 3q-1 & 3q-1 & \cdots & 3q-1 \\ 2q & 0 & 3q-1 & 3q-1 & \cdots & 3q-1 \\ 3q-1 & 3q-1 & 0 & 2(2q-1) & \cdots & 2(2q-1) \\ 3q-1 & 3q-1 & 2(2q-1) & 0 & \cdots & 2(2q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 3q-1 & 3q-1 & 2(2q-1) & 2(2q-1) & \cdots & 0 \end{pmatrix}.$$

This matrix can be written as,

$$DS(\overline{B_q}) = \begin{pmatrix} 2qJ_2 - 2qI_2 & (3q-1)J_{2\times 2q} \\ (3q-1)J_{2q\times 2} & 2(2q-1)J_{2q} - 2(2q-1)I_{2q} \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(\overline{B_q})}(\alpha) &= \mid \alpha I - DS(\overline{B_q}) \mid \\ &= \begin{vmatrix} (\alpha+2q)I_2 - 2qJ_2 & -(3q-1)J_{2\times 2q} \\ -(3q-1)J_{2q\times 2} & (\alpha+2(2q-1))I_{2q} - 2(2q-1)J_{2q} \end{vmatrix}. \end{aligned}$$

Now, by Lemma 1.2, we obtain,

$$\begin{aligned} P_{DS(\overline{B_q})}(\alpha) &= (\alpha+2q)^{2-1}(\alpha+2(2q-1))^{2q-1}((\alpha-2q)(\alpha-2(2q-1)^2) - 4q(3q-1)^2) \\ &= (\alpha+2q)(\alpha+2(2q-1))^{2q-1}(\alpha^2 - (2q+2(2q-1)^2)\alpha + 4q((2q-1)^2 \\ &\quad -(3q-1)^2), \end{aligned}$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of $\overline{B_q}$. □

Theorem 2.5. *The degree sum spectrum of the pentagonal snake graph PS_q is as follows:*
(1) $-4; 3q-2$ *times*
(2) $-8; q-3$ *times and*
(3) $2(5q-8) \pm 2\sqrt{(5q-8)^2 + 3q^2 + 25q - 30}$.
The degree sum energy of PS_q is:

$$\mathcal{E}_{DS}(PS_q) = 4(5q-8) + 4\sqrt{(5q-8)^2 + 3q^2 + 25q - 30}.$$

Proof. The degree sum matrix of PS_q is

$$DS(PS_q) = \begin{pmatrix} 0 & 4 & 4\cdots 4 & 6 & 6 & 6 & \cdots & 6 \\ 4 & 0 & 4\cdots 4 & 6 & 6 & 6 & \cdots & 6 \\ 4 & 4 & 0\cdots 4 & 6 & 6 & 6 & \cdots & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4 & 4 & 4\cdots 0 & 6 & 6 & 6 & \cdots & 6 \\ 6 & 6 & 6\cdots 6 & 0 & 8 & 8 & \cdots & 8 \\ 6 & 6 & 6\cdots 6 & 8 & 0 & 8 & \cdots & 8 \\ 6 & 6 & 6\cdots 6 & 8 & 8 & 0 & \cdots & 8 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 6 & 6 & 6\cdots 6 & 8 & 8 & 8 & \cdots & 0 \end{pmatrix}.$$

This matrix can be written as,

$$DS(PS_q) = \begin{pmatrix} 4J_{3q-1} - 4I_{3q-1} & 6J_{3q-1\times q-2} \\ 6J_{q-2\times 3q-1} & 8J_{q-2} - 8I_{q-2} \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(PS_q)}(\alpha) &= \mid \alpha I - DS(PS_q) \mid \\ &= \begin{vmatrix} (\alpha+4)I_{3q-1} - 4J_{3q-1} & -6J_{(3q-1)\times(q-2)} \\ -6J_{(q-2)\times(3q-1)} & (\alpha+8)I_{q-2} - 8J_{q-2} \end{vmatrix}. \end{aligned}$$

