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STRONG INCIDENCE DOMINATION INDEX IN FUZZY INCIDENCE GRAPHS

KAVYA R. NAIR[1*], M. S. SUNITHA[1], §

Abstract. This article introduces the concept of the domination index in fuzzy incidence graphs (FIGs) through the use of strong incidence domination. It explores several related notions, including fuzzy incidence irredundant set, fuzzy incidence independent set, fuzzy incidence independent dominating set, upper strong incidence domination number, strong incidence irredundance number, strong incidence upper irredundance number, strong incidence independent domination number and strong incidence independence number. The article examines inequalities involving these terms and introduces the concept of the strong incidence domination degree. It defines the strong incidence domination index in FIGs based on the domination degree of vertices and discusses bounds for the index. The study extends to complete FIGs, complete bipartite FIGs, fuzzy incidence cycles (FICs), fuzzy incidence trees (FITs), and the union and join of FIGs.

Keywords: upper strong incidence domination number, strong incidence irredundance number, strong incidence upper irredundance number, strong incidence independent domination number and strong incidence independence number, strong incidence domination degree, strong incidence domination index.

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1. Introduction

In the field of graph theory, FIGs are a captivating extension of traditional graphs, incorporating fuzzy set theory to manage uncertainty and vagueness often found in real-world problems. Unlike traditional graphs, which are based on a clear, binary relationship, many real-world connections are not strictly binary and can exist to varying degrees. The FIGs tackle this by allowing extra attributes of vertex-edge relationships and allowing vertices, edges, and pairs to have different levels of presence, represented by membership values between 0 and 1. This method effectively captures the uncertainty and imprecision in fields like social networks, biological systems, and decision-making processes, where relationships or interactions are not always well-defined. Dinesh [1] introduced the term FIG in 2016 and explored several of its properties. The concepts of connectivity and fuzzy

[1] Department of Mathematics, National Institute of Technology Calicut, Calicut-673 601, Kerala, India
e-mail: kavyarnair@gmail.com; ORCID: 0000-0002-7470-5789.
e-mail: sunitha@nitc.ac.in; ORCID: 0000-0002-0090-9252.
* corresponding author.

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end nodes were further developed by Mordeson et al. [2] Domination in graphs is a crucial concept in graph theory. A dominating set is a set of vertices D such that every vertex in $\mathcal{V} \setminus D$ has neighbor in D. This concept is important because it helps in understanding the control and influence within a network, optimizing resources, and solving problems like network coverage, facility location, and social network analysis. In the context of fuzzy incidence graphs, domination takes on additional significance due to the varying degrees of relationships between vertices. Here, domination helps in modeling and analyzing systems where connections are not binary but exist in degrees, allowing for more accurate and flexible solutions to real-world problems. There is a significant amount of research available on fuzzy incidence graphs and the domination in fuzzy incidence graphs in [1, 2, 3, 4, 5, 6]. Research on domination in vague and vague incidence graphs can be referred in [20, 21, 22, 23].

A topological index, also referred to as a molecular structure descriptor, is a numerical value that corresponds to the chemical composition and is utilized to correlate the chemical structure with various physical properties, chemical reactivity, or biological activity. Currently, numerous such indices exist in the literature. Two primary types of topological indices are distance-based and degree-based indices. The first distance-based index was the Wiener index W(G), introduced in 1974 by chemist Harry Wiener [8]. Following this, many indices emerged due to their applications in chemistry, such as the Zagreb indices, Randic, Harmonic, Gutman, and Schultz indices. Binu et al. [9] introduced Wiener and connectivity indices into fuzzy graphs. Other topological indices, including the modified and hyper-Wiener, Gutman, Schultz, Zagreb, Harmonic, and Randic indices in fuzzy graphs, are explored in other studies. Extensive research on indices can be found in the literature [11, 12, 13, 14, 10]. Most of these works have been extended to other types of fuzzy graphs, such as bipolar fuzzy graphs, fuzzy incidence graphs, and intuitionistic fuzzy graphs [7, 15, 16, 17].

Due to the significance of domination and topological indices in graph theory, it was relevant to merge these concepts to create a new topological index known as the domination index [18]. The domination index is the sum of domination degree of vertices. The domination degree of a vertex is the minimum cardinality of a minimal dominating containing that vertex. This idea was initially introduced in the context of graphs and fuzzy graphs [18, 19]. The current work extends this concept to fuzzy incidence graphs. This work integrates domination theory and topological indices by introducing a domination index in FIGs based on minimal strong incidence dominating sets. This concept is motivated by the broad applications of dominating sets across various domains.

A key application lies in facility allocation, where facilities are assigned to entities represented by vertices in a dominating set, ensuring complete coverage of the graphs. When a specific vertex must have a facility, it becomes necessary to find a dominating set that includes this vertex. To meet this requirement while maintaining minimality, identifying the smallest minimal dominating set containing the vertex is crucial. Although finding a minimum dominating set containing a specific vertex may not always be achievable, it is always possible to determine a minimal one, highlighting the flexibility and utility of this approach.

Furthermore, prioritizing certain vertices often depends on additional information derived from fuzzy graphs, and more precisely, from fuzzy incidence graphs, which provide a finer level of detail about vertex significance. This makes the extension of the domination index to fuzzy incidence graphs particularly important, as it allows for more accurate modeling and decision-making in scenarios where prioritization of vertices is necessary.

The article introduces the concept of the domination index in FIGs using strong incidence domination. Section 3 explores the notions of fuzzy incidence irredundant set, fuzzy incidence independent set, fuzzy incidence independent dominating set, upper strong incidence domination number, strong incidence irredundance number, strong incidence upper irredundance number, strong incidence independent domination number and strong incidence independence number. Inequalities involving these defined terms are examined. The idea of the strong incidence domination degree is introduced and illustrated. The strong incidence domination index is defined in FIGs using the strong incidence domination degree of vertices. Bounds for the index are discussed. The index is studied in complete fuzzy incidence graphs, complete bipartite FIGs, FICs, FITs, and the union and join of FIGs.

2. Preliminaries

The concepts and definitions are referred from [1, 2, 3, 4, 5, 6, 7].
In this article, the minimum operator is denoted by $\wedge$, and the maximum operator is denoted by $\vee$. An incidence graph(IG) is a triple $G = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ such that $\mathcal{V}$ is non-empty, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and $\mathcal{I} \subseteq \mathcal{V} \times \mathcal{E}$.
The set $\mathcal{I}$ consists of elements of the form (a, ab) such that $a \in \mathcal{V}$ and $ab \in \mathcal{E}$ and are called incidence pairs or simply pairs. An incidence subgraph, H of G is an IG having all its vertices, edges, and pairs in G. An incidence walk from a' to b' where $a', b' \in \mathcal{V} \cup \mathcal{E}$ consists of a sequence of vertices, edges and pairs starting at a' and ending at b'. An incidence path is an incidence walk with distinct vertices. A connected IG is such that each vertex is joined to every other vertex by a path. A maximally connected incidence subgraph of an IG is a component of the IG. Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. Let ψ and τ be fuzzy subsets of $\mathcal{V}$ and $\mathcal{E}$ respectively. Then $\mathcal{G} = (\mathcal{V}, \psi, \tau)$ is fuzzy graph(FG) of G if $\tau(cd) \leq \psi(c) \wedge \psi(d)$ for all $c, d \in \mathcal{V}$. Also, if $\xi(v', e') \leq \psi(v') \wedge \tau(e')$, for all $v' \in \mathcal{V}$ and $e' \in \mathcal{E}$, then ξ is the fuzzy incidence of G. And, $\tilde{\mathcal{G}} = (\psi, \tau, \xi)$ is called fuzzy incidence graph (FIG) of G. Here, ψ^*, τ^* and ξ^* are defined as $\psi^* = \{c \in \mathcal{V} : \psi(c) > 0\}$, $\tau^* = \{e' \in \mathcal{E} : \tau(e') > 0\}$, and $\xi^* = \{(c, cd) \in \mathcal{I} : \xi(c, cd) > 0\}$. If $|\psi^*| = 1$, then the FIG is called trivial. Let $cd \in \tau^*$, if $(c, cd), (d, cd) \in \xi^*$, then cd is an edge in $\tilde{\mathcal{G}}$. A FIG is connected if each vertex is connected to every other vertex by a path. A fuzzy incidence subgraph $\tilde{\mathcal{H}} = (\varphi, \eta, \zeta)$ of $\tilde{\mathcal{G}}$ is such that $\varphi \subseteq \psi, \eta \subseteq \tau,$ and $\zeta \subseteq \xi$ and $\tilde{\mathcal{H}}$ is a fuzzy incidence spanning subgraph of $\tilde{\mathcal{G}}$ if $\varphi = \psi$. If $\varphi = \psi, \eta = \tau,$ and $\zeta = \xi$ for elements in $\varphi^*, \eta^*, \zeta^*$ respectively, then $\tilde{\mathcal{H}}$ is a subgraph of $\tilde{\mathcal{G}}$. A complete fuzzy incidence graph (CFIG) $\tilde{\mathcal{G}}$ is such that $\xi(c, cd) = \wedge\{\psi(c), \tau(cd)\}$ for all $(c, cd) \in \mathcal{V} \times \mathcal{E}$ and $\tau(cd) = \psi(c) \wedge \psi(d)$ for all $(c, d) \in \mathcal{V} \times \mathcal{V}$. A pair (c, cd) is an effective pair if $\xi(c, cd) = \wedge\{\psi(c), \tau(cd)\}$. In a FIG $\tilde{\mathcal{G}}$, a path from s' to t' where $s', t' \in \psi^* \cup \tau^*$, is called an incidence path. The incidence strength of an incidence path is the minimum ξ values of pairs in the path. Here, $\xi^\infty(a, cd)$ or $ICONN_{\tilde{\mathcal{G}}}(a, cd)$ is denoted as the incidence strength of path from a to cd of greatest incidence strength. If $\tilde{\mathcal{G}}^* = (\psi^*, \tau^*, \xi^*)$ is a cycle, then $\tilde{\mathcal{G}}$ is a cycle. If in addition there exists no unique edge in $\tilde{\mathcal{G}}$ with least weight then, $\tilde{\mathcal{G}}$ is a fuzzy cycle(FC). A fuzzy incidence cycle(FIC) is a FC $\tilde{\mathcal{G}} = (\psi, \tau, \xi)$ such that there is no unique pair with the least weight. A FIG, $\tilde{\mathcal{G}} = (\psi, \tau, \xi)$ is a forest if (ψ^*, τ^*, ξ^*) is a forest and a tree if it is also connected. If the FIG, $\tilde{\mathcal{G}} = (\psi, \tau, \xi)$ has a fuzzy incidence spanning subgraph $\tilde{\mathcal{F}} = (\psi, \tau, \zeta)$ which is a forest and $\xi(c, cd) < \zeta^\infty(c, cd)$ for $(c, cd) \in \xi^* \setminus \zeta^*$, then $\tilde{\mathcal{G}}$ is called a fuzzy incidence forest (FIF) and a fuzzy incidence tree (FIT), if it is also connected. The spanning subgraph $\tilde{\mathcal{F}}$ is uniquely determined for FITs and is called unique maximum spanning tree. Let $\tilde{\mathcal{G}} = (\psi, \tau, \xi)$ be a FIG, $\xi'^\infty(a, cd)$ is the greatest incidence strength among the incidence strength of all paths from a to cd in $\tilde{\mathcal{G}} \setminus (a, cd)$.

