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ON (p, q)-FUZZY SUBGROUPS

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ABSTRACT. The (p, q)-fuzzy sets, which extend the concept of q-rung orthopair fuzzy sets, provide a broader framework for representing uncertainty. This article introduces the concept of (p, q)-fuzzy subgroups of finite groups and examines their fundamental properties. Additionally, it develops and analyzes key concepts such as (p, q)-fuzzy cosets, (p, q)-fuzzy normal subgroups, and (p, q)-fuzzy level subgroups, thereby providing deeper insights into the structure of (p, q)-fuzzy subgroups.

Keywords: (p, q)-Fuzzy set, (p, q)-fuzzy subgroup, (p, q)-fuzzy normal subgroup, (p, q)-fuzzy level subgroup.

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1. Introduction

Algebraic structures are fundamental to various areas of mathematics. Notable algebraic structures include groups, rings, fields, and vector spaces, which are explored in detail in [10]. The foundational contributions by Zadeh [22] on fuzzy subsets of a set and Rosenfeld [19] on fuzzy subgroups of a group resulted in the fuzzification of algebraic structures.

Anthony and Sherwood [2, 3] redefined the concept of fuzzy subgroups using triangular norms to enhance its applicability in various fuzzy scenarios. Das [9] introduced the notion of level subgroups of a fuzzy subgroup, which provided a characterization of all fuzzy subgroups of finite cyclic groups. Choudhury *et al.* [8] studied fuzzy homomorphisms between groups and examined their effects on fuzzy subgroups. Mukherjee and Bhattacharya [15, 16] contributed by defining fuzzy normal subgroups and fuzzy cosets, along with developing fuzzy analogs of some fundamental theorems in group theory. Kim [13] proposed the concept of fuzzy orders of elements in fuzzy subgroups and analyzed associated properties. Additionally, Kim [14] introduced the notion of order of a fuzzy subgroup based on fuzzy orders of its elements and proposed the notion of fuzzy p-subgroups.

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Atanassov [4] introduced intuitionistic fuzzy sets (IFSs) to model situations characterized by insufficient information regarding the membership degrees of elements in a universal set. IFSs incorporate the concept of indeterminacy, which arises from the hesitation between membership and non-membership degrees. Biswas [7] initiated the study of intuitionistic fuzzy subgroups, laying the foundation for further research in this domain. Subsequently, Hur *et al.* [11, 12] extended this work by introducing intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets, accompanied by a detailed study of their properties.

Yager [20] introduced Pythagorean fuzzy sets (PFSs) as an extension of IFSs, where the sum of the squares of membership and non-membership degrees is bounded by 1. PFSs can model situations involving uncertainty while ensuring that the inconsistency remains within a certain threshold. Bhunia *et al.* [6] formalized the concept of Pythagorean fuzzy subgroups and explored their structural properties in detail. Additionally, in [5], they presented Lagrange's theorem for Pythagorean fuzzy subgroups. Razaq *et al.* [17] conducted a comprehensive study on Pythagorean fuzzy cosets and Pythagorean fuzzy normal subgroups, while also introducing the notions of Pythagorean fuzzy homomorphism and Pythagorean fuzzy isomorphism. Yager [21] proposed q-rung orthopair fuzzy sets (q-ROFSs) as a generalization of PFSs. In q-ROFSs, the sum of the q^{th} powers of membership and non-membership degrees is bounded by 1, enabling these sets to handle greater inconsistency levels compared to PFSs. In [18], the authors defined the concept of q-rung orthopair fuzzy subgroups and examined their features in detail. Al-shami and Mhemdi [1] introduced a new extension of fuzzy sets, called (p,q)-fuzzy sets ((p,q)-FSs), which incorporates indeterminacy and generalizes q-ROFSs. This model is particularly applicable in real-world applications, where the relative importance of membership and non-membership degrees differs. Unlike q-ROFSs, which assign equal importance to membership and non-membership degrees, (p,q)-fuzzy sets provide greater flexibility by allowing distinct weights for these degrees. This article defines the notion of (p,q)-fuzzy subgroups of finite groups and examines their properties. Additionally, it defines the concepts of (p,q)-fuzzy cosets, (p,q)-fuzzy normal subgroups, and (p,q)-fuzzy level subgroups, while also presenting several significant findings related to these concepts.