Now, by Lemma 1.2, we obtain,

$$\begin{aligned} P_{DS(PS_q)}(\alpha) &= (\alpha+4)^{3q-1-1}(\alpha+8)^{q-2-1}((\alpha-4(3q-2))(\alpha-8(q-3))-36(q-2) \\ &\quad (3q-1)) \\ &= (\alpha+4)^{3q-2}(\alpha+8)^{q-3}((\alpha^2-4(5q-8))\alpha-4(3q^2+25q-30)), \end{aligned}$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of PS_q. □

Theorem 2.6. *The degree sum spectrum of the complement of the pentagonal snake graph $\overline{PS_q}$ is as follows:*

(1) $-4(2q-3); 3q-2$ *times*
(2) $-8(q-2); q-3$ *times and*
(3) $2(8q^2-23q+18) \pm 2\sqrt{(8q^2-23q+18)^2+(64q^3-285q^2+409q-190)}$.

The degree sum energy of $\overline{PS_q}$ is:

$$\mathcal{E}_{DS}(\overline{PS_q}) = 4(8q^2-23q+18) + 4\sqrt{(8q^2-23q+18)^2+(64q^3-285q^2+409q-190)}.$$

Proof. The degree sum matrix of $\overline{PS_q}$ is

$$DS(\overline{PS_q}) = \begin{pmatrix} 0 & 4(2q-3) & \cdots & 4(2q-3) & 2(4q-7) & 2(4q-7) & \cdots & 2(4q-7) \\ 4(2q-3) & 0 & \cdots & 4(2q-3) & 2(4q-7) & 2(4q-7) & \cdots & 2(4q-7) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4(2q-3) & 4(2q-3) & \cdots & 0 & 2(4q-7) & 2(4q-7) & \cdots & 2(4q-7) \\ 2(4q-7) & 2(4q-7) & \cdots & 2(4q-7) & 0 & 8(q-2) & \cdots & 8(q-2) \\ 2(4q-7) & 2(4q-7) & \cdots & 2(4q-7) & 8(q-2) & 0 & \cdots & 8(q-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2(4q-7) & 2(4q-7) & \cdots & 2(4q-7) & 8(q-2) & 8(q-2) & \cdots & 0 \end{pmatrix}.$$

This matrix can be written as,

$$DS(\overline{PS_q}) = \begin{pmatrix} 4(2q-3)J_{3q-1} - 4(2q-3)I_{3q-1} & 2(4q-7)J_{3q-1\times q-2} \\ 2(4q-7)J_{q-2\times 3q-1} & 8(q-2)J_{q-2} - 8(q-2)I_{q-2} \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(\overline{PS_q})}(\alpha) &= \mid \alpha I - DS(S_q) \mid \\ &= \begin{vmatrix} (\alpha+4(2q-3))I_q - 4(2q-3)J_q & -2(4q-7)J_{q\times q} \\ -2(4q-7)J_{q\times q} & (\alpha+8(q-2))I_q - 8(q-2)J_q \end{vmatrix}. \end{aligned}$$

Now, by Lemma 1.2, we obtain,

$$\begin{aligned} P_{DS(\overline{PS_q})}(\alpha) &= (\alpha+4(2q-3))^{3q-1-1}(\alpha+8(q-2))^{q-2-1}((\alpha-8(q-2)(q-3)) \\ &\quad (\alpha-4(2q-3)(3q-2)) - 4(4q-7)^2(q-2)(3q-1)) \\ &= (\alpha+4(2q-3))^{3q-2}(\alpha+8(q-2))^{q-3}(\alpha^2-4\alpha(8q^2-23q+18) \\ &\quad -4(64q^3-284q^2+409q-190)), \end{aligned}$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of $\overline{PS_q}$. □

Theorem 2.7. *The degree sum spectrum of the windmill graph W_q^3 is as follows:*

(1) $-2(q-1); 3q-4$ *times and*
(2) $(q-1)(3q-4) \pm \sqrt{(q-1)^2(3q-4)^2+48(q-1)^3}$.