If $\xi(c, cd) > \xi'^{\infty}(c, cd)$, then the pair (c, cd) is $\alpha-$ strong. Pair (c, cd) is $\beta-$ strong if $\xi(c, cd) = \xi'^{\infty}(c, cd)$, and is a $\delta-$ pair if $\xi(c, cd) < \xi'^{\infty}(c, cd)$. A strong pair is $\alpha-$ strong or $\beta-$ strong pair. A strong incidence path(SIP) is a path consisting of only strong pairs. A strong fuzzy incidence graph is a FIG consisting of only strong pairs. Two vertices c and d are called strong incidence neighbors (SIN) if (c, cd) and (d, cd) are strong. The strong incidence neighborhood of c, $N_{IS}(c) = \{d \in \mathcal{V} : d \text{is SIN of} c\}$. Vertex c dominates d if either $c = d$ or c is SIN of d. Isolated vertex x is such that $N_{IS}(x) = \phi$. A set $\tilde{\mathcal{D}} \subseteq \mathcal{V}$ in $\tilde{\mathcal{G}}$ is a strong incidence dominating set (SIDS) if for any $c \in V - \tilde{\mathcal{D}}$, $\exists$ some $d \in \tilde{\mathcal{D}}$ such that, c is a SIN of d. Here, $W(\tilde{\mathcal{D}})$ is the weight of SIDS, $\tilde{\mathcal{D}}$, defined as

$$W(\tilde{\mathcal{D}}) = \sum_{c \in \tilde{\mathcal{D}}} \xi(c, cd)$$

where $\xi(c, cd)$ is minimum weight of strong pairs at c and $d \in N_{IS}(c)$. The strong incidence domination number (SIDN), denoted as $\gamma_{IS}(\tilde{\mathcal{G}})$ or γ_{IS} is the minimum weight of the SIDSs in the FIG, $\tilde{\mathcal{G}}$. A minimum SIDS is a SIDS with minimum weight. A minimal SIDS(MSIDS) $\tilde{\mathcal{D}}$ is such that no other SIDS is properly contained by $\tilde{\mathcal{D}}$. The Wiener index $(\mathcal{WI})$ of FIG $\tilde{\mathcal{G}}$ is defined as:

$$\mathcal{WI}(\tilde{\mathcal{G}}) = \sum_{c, d \in \xi^*} \psi(c)\psi(d) d_s(c, d)$$

where $d_s(c, d)$ is the weight of the strong geodesics from c to d whose sum is minimum. Let $\tilde{\mathcal{G}_1} = (\psi_1, \tau_1, \xi_1)$ and $\tilde{\mathcal{G}_2} = (\psi_2, \tau_2, \xi_2)$ be two FIGs. Then the join of $\tilde{\mathcal{G}_1}$ and $\tilde{\mathcal{G}_2}$ denoted as $\tilde{\mathcal{G}_1} + \tilde{\mathcal{G}_2}$ is the FIG, $\tilde{\mathcal{G}} = (\psi, \tau, \xi)$ such that:

$$\psi(c) = \begin{cases} \psi_1(c) & if \quad c \in \tilde{\mathcal{G}_1} \\ \psi_2(c) & if \quad c \in \tilde{\mathcal{G}_2} \end{cases}$$

$$\tau(cd) = \begin{cases} \tau_1(cd) & if \quad c, d \in \tilde{\mathcal{G}_1} \\ \tau_2(cd) & if \quad c, d \in \tilde{\mathcal{G}_2} \\ \psi_1(c) \wedge \psi_2(d) & if \quad c \in \tilde{\mathcal{G}_1} \text{ and } d \in \tilde{\mathcal{G}_2} \end{cases}$$

$$\xi(c, cd) = \begin{cases} \xi_1(c, cd) & if \quad (c, cd) \in \tilde{\mathcal{G}_1} \\ \xi_2(c, cd) & if \quad (c, cd) \in \tilde{\mathcal{G}_2} \\ \psi_1(c) \wedge \psi_2(d) \wedge \xi_1(c, cc_i) & if \quad c, c_i \in \tilde{\mathcal{G}_1} \text{ and } d \in \tilde{\mathcal{G}_2} \\ & \qquad \text{where } c_i \in \tilde{\mathcal{G}_1} \end{cases}$$

3. Strong incidence domination index in fuzzy incidence graphs

The section begins with the definitions of fuzzy incidence irredundant set, fuzzy incidence independent set, fuzzy incidence independent dominating set, upper strong incidence domination number, strong incidence irredundance number, strong incidence upper irredundance number, strong incidence independent domination number, and strong incidence independence number. These terms are explained and illustrated using examples. Some properties of the defined sets are explored. The notion of strong incidence domination degree and strong incidence domination index are introduced. Some inequalities involving the defined parameters are discussed. The study is extended to FIGs such as complete fuzzy incidence graphs, complete bipartite fuzzy incidence graphs, FICs, FITs. Operations such as join and union are also considered in the study.

Definition 3.1. *Let $\tilde{\mathcal{G}}$ be a FIG. The private neighborhood of a vertex c with respect to set $\tilde{\mathcal{S}} \subseteq \mathcal{V}$ denoted as $P_{N_{IS}}[c, \tilde{\mathcal{S}}] = N_{IS}[c] \setminus \bigcup_{d \in \phi^* \setminus c} N_{IS}[d]$. A set $\tilde{\mathcal{S}}$ is called fuzzy incidence*

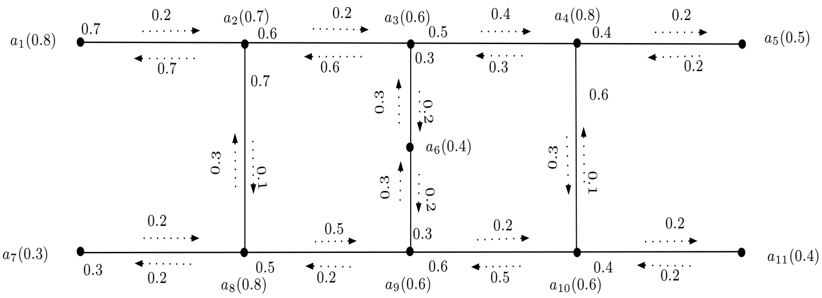

FIGURE 1. Illustration for $ir_{IS}, IR_{IS}, \gamma_{IS}, \Gamma_{IS}, i_{IS}$ and β_{IS}.