2. Preliminaries

This section reviews the fundamental concepts crucial for the development of subsequent sections. Throughout this article, let U represent a universal set and let G denote a group.

Definition 2.1. [19] *Let R be a fuzzy subset of* G *with membership function α_R, $\alpha_R : \mathrm{G} \to [0,1]$. Then R a fuzzy subgroup of* G *if for* $\mathrm{g}_1,\ \mathrm{g}_2 \in \mathrm{G}$, $\alpha_R(\mathrm{g}_1\mathrm{g}_2) \geq \min\{\alpha_R(\mathrm{g}_1), \alpha_R(\mathrm{g}_2)\}$ *and* $\alpha_R({\mathrm{g}_1}^{-1}) \geq \alpha_R(\mathrm{g}_1)$.

Definition 2.2. [7] *An intuitionistic fuzzy set,* $R = \{(\mathrm{g}, \alpha_R(\mathrm{g}), \beta_R(\mathrm{g})) : \mathrm{g} \in \mathrm{G}\}$ *on* G *is an intuitionistic fuzzy subgroup of* G *if, for* $\mathrm{g}_1,\ \mathrm{g}_2 \in \mathrm{G}$*:*

(1) $\alpha_R(\mathrm{g}_1\mathrm{g}_2) \geq \min\{\alpha_R(\mathrm{g}_1), \alpha_R(\mathrm{g}_2)\}$ *and* $\beta_R(\mathrm{g}_1\mathrm{g}_2) \leq \max\{\beta_R(\mathrm{g}_1), \beta_R(\mathrm{g}_2)\}$,
(2) $\alpha_R({\mathrm{g}_1}^{-1}) \geq \alpha_R(\mathrm{g}_1)$ *and* $\beta_R({\mathrm{g}_1}^{-1}) \leq \beta_R(\mathrm{g}_1)$.

Definition 2.3. [18] *A q-rung orthopair fuzzy set,* $R = \{(\mathrm{g}, \alpha_R(\mathrm{g}), \beta_R(\mathrm{g})) : \mathrm{g} \in \mathrm{G}\}$ *on* G *is a q-rung orthopair fuzzy subgroup of* G *if, for* $\mathrm{g}_1,\ \mathrm{g}_2 \in \mathrm{G}$*:*

(1) $(\alpha_R(\mathrm{g}_1\mathrm{g}_2))^q \geq \min\{(\alpha_R(\mathrm{g}_1))^q, (\alpha_R(\mathrm{g}_2))^q\}$, $(\beta_R(\mathrm{g}_1\mathrm{g}_2))^q \leq \max\{(\beta_R(\mathrm{g}_1))^q, (\beta_R(\mathrm{g}_2))^q\}$,
(2) $(\alpha_R({\mathrm{g}_1}^{-1}))^q \geq (\alpha_R(\mathrm{g}_1))^q$, $(\beta_R({\mathrm{g}_1}^{-1}))^q \leq (\beta_R(\mathrm{g}_1))^q$.

Definition 2.4. [1] *A (p,q)-fuzzy set* K *on* U *is defined as*

$$\mathrm{K} = \{(\mathrm{u}, \alpha_{\mathrm{K}}(\mathrm{u}), \beta_{\mathrm{K}}(\mathrm{u})) : \mathrm{u} \in \mathrm{U}\},$$

where $p, q \geq 1$*,* $\alpha_{\mathrm{K}} : \mathrm{U} \rightarrow [0,1]$ *is the membership function of* K*,* $\beta_{\mathrm{K}} : \mathrm{U} \rightarrow [0,1]$ *is the non-membership function of* K *and* $\alpha_{\mathrm{K}}(\mathrm{u})$*,* $\beta_{\mathrm{K}}(\mathrm{u})$ *satisfy* $0 \leq (\alpha_{\mathrm{K}}(\mathrm{u}))^p + (\beta_{\mathrm{K}}(\mathrm{u}))^q \leq 1$*.*