The degree sum energy of W_q^3 *is:*

$$\mathcal{E}_{DS}(W_q^3) = 2(q-1)(3q-4) + 2\sqrt{(q-1)^2(3q-4)^2+48(q-1)^3}.$$

Proof. The degree sum matrix of W_q^3 is

$$DS(W_q^3) = \begin{pmatrix} 0 & 4(q-1) & 4(q-1) & 4(q-1) & \cdots & 4(q-1) \\ 4(q-1) & 0 & 2(q-1) & 2(q-1) & \cdots & 2(q-1) \\ 4(q-1) & 2(q-1) & 0 & 2(q-1) & \cdots & 2(q-1) \\ 4(q-1) & 2(q-1) & 2(q-1) & 0 & \cdots & 2(q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4(q-1) & 2(q-1) & 2(q-1) & 2(q-1) & \cdots & 0 \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(W_q^3)}(\alpha) &= \mid \alpha I - DS(W_q^3) \mid \\ &= \begin{vmatrix} \alpha & -4(q-1) & -4(q-1) & -4(q-1) & \cdots & -4(q-1) \\ -4(q-1) & \alpha & -2(q-1) & -2(q-1) & \cdots & -2(q-1) \\ -4(q-1) & -2(q-1) & \alpha & -2(q-1) & \cdots & -2(q-1) \\ -4(q-1) & -2(q-1) & -2(q-1) & \alpha & \cdots & -2(q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -4(q-1) & -2(q-1) & -2(q-1) & -2(q-1) & \cdots & \alpha \end{vmatrix}. \end{aligned}$$

Applying the elementary row transformations $R_i + \dfrac{4(q-1)}{\alpha} R_1$, for $i = 2, 3, \cdots, 3q-1$, we obtain

$$P_{DS(W_q^3)} = \begin{vmatrix} \alpha & -4(q-1) & -4(q-1) & \cdots & -4(q-1) \\ 0 & \alpha - \frac{16(q-1)^2}{\alpha} & -2(q-1) - \frac{16(q-1)^2}{\alpha} & \cdots & -2(q-1) - \frac{16(q-1)^2}{\alpha} \\ 0 & -2(q-1) - \frac{16(q-1)^2}{\alpha} & \alpha - \frac{16(q-1)^2}{\alpha} & \cdots & -2(q-1) - \frac{16(q-1)^2}{\alpha} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -2(q-1) - \frac{16(q-1)^2}{\alpha} & -2(q-1) - \frac{16(q-1)^2}{\alpha} & \cdots & \alpha - \frac{16(q-1)^2}{\alpha} \end{vmatrix}.$$

Expanding the above determinant and applying Lemma 1.1, we obtain,

$$\begin{aligned} P_{DS(W_q^3)}(\alpha) &= \alpha \left(\alpha - \frac{16(q-1)^2}{\alpha} + 2(q-1) + \frac{16(q-1)^2}{\alpha} \right)^{3q-4} \\ &\quad \left(\alpha - \frac{16(q-1)^2}{\alpha} + (3q-4)\left(-2(q-1) - \frac{16(q-1)^2}{\alpha} \right) \right) \\ &= (\alpha + 2(q-1))^{3q-4} (\alpha^2 - (2(q-1)(3q-4)) - 48(q-1)^3), \end{aligned}$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of W_q^3. □

Theorem 2.8. *The degree sum spectrum of the complement of the windmill graph* $\overline{W_q^3}$ *is as follows:*

(1) $-4(q-1); 3q-4$ *times and*
(2) $2(q-1)(3q-4) \pm \sqrt{4(q-1)^2(3q-4)^2+12(q-1)^3}$.

The degree sum energy of $\overline{W_q^3}$ *is:*

$$\mathcal{E}_{DS}(\overline{W_q^3}) = 4(q-1)(3q-4) + 2\sqrt{4(q-1)^2(3q-4)^2 + 12(q-1)^3}.$$

Proof. The degree sum matrix of $\overline{W_q^3}$ is

$$DS(\overline{W_q^3}) = \begin{pmatrix} 0 & 2(q-1) & 2(q-1) & 2(q-1) & \cdots & 2(q-1) \\ 2(q-1) & 0 & 4(q-1) & 4(q-1) & \cdots & 2(q-1) \\ 2(q-1) & 4(q-1) & 0 & 4(q-1) & \cdots & 2(q-1) \\ 2(q-1) & 4(q-1) & 4(q-1) & 0 & \cdots & 2(q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2(q-1) & 4(q-1) & 4(q-1) & 4(q-1) & \cdots & 0 \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(\overline{W_q^3})}(\alpha) &= |\ \alpha I - DS(\overline{W_q^3})\ | \\ &= \begin{vmatrix} \alpha & -2(q-1) & -2(q-1) & \cdots & -2(q-1) \\ -2(q-1) & \alpha & -4(q-1) & \cdots & -4(q-1) \\ -2(q-1) & -4(q-1) & \alpha & \cdots & -4(q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -2(q-1) & -4(q-1) & -4(q-1) & \cdots & \alpha \end{vmatrix}. \end{aligned}$$