irredundant set (FIIRS) if for all $c \in \tilde{\mathcal{S}}$ $P_{N_{IS}}[c, \tilde{\mathcal{S}}] \neq \phi$. *A maximal fuzzy incidence irredundant set (MFIIRS)* $\tilde{\mathcal{S}}$ *is a FIIRS such that for any* $d \in \mathcal{V} \setminus \tilde{\mathcal{S}}$, $\tilde{\mathcal{S}} \cup \{d\}$ *is not a FIIRS. If any one pair between vertices c and d is not strong then c and d are fuzzy incidence independent. A fuzzy incidence independent set (FIIS)* $\tilde{\mathcal{M}}$, *is a set of vertices such that that every pair of vertex in* $\tilde{\mathcal{M}}$ *are fuzzy incidence independent. A fuzzy incidence independent dominating set (FIIDS) is a set that is both fuzzy incidence independent and SIDS.*
A SIDS $\tilde{\mathcal{S}}$ *is called maximal if* $\tilde{\mathcal{S}}$ *is not properly contained in any other SIDS. The maximum weight of all MSIDSs is called the upper strong incidence domination number denoted as* Γ_{IS}. *The minimum of weight of maximal fuzzy incidence irredundant sets of a FIG,* $\tilde{\mathcal{G}}$ *is called strong incidence irredundance number* $ir_{IS}(\tilde{\mathcal{G}})$. *The maximum of weight of fuzzy incidence irredundant sets of a FIG,* $\tilde{\mathcal{G}}$ *is called strong incidence upper irredundance number* $IR_{IS}(\tilde{\mathcal{G}})$. *The minimum of weight of fuzzy incidence independent dominating sets in FIG,* $\tilde{\mathcal{G}}$ *is called strong incidence independent domination number* $i_{IS}(\tilde{\mathcal{G}})$. *The maximum of weight of fuzzy incidence independent sets in FIG,* $\tilde{\mathcal{G}}$ *is called strong incidence independence number* $\beta_{IS}(\tilde{\mathcal{G}})$.

Example 3.2 gives the illustration of definitions in Definition 3.1.

Example 3.2. *For the FIG,* $\tilde{\mathcal{G}}$ *in Figure 1, a MFIIRS with minimum weight is* $\{a_2, a_3, a_8, a_9\}$. *Hence* $ir_{IS}(\tilde{\mathcal{G}}) = 0.8$. *A SIDS with minimum weight is* $\{a_2, a_8, a_6, a_4, a_{10}\}$ *with weight 1. Hence* $\gamma_{IS} = 1$. *Similarly* $i_{IS} = 1$. *Now, a FIIS with maximum weight is* $\{a_1, a_7, a_5, a_{11}, a_9, a_3\}$ *with weight 1.2. Hence* $\beta_{IS} = 1.2$. *Similarly the set* $\{a_1, a_7, a_5, a_{11}, a_9, a_3\}$ *is a MSIDS having maximum weight as well as a FIIRS having maximum weight. Therefore* $\Gamma_{IS} = IR_{IS} = 1.2$.

Theorem 3.3. [6] *Let* $\tilde{\mathcal{G}}$ *be a FIG without isolated vertices and* $\tilde{\mathcal{S}}$ *be a MSIDS in* $\tilde{\mathcal{G}}$. *Then* $\mathcal{V} \setminus \tilde{\mathcal{S}}$ *is a SIDS.*

Now, some properties of the sets defined in Definition 3.1 is discussed.

Theorem 3.4. *Let* $\tilde{\mathcal{S}}$ *be a FIIS in* $\tilde{\mathcal{G}}$. *Then* $\tilde{\mathcal{S}}$ *is maximal iff it is a SIDS.*

Proof. Suppose $\tilde{\mathcal{S}}$ be a maximal FIIS. Let u be any vertex in $\mathcal{V} \setminus \tilde{\mathcal{S}}$. Then $\tilde{\mathcal{S}} \cup \{u\}$ is not a FIIS. Therefore there exists a $v \in \tilde{\mathcal{S}}$ such that (u, uv) and (v, uv) are strong pairs which implies $\tilde{\mathcal{S}}$ is a SIDS.
Conversely suppose that $\tilde{\mathcal{S}}$ is a FIIS which is a SIDS. Clearly, since $\tilde{\mathcal{S}}$ is SIDS, for any vertex u in $\mathcal{V} \setminus \tilde{\mathcal{S}}$ there exists a $v \in \tilde{\mathcal{S}}$ such that (u, uv) and (v, uv) are strong pairs. Hence $\tilde{\mathcal{S}} \cup \{u\}$ is not a FIIS. Therefore $\tilde{\mathcal{S}}$ is maximal FIIS. Hence the result. □

Theorem 3.5. *Let* $\tilde{\mathcal{S}}$ *be a maximal FIIS. Then* $\tilde{\mathcal{S}}$ *is MSIDS.*

Proof. Suppose $\tilde{\mathcal{S}}$ be a maximal FIIS. From Theorem 3.4, $\tilde{\mathcal{S}}$ is a SIDS. Suppose $\tilde{\mathcal{S}}$ is not minimal. Then there exists a $v \in \tilde{\mathcal{S}}$ such that $\tilde{\mathcal{S}} \setminus \{v\}$ is a SIDS. This implies that there exists a $u \in \tilde{\mathcal{S}}$ which is a SIN of v. And this contradicts the fact that $\tilde{\mathcal{S}}$ is a FIIS. Hence $\tilde{\mathcal{S}}$ is a MSIDS. □

Theorem 3.6. *Let $\tilde{\mathcal{G}}$ be a FIG without isolated vertices. If $\tilde{\mathcal{S}}$ is a FIIRS, then $\mathcal{V} \setminus \tilde{\mathcal{S}}$ is a SIDS.*

Proof. Suppose $\tilde{\mathcal{G}}$ be a FIG without isolated vertices and $\tilde{\mathcal{S}}$ is a FIIRS. If $\mathcal{V} \setminus \tilde{\mathcal{S}}$ is not a SIDS, there exists a vertex $v \in \tilde{\mathcal{S}}$ that has no SIN in $\mathcal{V} \setminus \tilde{\mathcal{S}}$. Since $\tilde{\mathcal{S}}$ is a FIIRS, $P_{N_{IS}}[v, \tilde{\mathcal{S}}] = \{v\}$. It implies that v is isolated which is a contradiction. Hence $\mathcal{V} \setminus \tilde{\mathcal{S}}$ is a SIDS. □

Theorem 3.7. *Let $\tilde{\mathcal{S}}$ be a SIDS in FIG $\tilde{\mathcal{G}}$. Then $\tilde{\mathcal{S}}$ is a MSIDS iff it is a FIIRS.*

Proof. Suppose $\tilde{\mathcal{S}}$ be a MSIDS in $\tilde{\mathcal{G}}$. Assume $v \in \tilde{\mathcal{S}}$ and $P_{N_{IS}}[v, \tilde{\mathcal{S}}] = \phi$. It implies that v and its SINs are dominated by vertices of $\tilde{\mathcal{S}}$. Hence $\tilde{\mathcal{S}} \setminus \{v\}$ is also a SIDS which is a contradiction.
Conversely suppose that $\tilde{\mathcal{S}}$ is a SIDS which is also a FIIRS. Assume that $\tilde{\mathcal{S}}$ is not minimal. Then there exists a vertex $v \in \tilde{\mathcal{S}}$ such that $\tilde{\mathcal{S}} \setminus \{v\}$ is a SIDS. It implies that $P_{N_{IS}}[v, \tilde{\mathcal{S}}] = \phi$ which is a contradiction to the assumption. Hence the result. □

Theorem 3.8. *Let $\tilde{\mathcal{S}}$ be a MSIDS in FIG $\tilde{\mathcal{G}}$. Then $\tilde{\mathcal{S}}$ is MFIIRS.*

Proof. Suppose that $\tilde{\mathcal{S}}$ is a MSIDS in FIG $\tilde{\mathcal{G}}$. From Theorem 3.7, $\tilde{\mathcal{S}}$ is a FIIRS. Assume that $\tilde{\mathcal{S}}$ is not MFIIRS. Then there exists $v \in \mathcal{V} \setminus \tilde{\mathcal{S}}$ such that $\tilde{\mathcal{S}} \cup \{v\}$ is a FIIRS. It implies that $P_{N_{IS}}[v, \tilde{\mathcal{S}} \cup \{v\}] \neq \phi$, i.e, there exists a vertex w which is a private neighbor of v with respect to the set $\tilde{\mathcal{S}} \cup \{v\}$. Hence w is not dominated by any other vertex in $\tilde{\mathcal{S}}$ which is a contradiction to the fact that $\tilde{\mathcal{S}}$ is SIDS. Therefore $\tilde{\mathcal{S}}$ is MFIIRS. □

From Theorem 3.4,3.5,3.6,3.7, and 3.8, Proposition 3.9 follows.