In the subsequent section, $(\alpha_{\mathrm{K}}(\mathrm{u}))^p$ is denoted as $\alpha^p_{\mathrm{K}}(\mathrm{u})$, and $(\beta_{\mathrm{K}}(\mathrm{u}))^q$ as $\beta^q_{\mathrm{K}}(\mathrm{u})$.

3. (p, q)-fuzzy subgroups

In this section, let G be a finite group. This section elucidates the concept of a (p, q)-fuzzy subgroup of G and explores certain characteristics associated with it.

Definition 3.1. *For* $p, q \geq 1$*, let* $\mathrm{K} = \{(\mathrm{g}, \alpha_{\mathrm{K}}(\mathrm{g}), \beta_{\mathrm{K}}(\mathrm{g})) : \mathrm{g} \in \mathrm{G}\}$ *be a* (p, q)*-fuzzy set on* G*, then* K *is a* (p, q)*-fuzzy subgroup of* G *if, for* $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$*:*

(1) $\alpha^p_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}_1), \alpha^p_{\mathrm{K}}(\mathrm{g}_2)\}$ *and* $\beta^q_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2) \leq \max\{\beta^q_{\mathrm{K}}(\mathrm{g}_1), \beta^q_{\mathrm{K}}(\mathrm{g}_2)\}$*,*
(2) $\alpha^p_{\mathrm{K}}(\mathrm{g}_1{}^{-1}) = \alpha^p_{\mathrm{K}}(\mathrm{g}_1)$ *and* $\beta^q_{\mathrm{K}}(\mathrm{g}_1{}^{-1}) = \beta^q_{\mathrm{K}}(\mathrm{g}_1)$*.*

Note that, intuitionistic fuzzy subgroups of G and Pythagorean fuzzy subgroups of G are (p, q)-fuzzy subgroups of G for $p = q = 1$ and $p = q = 2$, respectively.

Example 3.1. *Let* $p = 4$*,* $q = 3$ *and* $\mathrm{G} = (\mathbb{Z}_4, +_4)$*, where* $+_4$ *denotes addition modulo* 4*. Let*

$$\mathrm{K} = \{(0, 0.8, 0.1), (1, 0.6, 0.3)(2, 0.7, 0.2), (3, 0.6, 0.3)\}.$$

K *is a* $(4, 3)$*- fuzzy set defined on* $\mathbb{Z}_4$*. Here, the membership and non-membership degrees of each element in* $\mathbb{Z}_4$ *satisfies the conditions enlisted in Definition 3.1; hence, it is a* $(4, 3)$*-fuzzy subgroup of* $\mathbb{Z}_4$*.*

Theorem 3.1. *Let* K *be a* (p, q)*-fuzzy subgroup of* G*. Then* $\alpha^p_{\mathrm{K}}(\mathrm{g}^m) \geq \alpha^p_{\mathrm{K}}(\mathrm{g})$ *and* $\beta^q_{\mathrm{K}}(\mathrm{g}^m) \leq \beta^q_{\mathrm{K}}(\mathrm{g})$ *for* $m \in \mathbb{N}$*.*

Proof. By Definition 3.1,

$$\alpha^p_{\mathrm{K}}(\mathrm{g}^2) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}), \alpha^p_{\mathrm{K}}(\mathrm{g})\} = \alpha^p_{\mathrm{K}}(\mathrm{g})$$

and

$$\beta^q_{\mathrm{K}}(\mathrm{g}^2) \leq \max\{\beta^q_{\mathrm{K}}(\mathrm{g}), \beta^q_{\mathrm{K}}(\mathrm{g})\} = \beta^q_{\mathrm{K}}(\mathrm{g}).$$