Applying the elementary row transformations $R_i + \dfrac{2(q-1)}{\alpha}R_1$, for $i = 2, 3, \cdots, 3q-1$.

we obtain

$$P_{DS(\overline{W_q^3})}(\alpha) = \begin{vmatrix} \alpha & -2(q-1) & -2(q-1) & \cdots & -2(q-1) \\ 0 & \alpha - \frac{4(q-1)^2}{\alpha} & -4(q-1) - \frac{4(q-1)^2}{\alpha} & \cdots & -4(q-1) - \frac{4(q-1)^2}{\alpha} \\ 0 & -4(q-1) - \frac{4(q-1)^2}{\alpha} & \alpha - \frac{4(q-1)^2}{\alpha} & \cdots & -4(q-1) - \frac{4(q-1)^2}{\alpha} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -4(q-1) - \frac{4(q-1)^2}{\alpha} & -4(q-1) - \frac{4(q-1)^2}{\alpha} & \cdots & \alpha - \frac{4(q-1)^2}{\alpha} \end{vmatrix}.$$

By following steps similar to those in the proof of Theorem 2.7, we obtain the characteristic polynomial of $DS(\overline{W_q^3})$,

$$P_{DS(\overline{W_q^3})}(\alpha) \quad = \quad (\alpha + 4(q-1))^{3q-4}(\alpha^2 - 4(q-1)(3q-4) - 12(q-1)^3),$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of $\overline{W_q^3}$. □

Theorem 2.9. *The degree sum spectrum of the friendship graph* F_q *is as follows:*

(1) $-4; 2q-1$ *times and*
(2) $2(2q-1) \pm \sqrt{4(1-2q)^2 + 2q(2q+2)^2}$.

The degree sum energy of F_q *is:*

$$\mathcal{E}_{DS}(F_q) = 4(2q-1) + 2\sqrt{4(1-2q)^2 + 2q(2q+2)^2}.$$

Proof. The degree sum matrix of F_q is

$$DS(F_q) = \begin{pmatrix} 0 & 2q+2 & 2q+2 & 2q+2 & \cdots & 2q+2 \\ 2q+2 & 0 & 4 & 4 & \cdots & 4 \\ 2q+2 & 4 & 0 & 4 & \cdots & 4 \\ 2q+2 & 4 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2q+2 & 4 & 4 & 4 & \cdots & 0 \end{pmatrix}.$$

The characteristic polynomial of the above matrix is,

$$\begin{aligned} P_{DS(F_q)}(\alpha) &= \mid \alpha I - DS(F_q) \mid \\ &= \begin{vmatrix} \alpha & -(2q+2) & -(2q+2) & \cdots & -(2q+2) \\ -(2q+2) & \alpha & -4 & \cdots & -4 \\ -(2q+2) & -4 & \alpha & \cdots & -4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(2q+2) & -4 & -4 & \cdots & \alpha \end{vmatrix}. \end{aligned}$$

Applying the elementary row transformations $R_i + \dfrac{(2q+2)}{\alpha} R_1,$ for $i = 2, 3, \cdots, 2q+1,$ we obtain

$$P_{DS(F_q)} = \begin{vmatrix} \alpha & -(2q+2) & -(2q+2) & \cdots & -(2q+2) \\ 0 & \alpha - \frac{(2q+2)^2}{\alpha} & -4 - \frac{(2q+2)^2}{\alpha} & \cdots & -4 - \frac{(2q+2)^2}{\alpha} \\ 0 & -4 - \frac{(2q+2)^2}{\alpha} & \alpha - \frac{(2q+2)^2}{\alpha} & \cdots & -4 - \frac{(2q+2)^2}{\alpha} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -4 - \frac{(2q+2)^2}{\alpha} & -4 - \frac{(2q+2)^2}{\alpha} & \cdots & \alpha - \frac{(2q+2)^2}{\alpha} \end{vmatrix}.$$