Proposition 3.9. *For a FIG $\tilde{\mathcal{G}}$, $ir_{IS} \leq \gamma_{IS} \leq i_{IS} \leq \beta_{IS} \leq \Gamma_{IS} \leq IR_{IS}$.*

Example 3.10. *For FIG in Figure 1, $\tilde{\mathcal{S}}_1 = \{a_1, a_7, a_5, a_{11}, a_9, a_3\}$ is a FIIRS with maximum weight. Now, $\mathcal{V} \setminus \tilde{\mathcal{S}}_1 = \{a_2, a_4, a_8, a_{10}, a_6\}$ is a SIDS. Similarly $\tilde{\mathcal{S}}_2 = \{a_2, a_8, a_4, a_6, a_{10}\}$ is a minimum SIDS, hence a MSIDS. Clearly, $P_{N_{IS}}[a_i, \tilde{\mathcal{S}}_2] \neq \phi, \forall a_i \in \tilde{\mathcal{S}}_2$. Therefore $\tilde{\mathcal{S}}_2$ is a FIIRS. Now, if a_1 is added to $\tilde{\mathcal{S}}_2$ then, $P_{N_{IS}}[a_2, \tilde{\mathcal{S}}_2 \cup \{a_1\}] = \phi$. Similarly if a_7 is added to $\tilde{\mathcal{S}}_2$ then, $P_{N_{IS}}[a_8, \tilde{\mathcal{S}}_2 \cup \{a_7\}] = \phi$. And if a_5 or a_{11} is added $P_{N_{IS}}[a_4, \tilde{\mathcal{S}}_2 \cup \{a_5\}]$ and $P_{N_{IS}}[a_{10}, \tilde{\mathcal{S}}_2 \cup \{a_{11}\}]$ will be empty respectively. By the same argument if a_3 or a_9 is added to $\tilde{\mathcal{S}}_2$ then, $P_{N_{IS}}[a_6, \tilde{\mathcal{S}}_2 \cup \{a_i\}] = \phi, i = 3$ or 9. Hence $\tilde{\mathcal{S}}_2$ is a MFIIRS.*

Theorem 3.11. *Let $\tilde{\mathcal{G}}$ be a FIG and u be a vertex in $\tilde{\mathcal{G}}$ then, there always exists a MSIDS containing u.*

Definition 3.12 introduces the concept of the domination degree of a vertex in FIG. Here the notion of SID is used.

Definition 3.12. *Let $\tilde{\mathcal{G}}$ be a FIG and $u \in \mathcal{V}(\tilde{\mathcal{G}})$. The strong incidence domination degree (SIDD) of vertex u is minimum weight of MSIDSs containing u, denoted as ${}_{sid}d_{\tilde{\mathcal{G}}}(u)$ or simply ${}_{sid}d(u)$. Hence,*

$$ {}_{sid}d(u) = \wedge\{\mathcal{W}(\tilde{\mathcal{D}}) \,|\, \tilde{\mathcal{D}} \text{ is a MSIDS containing } u\}. $$

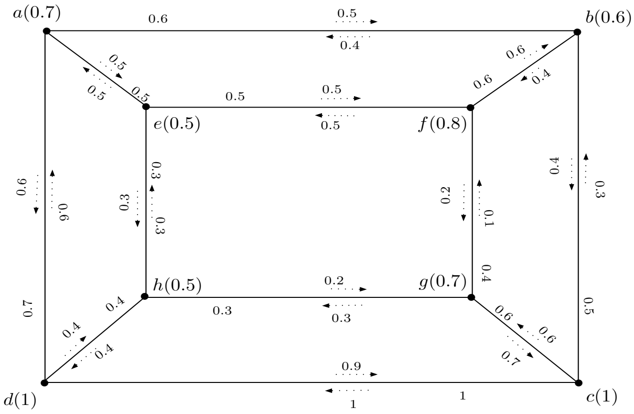

FIGURE 2. Illustration of SIDD

The maximum and minimum SIDD of a FIG can accordingly defined as

$$_{sid}\delta(\tilde{\mathcal{G}}) = \wedge \{_{sid}d(u) : u \in \mathcal{V}(\tilde{\mathcal{G}})\}$$

$$_{sid}\Delta(\tilde{\mathcal{G}}) = \vee \{_{sid}d(u) : u \in \mathcal{V}(\tilde{\mathcal{G}})\}.$$

Example 3.13 is the illustration of Definition 3.12.

Example 3.13. *Consider the FIG in Figure 2. All pairs except* $(c, bc), (g, fg)$ *and* (h, hg) *are strong pairs. For vertex a the MSIDS containing a with minimum weight is* $\{a, e, c\}$ *and the weight of the MSIDS is* $0.5 + 0.3 + 0.6 = 1.4$. *Hence* $_{sid}d(a) = 1.4$. *For vertex b the MSIDS containing b with minimum weight is* $\{b, h, c\}$ *and the weight of the MSIDS is* $0.4 + 0.3 + 0.6 = 1.3$. *Hence* $_{sid}d(b) = 1.3$. *By the same argument* $_{sid}d(c) =_{sid} d(e) =_{sid} d(h) = 1.3$, $_{sid}d(d) = 1.7$, $_{sid}d(f) = 1.5$ *and* $_{sid}d(g) = 1.4$. *Therefore,* $_{sid}\delta(\tilde{\mathcal{G}}) = 1.3$ *and* $_{sid}\Delta(\tilde{\mathcal{G}}) = 1.7$.

From the definition of SIDD of a vertex Proposition 3.14 follows.

Proposition 3.14. *Let* $\tilde{\mathcal{G}}$ *be a FIG. Then*

$$ir_{IS} \leq \gamma_{IS} \leq_{sid} d(u) \leq \Gamma_{IS} \leq IR_{IS}$$

$\forall u \in \mathcal{V}(\tilde{\mathcal{G}})$.

Definition 3.15. *A FIG,* $\tilde{\mathcal{G}}$ *is w-strong incidence domination regular fuzzy incidence graph (w-SIDRFIG) or simply SIDRFIG if every vertex in* $\tilde{\mathcal{G}}$ *has equal SIDD, i.e, if for every* $u \in \mathcal{V}(\tilde{\mathcal{G}})$, $_{sid}d(u) = w$.

Example 3.16. *For the FIG in Figure 3, the pair* (d, df) *is a* $\delta-$ *pair. Here to dominate vertex a either a or d is required. The case is similar for vertices b and c. Hence, a SIDS contains at least 3 vertices. Suppose it is required to find a MSIDS with the least weight consisting of vertex a, then the set* $\{a, e, f\}$ *is a required set with weight 0.3. Hence,* $_{sid}d(a) = 0.3$. *A MSIDS containing vertex b with the least weight is* $\{b, d, f\}$, *with weight 0.3. Similarly for all vertices, the SIDD is 0.3. Hence the given FIG is* $0.3-$ *SIDRFIG.*

Theorem 3.17. *Let* $\tilde{\mathcal{G}}$ *be a FIG. If u is a vertex not dominated by a MFIIRS* $\tilde{\mathcal{M}}$. *Then for some* $x \in \tilde{\mathcal{M}}$,

(1) $P_{N_{IS}}[x, \tilde{\mathcal{M}}] \subseteq N_{IS}(u)$.

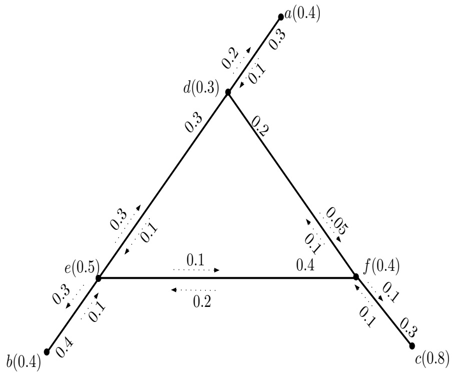

FIGURE 3. Strong incidence domination regular fuzzy incidence graph.

(2) *For* $x_1, x_2 \in P_{N_{IS}}[x, \tilde{\mathcal{M}}]$, *such that* $x_1 \neq x_2$, *either* (x_1, x_1x_2) *and* (x_2, x_1x_2) *are strong pairs or* $\exists\ a_i \in \tilde{\mathcal{M}} \setminus \{x\}$, *for* $i = 1, 2$ *such that* x_i *is SIN of each vertex in* $P_{N_{IS}}[a_i, \tilde{\mathcal{M}}]$.