The proof follows by mathematical induction on m. □

Corollary 3.1. *In Definition 3.1, since* G *is a finite group, condition* 1 *implies condition* 2*.*

Proof. Let $\mathrm{g} \in \mathrm{G}$ with order of g, $o(\mathrm{g}) = n$, then $\mathrm{g}^{-1} = \mathrm{g}^{\mathrm{n}-1}$.

$$\alpha^p(\mathrm{g}^{-1}) = \alpha^p_{\mathrm{K}}(\mathrm{g}^{n-1}) \geq \alpha^p_{\mathrm{K}}(\mathrm{g}).$$

For $\mathrm{g} = \mathrm{g}^{-1}$, it follows that

$$\alpha^p_{\mathrm{K}}(\mathrm{g}) \geq \alpha^p_{\mathrm{K}}(\mathrm{g}^{-1}).$$

Thus, $\alpha^p_{\mathrm{K}}(\mathrm{g}) = \alpha^p_{\mathrm{K}}(\mathrm{g}^{-1})$.
Similarly, for non-membership degree, $\beta^q_{\mathrm{K}}(\mathrm{g}) = \beta^q_{\mathrm{K}}(\mathrm{g}^{-1})$. □

Theorem 3.2. *Let* K *be a* (p, q)*-fuzzy group of* G*,* e *be the identity element in* G*, then* $\alpha^p_{\mathrm{K}}(\mathrm{e}) \geq \alpha^p_{\mathrm{K}}(\mathrm{g})$ *and* $\beta^q_{\mathrm{K}}(\mathrm{e}) \leq \beta^q_{\mathrm{K}}(\mathrm{g})$ *for* $\mathrm{g} \in \mathrm{G}$*.*

Proof. Let $\mathrm{g} \in \mathrm{G}$, $\alpha^p_{\mathrm{K}}(\mathrm{e}) = \alpha^p_{\mathrm{K}}(\mathrm{g}\mathrm{g}^{-1}) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}), \alpha^p_{\mathrm{K}}(\mathrm{g}^{-1})\} = \alpha^p_{\mathrm{K}}(\mathrm{g})$. Analogously, it can be shown that $\beta^q_{\mathrm{K}}(\mathrm{e}) \leq \beta^q_{\mathrm{K}}(\mathrm{g})$. □

Theorem 3.3. *Let* K *be a* (p,q)*-fuzzy set on* G*.* K *is a* (p,q)*-fuzzy subgroup of* G *if and only if* $\alpha^p_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}_1), \alpha^p_{\mathrm{K}}(\mathrm{g}_2)\}$ *and* $\beta^q_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) \leq \max\{\beta^q_{\mathrm{K}}(\mathrm{g}_1), \beta^q_{\mathrm{K}}(\mathrm{g}_2)\}$ *for* $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$*.*

Proof. Let K be a (p,q)-fuzzy set on G. Suppose K is a (p,q)-fuzzy subgroup of G, then for $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$,

$$\alpha^p_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}_1), \alpha^p_{\mathrm{K}}(\mathrm{g}_2{}^{-1})\} = \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}_1), \alpha^p_{\mathrm{K}}(\mathrm{g}_2)\}$$

and

$$\beta^q_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) \leq \max\{\beta^q_{\mathrm{K}}(\mathrm{g}_1), \beta^q_{\mathrm{K}}(\mathrm{g}_2{}^{-1})\} = \max\{\beta^q_{\mathrm{K}}(\mathrm{g}_1), \beta^q_{\mathrm{K}}(\mathrm{g}_2)\}.$$