By following steps similar to those in the proof of Theorem 2.7, we obtain the characteristic polynomial of $DS(F_q)$,

$$P_{DS(F_q)} = (\alpha+4)^{2q-1}(\alpha^2 + (4-8q)\alpha - 2q(2q+2)^2),$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of F_q. □

Theorem 2.10. *The degree sum spectrum of the complement of the friendship graph* $\overline{F_q}$ *is as follows:*

(1) $-4(q-1); 2q-1$ *times and*

(2) $2(q-1)(2q-1) \pm 2\sqrt{(q-1)^2(2q-1)^2 + 2q(q-1)^2}$.

The degree sum energy of $\overline{F_q}$ *is:*

$$\mathcal{E}_{DS}(\overline{F_q}) = 4(q-1)(2q-1) + 4\sqrt{(q-1)^2(2q-1)^2 + 2q(q-1)^2}.$$

Proof. The degree sum matrix of $\overline{F}_q$ is

$$DS(\overline{F}_q) = \begin{pmatrix} 0 & 2(q-1) & 2(q-1) & 2(q-1) & \cdots & 2(q-1) \\ 2(q-1) & 0 & 4(q-1) & 4(q-1) & \cdots & 2(q-1) \\ 2(q-1) & 4(q-1) & 0 & 4(q-1) & \cdots & 2(q-1) \\ 2(q-1) & 4(q-1) & 4(q-1) & 0 & \cdots & 2(q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2(q-1) & 4(q-1) & 4(q-1) & 4(q-1) & \cdots & 0 \end{pmatrix}.$$

The characteristic polynomial ofthe above matrix is,

$$\begin{aligned} P_{DS(\overline{F}_q)}(\alpha) &= \mid \alpha I - DS(\overline{F}_q) \mid \\ &= \begin{vmatrix} \alpha & -2(q-1) & -2(q-1) & \cdots & -2(q-1) \\ -2(q-1) & \alpha & -4(q-1) & \cdots & -4(q-1) \\ -2(q-1) & -4(q-1) & \alpha & \cdots & -4(q-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -2(q-1) & -4(q-1) & -4(q-1) & \cdots & \alpha \end{vmatrix}. \end{aligned}$$

Applying the elementary row transformations $R_i + \dfrac{2(q-1)}{\alpha} R_1$, for $i = 2, 3, \cdots, 2q+1$, we obtain

$$P_{DS(\overline{F}_q)} = \begin{vmatrix} \alpha & -2(q-1) & -2(q-1) & \cdots & -2(q-1) \\ 0 & \alpha - \frac{4(q-1)^2}{\alpha} & -4(q-1) - \frac{4(q-1)^2}{\alpha} & \cdots & -4(q-1) - \frac{4(q-1)^2}{\alpha} \\ 0 & -4(q-1) - \frac{4(q-1)^2}{\alpha} & \alpha - \frac{4(q-1)^2}{\alpha} & \cdots & -4(q-1) - \frac{4(q-1)^2}{\alpha} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -4(q-1) - \frac{4(q-1)^2}{\alpha} & -4(q-1) - \frac{4(q-1)^2}{\alpha} & \cdots & \alpha - \frac{4(q-1)^2}{\alpha} \end{vmatrix}.$$

By following steps similar to those in the proof of Theorem 2.7, we obtain the characteristic polynomial of $DS(\overline{F}_q)$,

$$P_{DS(\overline{F}_q)} = (\alpha + 4(q-1))^{2q-1}(\alpha^2 + 4(q-1)(1-2q)\alpha - 8q(q-1)^2),$$

which gives the required degree sum spectrum. By summing these values, we obtain the degree sum energy of $\overline{F}_q$. □

3. Conclusions

In this paper, we have examined the spectra and energy concepts associated with the degree sum matrix of graphs and their complements. This investigation can be further extended to other classes of non-regular graphs for potential applications.

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