Proof. Let $\tilde{\mathcal{G}}$ be a FIG and $\tilde{\mathcal{M}}$ be a MFIIRS in $\tilde{\mathcal{G}}$. Let u be the vertex not dominated by $\tilde{\mathcal{M}}$. Then $\tilde{\mathcal{M}} \cup \{u\}$ is a fuzzy incidence redundant set. Since u is not dominated by $\tilde{\mathcal{M}}$ $u \in P_{N_{IS}}[u, \tilde{\mathcal{M}} \setminus \{u\}]$. Hence for some $x \in \tilde{\mathcal{M}}$, $P_{N_{IS}}[x, \tilde{\mathcal{M}} \cup \{u\} = \phi$. It implies that $N_{IS}[x] \setminus N_{IS}[\{\tilde{\mathcal{M}} \cup \{u\}\} \setminus \{x\}] = \phi$. Therefore, $N_{IS}[x] \subseteq N_{IS}[\tilde{\mathcal{M}} \setminus \{x\}] \cup N_{IS}[u]$, i.e, $N_{IS}[x] \setminus N_{IS}[\tilde{\mathcal{M}} \setminus \{x\}] \subseteq N_{IS}[u]$. Since $u \notin P_{N_{IS}}[x, \tilde{\mathcal{M}}]$, it follows that $P_{N_{IS}}[x, \tilde{\mathcal{M}}] = N_{IS}[x] \setminus N_{IS}[\tilde{\mathcal{M}} \setminus \{x\}] \subseteq N_{IS}(u)$.
Now, let $x_1 \neq x_2$ be two vertices in $P_{N_{IS}}[x, \tilde{\mathcal{M}}]$ such that either (x_1, x_1x_2) or (x_2, x_1x_2) is a non strong pair. Also, suppose for all $a_i \in \tilde{\mathcal{M}} \setminus \{x\}$ there exists $b_i \in P_{N_{IS}}[a_i, \tilde{\mathcal{M}}]$ such that either (x_1, x_1b_i) or (b_i, x_1b_i) is a non- strong pair. Now, consider the set $\tilde{\mathcal{M}} \cup \{x_1\}$. The following are the observations:

$$x_2 \in P_{N_{IS}}[x, \tilde{\mathcal{M}} \cup \{x_1\}]$$
$$u \in P_{N_{IS}}[x_1, \{x_1\} \cup \tilde{\mathcal{M}}]\ [\text{ since } x_1, x_2 \in P_{N_{IS}}[x, \tilde{\mathcal{M}}] \subseteq N(u)]$$
$$\text{for each } a_i \in \tilde{\mathcal{M}} \setminus \{x\},\ b_i \in P_{N_{IS}}[a_i, \tilde{\mathcal{M}} \cup \{x_1\}].$$

The observations imply that the set $\tilde{\mathcal{M}} \cup \{x_1\}$ is a FIIRS which is a contradiction. □

Definition 3.18 is the definition of strong incidence domination index of a FIG. Example 3.19 is the illustration for Definition 3.18. Theorem 3.20, Propositions 3.21, and 3.30 give some bounds for SIDI of a FIG.

Definition 3.18. *The strong incidence domination index (SIDI) of a FIG,* $\tilde{\mathcal{G}}$ *is the sum of SIDD of vertices of* $\tilde{\mathcal{G}}$, *i.e,*

$$SIDI(\tilde{\mathcal{G}}) = \sum_{u \in \mathcal{V}(\tilde{\mathcal{G}})} {}_{sid}d(u).$$

Example 3.19. *For the FIG in Figure 2, 4 vertices have SIDD 1.3, 2 vertices have SIDD 1.4, and one vertex each with SIDD 1.5 and 1.7. Hence the SIDI of FIG in Figure 2 is* $1.3 \times 4 + 1.4 \times 2 + 1.5 + 1.7 = 11.2$.

Theorem 3.20. *For a FIG $\tilde{\mathcal{G}}$ with SIDN γ_{IS} and USIDN Γ_{IS},*

$$\gamma_{IS}(\tilde{\mathcal{G}}) \leq \frac{SIDI(\tilde{\mathcal{G}})}{n} \leq \Gamma_{IS}(\tilde{\mathcal{G}}),$$

where $n = |\psi *|$.

Proof. From Theorem 3.14, $\gamma_{IS}(\tilde{\mathcal{G}}) \leq {}_{sid}d(v) \leq \Gamma_{IS}(\tilde{\mathcal{G}}), \forall v \in \mathcal{V}(\tilde{\mathcal{G}})$. Hence, $\sum_{v\in\mathcal{V}(\tilde{\mathcal{G}})} \gamma_{IS}(\tilde{\mathcal{G}}) \leq \sum_{v\in\mathcal{V}(\tilde{\mathcal{G}})} {}_{sid}d(v) \leq \sum_{v\in\mathcal{V}(\tilde{\mathcal{G}})} \Gamma_{IS}(\tilde{\mathcal{G}})$. Therefore, $n\gamma_{IS}(\tilde{\mathcal{G}}) \leq SIDI(\tilde{\mathcal{G}}) \leq n\Gamma_{IS}(\tilde{\mathcal{G}})$. □

From Proposition 3.14, Proposition 3.21 follows:

Proposition 3.21. *Let $\tilde{\mathcal{G}}$ be a FIG with $|\psi^*| = n$, then*

$$ir_{IS} \leq \gamma_{IS} \leq \frac{SIDI(\tilde{\mathcal{G}})}{n} \leq \Gamma_{IS} \leq IR_{IS}.$$

The SIDI is studied in CFIG, CBFIG, FICs, FITs in Theorem 3.22, 3.24, 3.26, and 3.27.

Theorem 3.22. *Let $\tilde{\mathcal{G}}$ be a CFIG with n vertices. Then,*

$$SIDI(\tilde{\mathcal{G}}) = n\psi(v),$$

where v is the vertex with least weight, and $n = |\psi^*|$.

Proof. Let $\tilde{\mathcal{G}}$ be a FIG with $|\psi^*| = n$. Let v be the vertex with the least weight. Then any pair of the form (v, vu) or (u, vu) for any other vertex $u \in \psi^*$ will have weight $\psi(v)$. Since $\tilde{\mathcal{G}}$ is a CFIG every vertex $u \in \psi^*$ has a pair of weight $\psi(v)$ incident at u. Also, each set $\{u\}$, $u \in \psi^*$ is a MSIDS. Hence the SIDD ${}_{sid}d(u)$ is $\psi(v)$, $\forall u \in \psi^*$. Therefore, $SIDI(\tilde{\mathcal{G}}) = n\psi(v)$. □

Definition 3.23. *A FIG $\tilde{\mathcal{G}} = (\psi, \tau, \xi)$ is said to be $k-$partite if the vertex set $\mathcal{V}$ can be partitioned into r non-empty sets $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_r$ such that $\tau(uv) = 0$ if $u, v \in \mathcal{V}_i$, for $i = 1, 2, ..., r$. Also, if $\tau(uv) = \psi(u)\wedge\psi(v)$, $\xi(u, uv) = \psi(u)\wedge\tau(uv)$ and $\xi(v, uv) = \psi(v)\wedge\tau(uv)$ for all $u \in \mathcal{V}_i$ and $v \in V_j, i \neq j$, then $\tilde{\mathcal{G}}$ is called a complete $k-$ partite FIG. A complete bipartite fuzzy incidence graph (CBFIG) is a complete 2-partite FIG.*

Theorem 3.24. *Let $\tilde{\mathcal{G}}$ be a complete k-partite FIG with partitions $M_1, M_2, ..., M_k$ with $|M_1| > 1, |\ |M_2| > 1, ..., |M_k| > 1$. Let the vertices in each partition be labeled as $u_{11}, u_{12}, ...u_{1m_1}$, $u_{21}, u_{22}, ...u_{2m_2}$, ...,$u_{k1}, u_{k2}, ...u_{km_k}$ respectively such that $\psi(u_{11}) \leq \psi(u_{12}) \leq ... \leq \psi(u_{1m_1})$, $\psi(u_{21}) \leq \psi(u_{22}) \leq ... \leq \psi(u_{2m_2})$,..., $\psi(u_{k1}) \leq \psi(u_{k2}) \leq ... \leq \psi(u_{km_k})$. Then*

$$SIDI(\tilde{\mathcal{G}}) = [2(m_2 + m_3 + ... + m_r + 1) + (m_1 - 1)]\psi(u_{11}) + \psi(u_{12}) + ... + \psi(u_{1r-1}) + (m_1 - (r-1))\psi(u_i)$$

where u_{11} is the vertex with least weight and u_i is the r^{th} minimum vertex in M_n $n \neq 1$ such that $\psi(u_{1r}) \leq \psi(u_i) \leq \psi(u_{1r+1})$ or $\psi(u_{1r-1}) < \psi(u_i) \leq \psi(u_{1r})$.

Proof. Let $\tilde{\mathcal{G}}$ be a complete k-partite FIG with partitions $M_1, M_2, ..., M_k$ with $|M_1| > 1, |\ |M_2| > 1, ..., |M_k| > 1$. The least weight vertex u_{11} belongs to M_1. To dominate the entire vertex set only two vertices from different partitions are required. Now, any pair of the form $(u_{11}, u_{11}u_j)$ or $(u_j, u_{11}u_j)$, for any $u_j \notin M_1$ will have weight $\psi(u_{11})$. Hence for vertex u_{11} and u_j the set $\{u_{11}, u_j\}$ is a MSIDS having the least weight $2\psi(u_{11})$.