Conversely, let $\alpha^p_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}_1), \alpha^p_{\mathrm{K}}(\mathrm{g}_2)\}$ and $\beta^q_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) \leq \max\{\beta^q_{\mathrm{K}}(\mathrm{g}_1), \beta^q_{\mathrm{K}}(\mathrm{g}_2)\}$ for $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$. Then

$$\alpha^p_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2) = \alpha^p_{\mathrm{K}}(\mathrm{g}_1(\mathrm{g}_2{}^{-1})^{-1}) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}_1), \alpha^p_{\mathrm{K}}(\mathrm{g}_2{}^{-1})\} = \min\{\alpha^p_{\mathrm{K}}(\mathrm{g}_1), \alpha^p_{\mathrm{K}}(\mathrm{g}_2)\}.$$

Similarly, it can be proved that $\beta^q_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2) \leq \max\{\beta^q_{\mathrm{K}}(\mathrm{g}_1), \beta^q_{\mathrm{K}}(\mathrm{g}_2)\}$.

Now consider $\alpha^p_{\mathrm{K}}(\mathrm{g}_1{}^{-1})$,

$$\alpha^p_{\mathrm{K}}(\mathrm{g}_1{}^{-1}) = \alpha^p_{\mathrm{K}}(\mathrm{e}\mathrm{g}_1{}^{-1}) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{e}), \alpha^p_{\mathrm{K}}(\mathrm{g}_1)\} = \alpha^p_{\mathrm{K}}(\mathrm{g}_1), \tag{1}$$

which gives

$$\alpha^p_{\mathrm{K}}(\mathrm{g}_1) = \alpha^p_{\mathrm{K}}((\mathrm{g}_1{}^{-1})^{-1}) \geq \alpha^p_{\mathrm{K}}(\mathrm{g}_1{}^{-1}). \tag{2}$$

From (1) and (2), $\alpha^p_{\mathrm{K}}(\mathrm{g}_1{}^{-1}) = \alpha^p_{\mathrm{K}}(\mathrm{g}_1)$. Similarly, it can be proved that $\beta^q_{\mathrm{K}}(\mathrm{g}_1{}^{-1}) = \beta^q_{\mathrm{K}}(\mathrm{g}_1)$. Hence, K is a (p,q)-fuzzy subgroup of G. □

Theorem 3.4. *Let* K_1 *and* K_2 *be two* (p,q)*-fuzzy subgroups of* G*. Then* $\mathrm{K}_1 \cap \mathrm{K}_2$ *is a* (p,q)*-fuzzy subgroup of* G*.*

Proof. For $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$,

$$\begin{aligned}
\alpha^p_{\mathrm{K}_1\cap\mathrm{K}_2}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) &= \min\{\alpha^p_{\mathrm{K}_1}(\mathrm{g}_1\mathrm{g}_2{}^{-1}), \alpha^p_{\mathrm{K}_2}(\mathrm{g}_1\mathrm{g}_2{}^{-1})\} \\
&\geq \min\{\min\{\alpha^p_{\mathrm{K}_1}(\mathrm{g}_1), \alpha^p_{\mathrm{K}_1}(\mathrm{g}_2{}^{-1})\}, \min\{\alpha^p_{\mathrm{K}_2}(\mathrm{g}_1), \alpha^p_{\mathrm{K}_2}(\mathrm{g}_2{}^{-1})\}\} \\
&= \min\{\min\{\alpha^p_{\mathrm{K}_1}(\mathrm{g}_1), \alpha^p_{\mathrm{K}_1}(\mathrm{g}_2)\}, \min\{\alpha^p_{\mathrm{K}_2}(\mathrm{g}_1), \alpha^p_{\mathrm{K}_2}(\mathrm{g}_2)\}\} \\
&= \min\{\min\{\alpha^p_{\mathrm{K}_1}(\mathrm{g}_1), \alpha^p_{\mathrm{K}_2}(\mathrm{g}_1)\}, \min\{\alpha^p_{\mathrm{K}_1}(\mathrm{g}_2), \alpha^p_{\mathrm{K}_2}(\mathrm{g}_2)\}\} \\
&= \min\{\alpha^p_{\mathrm{K}_1\cap\mathrm{K}_2}(\mathrm{g}_1), \alpha^p_{\mathrm{K}_1\cap\mathrm{K}_2}(\mathrm{g}_2)\}.
\end{aligned}$$

Similarly, it can be shown that $\beta^q_{\mathrm{K}_1\cap\mathrm{K}_2}(\mathrm{g}_1\mathrm{g}_2{}^{-1}) \leq \max\{\beta^q_{\mathrm{K}_1\cap\mathrm{K}_2}(\mathrm{g}_1), \beta^q_{\mathrm{K}_1\cap\mathrm{K}_2}(\mathrm{g}_2)\}$.
Hence, $\mathrm{K}_1 \cap \mathrm{K}_2$ is a (p,q)-fuzzy subgroup of G. □

Theorem 3.5. *Let* K *be a* (p,q)*-fuzzy subgroup of* G *and* e *be the identity element in* G*. Then* $\alpha^p_{\mathrm{K}}(\mathrm{xg}) = \alpha^p_{\mathrm{K}}(\mathrm{g})$ *and* $\beta^q_{\mathrm{K}}(\mathrm{xg}) = \beta^q_{\mathrm{K}}(\mathrm{g})$ *for all* $\mathrm{g} \in \mathrm{G}$ *if and only if* $\alpha^p_{\mathrm{K}}(\mathrm{x}) = \alpha^p_{\mathrm{K}}(\mathrm{e})$ *and* $\beta^q_{\mathrm{K}}(\mathrm{x}) = \beta^q_{\mathrm{K}}(\mathrm{e})$*.*

Proof. Suppose $\alpha^p_{\mathrm{K}}(\mathrm{xg}) = \alpha^p_{\mathrm{K}}(\mathrm{g})$ and $\beta^q_{\mathrm{K}}(\mathrm{xg}) = \beta^q_{\mathrm{K}}(\mathrm{g})$ for all $\mathrm{g} \in \mathrm{G}$. Specifically, for $\mathrm{g} = \mathrm{e}$, it follows that

$$\alpha^p_{\mathrm{K}}(\mathrm{x}) = \alpha^p_{\mathrm{K}}(\mathrm{e}), \beta^q_{\mathrm{K}}(\mathrm{x}) = \beta^q_{\mathrm{K}}(\mathrm{e}).$$

Suppose $\alpha^p_{\mathrm{K}}(\mathrm{x}) = \alpha^p_{\mathrm{K}}(\mathrm{e})$ and $\beta^q_{\mathrm{K}}(\mathrm{x}) = \beta^q_{\mathrm{K}}(\mathrm{e})$. Since $\alpha^p_{\mathrm{K}}(\mathrm{g}) \leq \alpha^p_{\mathrm{K}}(\mathrm{e})$ for all $\mathrm{g} \in \mathrm{G}$,

$$\alpha^p_{\mathrm{K}}(\mathrm{xg}) \geq \min\{\alpha^p_{\mathrm{K}}(\mathrm{x}), \alpha^p_{\mathrm{K}}(\mathrm{g})\} = \min\{\alpha^p_{\mathrm{K}}(\mathrm{e}), \alpha^p_{\mathrm{K}}(\mathrm{g})\} = \alpha^p_{\mathrm{K}}(\mathrm{g}) \text{ for all } \mathrm{g} \in \mathrm{G}.$$

Also,

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}) = \alpha_{\mathrm{K}}^{p}(\mathrm{x}^{-1}\mathrm{xg}) \geq \min\{\alpha_{\mathrm{K}}^{p}(\mathrm{x}), \alpha_{\mathrm{K}}^{p}(\mathrm{xg})\} = \alpha_{\mathrm{K}}^{p}(\mathrm{xg}) \text{ for all } \mathrm{g} \in \mathrm{G}.$$

Hence, $\alpha_{\mathrm{K}}^{p}(\mathrm{g}) = \alpha_{\mathrm{K}}^{p}(\mathrm{xg})$ for all $\mathrm{g} \in \mathrm{G}$. Similarly, $\beta_{\mathrm{K}}^{q}(\mathrm{xg}) = \beta_{\mathrm{K}}^{q}(\mathrm{g})$ for all $\mathrm{g} \in \mathrm{G}$. □