Now consider the vertices in M_1. Suppose the next minimum weight vertex is u_{12}, then weight of the pair at u_{12} is $\psi(u_{12})$. Also, $\{u_{12}, u_i\}$ is a MSIDS containing u_{12} having weight $\psi(u_{12}) + \psi(u_{11})$. Similarly the weight of pair at u_{13},...,u_{1r-1} is $\psi(u_{13}), ..., \psi(u_{1r-1})$ respectively. Suppose the r^{th} minimum vertex $u_i \in M_n$, $n \neq 1$. Then the weight of the pair incident at vertices $u_{1r}, u_{1r+1}, ..., u_{1m_1}$ is $\psi(u_i)$. Also, the set $\{u_{1l}, u_i\}$, $l \in \{r, r+1, ..., m_1\}$ is a MSIDS containing u_{1l}, $l \in \{r, r+1, ...m_1\}$ with the least weight $\psi(u_i) + \psi(u_1 1)$. And hence

$$SIDI(\tilde{\mathcal{G}}) = [2(m_2 + m_3 + ... + m_r + 1) + (m_1 - 1)]\psi(u_{11}) + \psi(u_{12}) + ... \\ + \psi(u_{1r-1}) + (m_1 - (r-1))\psi(u_i).$$

□

Definition 3.25. *A FIC is $\beta-$saturated if every vertex has a $\beta-$pair incident to it.*

Theorem 3.26. *Let $\tilde{\mathcal{G}}$ be a $\beta-$saturated FIC, such that the weight of the $\beta-$pair is ϖ. Then*

$$SIDI(\tilde{\mathcal{G}}) = n\left\lceil\frac{n}{3}\right\rceil\varpi,$$

where $n = |\psi^*|$.

Proof. Let $\tilde{\mathcal{G}}$ be a $\beta-$saturated FIC. In a FIC the weight of the $\beta-$pair will always be less than the weight of an $\alpha-$pair. Hence the weight contributed by each vertex to the MSIDS is the weight of the $\beta-$pair which is ϖ. The domination number of a cycle on n vertices is $\left\lceil\frac{n}{3}\right\rceil$. The SIDD of each vertex is $\left\lceil\frac{n}{3}\right\rceil \times \varpi$. Hence $SIDI(\tilde{\mathcal{G}}) = n\left\lceil\frac{n}{3}\right\rceil\varpi$. □

Theorem 3.27. *Let $\tilde{\mathcal{G}}$ be a FIG. If (u, uv) is a $\delta-$ pair, then $SIDI(\tilde{\mathcal{G}}) = SIDI(\tilde{\mathcal{G}} \setminus (u, uv))$.*

Proof. In a FIG a vertex u dominates vertex v if both (u, uv) and (v, uv) are strong pairs. Also the weight of a pair (u, uv) is contributed to a SIDS only if u and v are SIN. Therefore deletion of a $\beta-$pair will not affect the SID of the FIG. Hence, if (u, uv) is a $\beta-$pair $SIDI(\tilde{\mathcal{G}}) = SIDI(\tilde{\mathcal{G}} \setminus (u, uv))$. □

Moreover if (u, uv) is a $\beta-$ pair $SIDI(\tilde{\mathcal{G}}) = SIDI(\tilde{\mathcal{G}} \setminus \{(u, uv), (v, uv)\})$.

Corollary 3.28. *Let $\tilde{\mathcal{G}}$ be a FIT such that the underlying graph is not a tree. Then there exists at least one pair $(u, uv) \in \xi^*$ such that $SIDI(\tilde{\mathcal{G}} \setminus (u, uv) = SIDI(\tilde{\mathcal{G}})$.*

Proof. Let $\tilde{\mathcal{G}}$ be a FIT such that the underlying graph is not a tree. Hence $\tilde{\mathcal{G}}$ contains $\alpha-$pairs and at least one $\delta-$pair. Suppose that the pair (u, uv) is one $\delta-$ pair. Then by Theorem 3.27, $SIDI(\tilde{\mathcal{G}} \setminus (u, uv) = SIDI(\tilde{\mathcal{G}})$. Hence there exists at least one pair (u, uv) such that $SIDI(\tilde{\mathcal{G}} \setminus (u, uv) = SIDI(\tilde{\mathcal{G}})$. □

Corollary 3.29. *Let $\tilde{\mathcal{G}}$ be a FIT, if $\tilde{\mathcal{T}}$ is the unique maximum spanning tree of $\tilde{\mathcal{G}}$, then $SIDI(\tilde{\mathcal{G}}) = SIDI(\tilde{\mathcal{T}})$.*

Proof. The unique maximum spanning tree $\tilde{\mathcal{T}}$ is obtained from $\tilde{\mathcal{G}}$ by deleting the pair (u, uv) of a cycle such that $\xi(u, uv) < ICONN_{\tilde{\mathcal{G}}\setminus(u,uv)}(u, uv)$, i.e, by deleting all the $\delta-$ pairs of $\tilde{\mathcal{G}}$. By Theorem 3.27, the deletion of a $\delta-$ pair does not affect the SIDI of $\tilde{\mathcal{G}}$. Applying Theorem 3.27 to each $\delta-$ pair in $\tilde{\mathcal{G}}$, the SIDI of $\tilde{\mathcal{T}}$ is obtained as $SIDI(\tilde{\mathcal{G}}) = SIDI(\tilde{\mathcal{T}})$. □

Proposition 3.30. *Let $\tilde{\mathcal{G}}$ be a connected SFIG, then $SIDI(\tilde{\mathcal{G}}) \leq |\psi^*|\mathcal{WI}(\tilde{\mathcal{G}})$, where $\psi(v) = 1\,\forall v \in \psi^*$ and $|\psi^*| > 1$*

Proof. Let $\tilde{\mathcal{G}}$ be a connected SFIG such that $\psi(v) = 1\,\forall v \in \psi^*$. Let v be any vertex in ψ^* and $\tilde{\mathcal{D}}_u$ be a MSIDS containing v with the least weight.
Suppose that $\tilde{\mathcal{D}}_v = \{v, v_1, v_2, ..., v_n\}$. Since $\tilde{\mathcal{D}}_v$ is minimal each vertex in $\tilde{\mathcal{D}}_v$ has a private neighbor, i.e, for each $x \in \tilde{\mathcal{D}}_v$, either there exists y such that $y \in P_{N_{IS}}[x, \tilde{\mathcal{D}}_v]$ or $P_{N_{IS}}[x, \tilde{\mathcal{D}}_v] = \{x\}$. Among $v, v_1, v_2, ...v_k$ let $x_1, x_2, ..., x_l$ be vertices having at least one distinct vertex say $z_1, z_2, ...z_l$ respectively in the private neighborhood and let $y_1, y_2, ...y_{l'}$ be vertices such that $P_{N_{IS}}[y_i, \tilde{\mathcal{D}}_v] = \{y_i\}$, $l + l' = k + 1, i = 1, 2, ..., l'$. For the vertices in $\tilde{\mathcal{D}}$ the minimum of weight of pairs incident at each vertex is contributed to the weight of $\tilde{\mathcal{D}}$. The minimum of weight of pairs at x_i is less than or equal to $(x_i, x_i z_i)$. Now, consider any two vertices among $y_1, y_2, ..., y_{l'}$. Any path from y_i to y_j, $1 \leq i, j \leq l', i \neq j$ contains at least one vertex $y_{i'}$ such that $y_{i'} \notin N_{IS}[z]$, $z \in \{z_1, z_2, ..., z_l\}$. Therefore, $\exists$ at least two edges in each path. Suppose there is only one vertex y such that $P_{N_{IS}}[y, \tilde{\mathcal{D}}_v] = \{y\}$. Then y is isolated in $\tilde{\mathcal{G}}[\tilde{\mathcal{D}}_v]$. Since $\tilde{\mathcal{G}}$ is connected, consider any path from y to v_i, then $\exists$ at least two edges in such paths. And the pair weight contributed by y is distinct from $\psi(x_i z_i)$. Hence ${}_{sid}d(v) \leq \mathcal{WI}(\tilde{\mathcal{G}})$. □

Definition 3.31. *Two FIGs, $\tilde{\mathcal{G}}_1 = (\psi_1, \tau_1, \xi_1)$ and $\tilde{\mathcal{G}}_2 = (\psi_2, \tau_2, \xi_2)$ are fuzzy incidence isomorphic if there exists a bijective map $\xi : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\psi_1(u) = \psi_2(\xi(u))$, $\tau_1(uv) = \tau_2(\xi(u)\xi(v))$, and $\xi_1(u, uv) = \xi_2(\xi(u), \xi(u)\xi(v))\ \forall u, v \in \mathcal{V}_1$.*

Theorem 3.32. *Let $\tilde{\mathcal{G}}_1 = (\psi_1, \tau_1, \xi_1)$ and $\tilde{\mathcal{G}}_2 = (\psi_2, \tau_2, \xi_2)$ be fuzzy incidence isomorphic. Then $SIDI(\tilde{\mathcal{G}}_1) = SIDI(\tilde{\mathcal{G}}_2)$.*

Proof. Let $\tilde{\mathcal{G}}_1 = (\psi_1, \tau_1, \xi_1)$ and $\tilde{\mathcal{G}}_2 = (\psi_2, \tau_2, \xi_2)$ be fuzzy incidence isomorphic and ξ be the bijection from $\mathcal{V}_1$ to $\mathcal{V}_2$ such that $\psi_1(u) = \psi_2(\xi(u))$, $\tau_1(uv) = \tau_2(\xi(u)\xi(v))$, and $\xi_1(u, uv) = \xi_2(\xi(u), \xi(u)\xi(v))\ \forall u, v \in \mathcal{V}_1$. Since $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ are isomorphic the weight of pairs incident at v and $\xi(v)$ are same. Hence, the minimum of weight of pairs incident at u and $\xi(v)$ are same. Hence ${}_{sid}d(v) =_{sid} d(\xi(v))$.

$$\begin{aligned} SIDI(\tilde{\mathcal{G}}_1) &= \sum_{u \in \mathcal{V}(\tilde{\mathcal{G}}_1)} {}_{sid}d(u) \\ &= \sum_{\xi(u) \in \mathcal{V}(\tilde{\mathcal{G}}_2)} {}_{sid}d(\xi(u)) \\ &= SIDI(\tilde{\mathcal{G}}_2) \end{aligned}$$

Therefore, $SIDI(\tilde{\mathcal{G}}_1) = SIDI(\tilde{\mathcal{G}}_2)$. □

Now, SIDI is explored in union and join of FIGs in Theorem 3.34, 3.35, and 3.36.

Definition 3.33. *Let $\tilde{\mathcal{G}}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{I}_1, \psi_1, \tau_1, \xi_1)$ and $\tilde{\mathcal{G}}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{I}_2, \psi_2, \tau_2, \xi_2)$ be two FIGs. The the union $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_1 \cup \tilde{\mathcal{G}}_2$ is a FIG with $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ and $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$.*

Here $\mathcal{V}_1 \cap \mathcal{V}_2 = \phi$. *Also* $\psi = \psi_1 \cup \psi_2$, $\tau = \tau_1 \cup \tau_2$ *and* $\xi = \xi_1 \cup \xi_2$ *are defined as follows:*

$$\psi(a) = \begin{cases} \psi_1(a) & \text{if } a \in \mathcal{V}_1 \\ \psi_2(a) & \text{if } a \in \mathcal{V}_2 \end{cases}$$

$$\tau(ab) = \begin{cases} \tau_1(ab) & \text{if } ab \in \mathcal{E}_1 \\ \tau_2(ab) & \text{if } ab \in \mathcal{E}_2 \end{cases}$$

$$\xi(a, ab) = \begin{cases} \xi_1(a, ab) & \text{if } (a, ab) \in \mathcal{I}_1 \\ \xi_2(a, ab) & \text{if } (a, ab) \in \mathcal{I}_2 \end{cases}$$

Theorem 3.34. *Let* $\tilde{\mathcal{G}} = \bigcup_{i=1}^{m} \tilde{\mathcal{G}}_i$ *be the disjoint union of FIGs* $\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, ..., \tilde{\mathcal{G}}_m$*, then*

$$SIDI(\tilde{\mathcal{G}}) = \sum_{i=1}^{m} SIDI(\tilde{\mathcal{G}}_i) + \sum_{i=1}^{m} \big(n \setminus |\mathcal{V}(\tilde{\mathcal{G}}_i)|\big)\gamma_{IS}(\tilde{\mathcal{G}}_i),$$

where $n = |\mathcal{V}(\tilde{\mathcal{G}})|$ *and* $|\mathcal{V}(\tilde{\mathcal{G}}_i)|$ *is the number of vertices in each component* $\tilde{\mathcal{G}}_i$.

Proof. Let $\tilde{\mathcal{G}} = \bigcup_{i=1}^{m} \tilde{\mathcal{G}}_i$ be the disjoint union of FIGs $\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, ..., \tilde{\mathcal{G}}_m$. Consider any component $\tilde{\mathcal{G}}_j$ of $\tilde{\mathcal{G}}$ and let v be a vertex of $\tilde{\mathcal{G}}_j$. A SIDS of $\tilde{\mathcal{G}}$ is a disjoint union of SIDSs of each component $\tilde{\mathcal{G}}_i$. Let $\tilde{\mathcal{D}}_v$ be the MSIDS corresponding to ${}_{sid}d(v)$. For all other components other than $\tilde{\mathcal{G}}_j$, the SIDS contributed from each $\tilde{\mathcal{G}}_j$ is the minimum SIDS since it is the least weight SIDS satisfying the minimality condition. And, the SIDS contributed from $\tilde{\mathcal{G}}_j$ is the SIDS corresponding to ${}_{sid}d(v)$ in $\tilde{\mathcal{G}}_j$. Hence, by adding ${}_{sid}d(v)$ of each vertex in $\tilde{\mathcal{G}}_j$, the $SIDI(\tilde{\mathcal{G}}_j)$ is obtained with the SIDN of each other components $\tilde{\mathcal{G}}_j$ being added $n \setminus |\mathcal{V}(\tilde{\mathcal{G}}_j)|$ times. Therefore

$$\begin{aligned}
&SIDI(\tilde{\mathcal{G}}) \\
&= \sum_{i=1}^{m} SIDI(\tilde{\mathcal{G}}_i) + \mathcal{V}(\tilde{\mathcal{G}}_1)(\gamma_{IS}(\tilde{\mathcal{G}}_2) + \gamma_{IS}(\tilde{\mathcal{G}}_3) + ... + \gamma_{IS}(\tilde{\mathcal{G}}_m)) + \mathcal{V}(\tilde{\mathcal{G}}_2)(\gamma_{IS}(\tilde{\mathcal{G}}_1) + \\
&\qquad \gamma_{IS}(\tilde{\mathcal{G}}_3) + ... + \gamma_{IS}(\tilde{\mathcal{G}}_m)) + ... + \mathcal{V}(\tilde{\mathcal{G}}_m)(\gamma_{IS}(\tilde{\mathcal{G}}_1) + \gamma_{IS}(\tilde{\mathcal{G}}_2) + ... + \gamma_{IS}(\tilde{\mathcal{G}}_{m-1})) \\
&= \sum_{i=1}^{t} SDI(\tilde{\mathcal{G}}_i) + \gamma_{IS}(\tilde{\mathcal{G}}_1)(\mathcal{V}(\tilde{\mathcal{G}}_2) + \mathcal{V}(\tilde{\mathcal{G}}_3) + ...\mathcal{V}(\tilde{\mathcal{G}}_m)) + \gamma_{IS}(\tilde{\mathcal{G}}_2)(\mathcal{V}(\tilde{\mathcal{G}}_1) + \mathcal{V}(\tilde{\mathcal{G}}_3) + \\
&\qquad ...\mathcal{V}(\tilde{\mathcal{G}}_m)) + ... + \gamma_{IS}(\tilde{\mathcal{G}}_m)(\mathcal{V}(\tilde{\mathcal{G}}_1) + \mathcal{V}(\tilde{\mathcal{G}}_2) + ...\mathcal{V}(\tilde{\mathcal{G}}_{m-1})) \\
&= \sum_{i=1}^{t} SDI(\tilde{\mathcal{G}}_i) + \sum_{i=1}^{m} \big(n \setminus |\mathcal{V}(\tilde{\mathcal{G}}_i)|\big)\gamma_{IS}(\tilde{\mathcal{G}}_i).
\end{aligned}$$

□

Theorem 3.35. *Let* $\tilde{\mathcal{G}}_1$ *and* $\tilde{\mathcal{G}}_2$ *be two FIGs with* m *and* n *vertices respectively such that the join* $\tilde{\mathcal{G}}_1 + \tilde{\mathcal{G}}_2$ *is strong. Then,*

$$SIDI(\tilde{\mathcal{G}}) = \sum_{\substack{u_i \in V(\tilde{\mathcal{G}}_1) \\ 1 \le i \le m}} \wedge\{\mathcal{W}(\{u_i, v\}), \mathcal{W}(\tilde{\mathcal{D}}_{u_i})\} + \sum_{\substack{v_i \in V(\tilde{\mathcal{G}}_2) \\ 1 \le i \le n}} \wedge\{\mathcal{W}(\{u, v_i\}), \mathcal{W}(\tilde{\mathcal{D}}_{v_i})\}$$

where v *in the first summation is vertex in* $\tilde{\mathcal{G}}_2$ *having the least weight pair incident at it and* u *in the second summation is vertex in* $\tilde{\mathcal{G}}_1$ *having the least weight pair incident at* *it. And* $\tilde{\mathcal{D}}_{u_i}$ *is any MSIDS containing* u_i *in* $\tilde{\mathcal{G}}_1$ *with least weight and* $\tilde{\mathcal{D}}_{v_i}$ *is any MSIDS containing* v_i *in* $\tilde{\mathcal{G}}_2$ *with least weight.*

Proof. Let $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ be two FIGs with m and n vertices respectively such the join $\tilde{\mathcal{G}}_1+\tilde{\mathcal{G}}_2$ is strong. Let $\{u_1, u_2, ..., u_m\}$ be the vertices of $\tilde{\mathcal{G}}_1$. Let $\tilde{\mathcal{D}}_{u_i}, 1 \leq i \leq m$ be a MSIDS in $\tilde{\mathcal{G}}_1$ containing u_i with least weight. Also assume that v is a vertex in $\tilde{\mathcal{G}}_2$ having the least weight pair incident at it. If $W(\tilde{\mathcal{D}}_{u_i}) < W(\{u_i, v\})$ then, since the join is strong $\tilde{\mathcal{D}}_{u_i}$ can be considered as the MSIDS containing u_i in $\tilde{\mathcal{G}}$ also. If $W(\tilde{\mathcal{D}}_{u_i}) > W(\{u_i, v\})$ then, $\{u_i, v\}$ is a MSIDS with least weight in $\tilde{\mathcal{G}}$. The argument follows for the vertices in $\tilde{\mathcal{G}}_2$. The case is similar for $\tilde{\mathcal{G}}_2$ also. Hence, the

$$SIDI(\tilde{\mathcal{G}}) = \sum_{\substack{u_i \in V(\tilde{\mathcal{G}}_1) \\ 1 \leq i \leq m}} \wedge\{W(\{u_i, v\}), W(\tilde{\mathcal{D}}_{u_i})\} + \sum_{\substack{v_i \in V(\tilde{\mathcal{G}}_2) \\ 1 \leq i \leq n}} \wedge\{W(\{u, v_i\}), W(\tilde{\mathcal{D}}_{v_i})\}$$

□

Theorem 3.36. *Let* $\tilde{\mathcal{G}}_1$ *and* $\tilde{\mathcal{G}}_2$ *be two SFIGs with* m *and* n *vertices respectively such that the join* $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_1 + \tilde{\mathcal{G}}_2$ *is strong. Then,* $SIDI(\tilde{\mathcal{G}}) \leq SIDI(\tilde{\mathcal{G}}_1) + SIDI(\tilde{\mathcal{G}}_2)$.

Proof. Let $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ be two SFIGs with m and n vertices respectively such that join $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_1 + \tilde{\mathcal{G}}_2$ is strong. Since the join is strong, every MSIDS of $\tilde{\mathcal{G}}_1$ or $\tilde{\mathcal{G}}_2$ is a SIDS of $\tilde{\mathcal{G}}$. Hence the MSIDS containing vertex $a, a \in \mathcal{V}(\tilde{\mathcal{G}}_1)$ or $(\mathcal{V}(\tilde{\mathcal{G}}_2))$ considered while calculating the $SIDI$ of $\tilde{\mathcal{G}}_1$ or $(\tilde{\mathcal{G}}_2)$ is also a SIDS containing a in $\tilde{\mathcal{G}}$. Also, since the join is strong all sets of the form $\{u, v\}$ where $u \in (\tilde{\mathcal{G}}_1)$ and $v \in (\tilde{\mathcal{G}}_2)$ are also SIDSs in $\tilde{\mathcal{G}}$. Since the ${}_{sid}d(a)$ of a vertex a is the minimum of weight of MSIDSs containing a, ${}_{sid}d_{\tilde{\mathcal{G}}}(a) \leq_{sid} d_{\tilde{\mathcal{G}}_i}(a), i = 1, 2$. Also, since the vertices of $\tilde{\mathcal{G}}$ is the union of vertices of $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$, it follows that $SIDI(\tilde{\mathcal{G}}) \leq SIDI(\tilde{\mathcal{G}}_1) + SIDI(\tilde{\mathcal{G}}_2)$. □

4. Application

Suppose a state is suddenly dealing with an outbreak of a contagious disease. The government is trying to provide the necessary resources for testing in labs located in various parts of the city. If a part of the city is highly active, with significant incoming and outgoing of people, it has a higher chance of becoming a containment zone. Therefore, these areas require more testing resources than others. By using the strong incidence domination degree, it can be determined where resources should be allocated to ensure that labs in the most likely containment zones receive the necessary support. This method also ensures that people from all other parts of the city benefit, either directly or through labs in the nearby areas. Consider a FIG model of a city as in Figure 4. Let the vertices represent prominent parts of the city, with the vertex weight corresponding to the population of each part, assumed to be one for simplicity. The edges represent the connectivity between these parts, and their weights indicate the number of people testing for the contagious disease between the connected parts. A pair weight (b_1, b_1b_2) represents the number of people between parts b_1 and b_2 testing if resources are provided at the labs in part b_1. Suppose there is a condition that the labs in part b_1 must receive all the resources, as b_1 is a very prominent and high-risk area of the city. To meet this requirement while ensuring that people from other parts also benefit, we need to find a MSIDS containing b_1. The weight of the MSIDS represents the minimum number of people testing at the corresponding labs in the MSIDS.

Here a MSIDS containing b_1 with the least weight is $\{b_1, b_7\}$ with weight 0.6 which means if the resources are provided at b_1 and b_7 the minimum number of people testing from

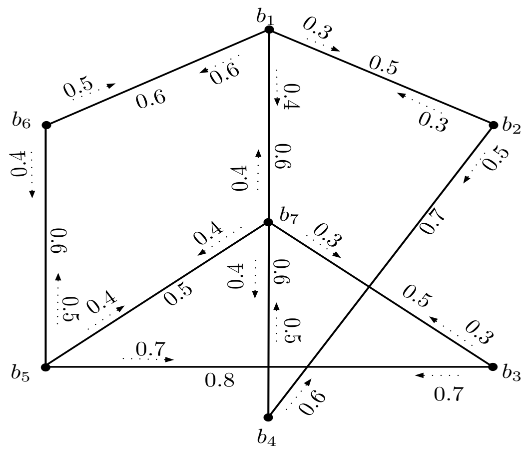

FIGURE 4. FIG model of a city.

b_1 and b_7 is 0.6. However, it is always necessary to ensure that the maximum number of people benefit from the provided resources. Therefore, we also need to consider other possibilities. The set $\{b_1, b_4, b_5\}$ is a MSIDS containing b_1 with maximum weight 1.2. Hence if the resources are provided at b_1, b_4, and b_5 maximum number of people will get the benefit.

The concept of SIDD extends naturally to various practical applications, such as facility allocation and network security. For instance, in a data flow network, vertices represent components like servers, routers, and their weights correspond to their data capacities. Edges represent the data flow between these components, with edge weights indicating normalized data flow. The pair weight between vertex u and edge uv signifies the data flow from u to v. In scenarios where network protection is critical, and specific nodes, such as u, demand maximum security, SIDD proves invaluable. By identifying the MSIDS that contains u, monitoring devices can be strategically placed at these nodes. This ensures comprehensive network protection while SIDD quantifies the minimum data safeguarded.

Thus, the concept of SIDD is broadly applicable to any situation requiring domination-based strategies with an added priority condition for specific vertices. It provides a robust framework for addressing practical problems like network monitoring, resource allocation, and secure communication.

5. Conclusion

The article integrates two fundamental concepts in graph theory- domination and topological indices through the introduction of the domination index in FIGs defined using MSIDSs. The extension to FIGs, effectively addresses ambiguity and uncertainty, enhancing its applicability to real-world problems such as facility allocation. The ability to prioritize specific vertices using information derived from FIGs underscores the flexibility and significance of this approach. The incorporation of parameters like strong incidence domination and their associated bounds enhances the analytical capabilities within FIG theory, enabling a deeper understanding of relationships and dependencies in systems characterized by imprecision.

The article studies the concept of the domination index within FIGs, particularly through the framework of strong incidence domination. It extends traditional graph-related terms in domination and irredundance, such as the fuzzy incidence irredundant set, fuzzy incidence independent set, fuzzy incidence independent dominating set, upper strong incidence domination number, strong incidence irredundance number, strong incidence upper irredundance number, strong incidence independent domination number, and strong incidence independence number. The study also examines inequalities and bounds for these parameters, offering valuable insights into their mathematical properties and potential applications.

Extending the analysis, the study applies these concepts to various FIG structures, including CFIGs, CBFIGs, FICs, FITs, and their unions and joins. This comprehensive approach provides a robust framework for future research in fuzzy graph theory while laying the groundwork for practical applications in graph-based models involving uncertainty and prioritization.

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6. Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Kavya R. Nair received her BSc degree in Mathematics from Sacred Heart College, Thevara, Ernakulam, Kerala, India in 2016 and MSc in Mathematics from Cochin University of Science and Technology, Ernakulam, Kerala, India in 2018. Currently she is doing research at National Institute of Technology, Calicut, Kerala, India. Her research interests include Discrete Mathematics, Graph theory and Fuzzy Graph Theory.

Sunitha M S is currently working as a Professor of Mathematics in National Institute of Technology Calicut, Kerala, India. She is a Life Member of Academy of Discrete Mathematics and Applications (ADMA), India. Her research interest include Fuzzy Graph Theory and Algebraic Graph Theory.