Theorem 3.6. *Let* K *be a* (p,q)*-fuzzy subgroup of* G *and* e *be the identity element in* G. *Then* $\mathrm{H} = \{\mathrm{g} \in \mathrm{G} : \alpha_{\mathrm{K}}^{p}(\mathrm{g}) = \alpha_{\mathrm{K}}^{p}(\mathrm{e}), \beta_{\mathrm{K}}^{q}(\mathrm{g}) = \beta_{\mathrm{K}}^{q}(\mathrm{e})\}$ *is a subgroup of* G.

Proof. By the definition of H, it follows that $\mathrm{e} \in \mathrm{H}$. Hence, H is a non-empty subset of G. Let $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{H}$, then $\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1) = \alpha_{\mathrm{K}}^{p}(\mathrm{e}) = \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2)$ and $\beta_{\mathrm{K}}^{q}(\mathrm{g}_1) = \beta_{\mathrm{K}}^{q}(\mathrm{e}) = \beta_{\mathrm{K}}^{q}(\mathrm{g}_2)$. Then

$$\begin{aligned}\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1{\mathrm{g}_2}^{-1}) &\geq \min\left[(\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1), \alpha_{\mathrm{K}}^{p}({\mathrm{g}_2}^{-1})\right] \\ &= \min\left[\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1), \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2)\right] \\ &= \min\left[\alpha_{\mathrm{K}}^{p}(\mathrm{e}), \alpha_{\mathrm{K}}^{p}(\mathrm{e})\right] = \alpha_{\mathrm{K}}^{p}(\mathrm{e}).\end{aligned}$$

By Theorem 3.2, $\alpha_{\mathrm{K}}^{p}(\mathrm{e}) \geq \alpha_{\mathrm{K}}^{p}(\mathrm{g}_1{\mathrm{g}_2}^{-1})$. Therefore, $\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1{\mathrm{g}_2}^{-1}) = \alpha_{\mathrm{K}}^{p}(\mathrm{e})$. Similarly, it can be shown that $\beta_{\mathrm{K}}^{q}(\mathrm{g}_1{\mathrm{g}_2}^{-1}) = \beta_{\mathrm{K}}^{q}(\mathrm{e})$. Thus, $\mathrm{g}_1{\mathrm{g}_2}^{-1} \in \mathrm{H}$, which proves that H is a subgroup of G. □

Theorem 3.7. *Let* K *be a* (p,q)*-fuzzy subgroup of* G. *Then there exists an element* $\mathrm{x} \in \mathrm{G}$ *such that* $\alpha_{\mathrm{K}}^{p}(\mathrm{x}) \leq \alpha_{\mathrm{K}}^{p}(\mathrm{g})$ *and* $\beta_{\mathrm{K}}^{q}(\mathrm{x}) \geq \beta_{\mathrm{K}}^{q}(\mathrm{g})$ *for all* $\mathrm{g} \in \mathrm{G}$.

Proof. Let A denote the set of all elements in G with the least membership degree, and B denote the set of all elements in G with the greatest non-membership degree, i.e.,

$$A = \{\mathrm{g}' \in \mathrm{G} : \alpha_{\mathrm{K}}^{p}(\mathrm{g}') \leq \alpha_{\mathrm{K}}^{p}(\mathrm{g}) \ \forall \ \mathrm{g} \in \mathrm{G}\}$$
$$B = \{\mathrm{g}'' \in \mathrm{G} : \beta_{\mathrm{K}}^{q}(\mathrm{g}'') \geq \beta_{\mathrm{K}}^{q}(\mathrm{g}) \ \forall \ \mathrm{g} \in \mathrm{G}\}$$

To prove $A \cap B \neq \emptyset$. Let $\mathrm{g}' \in A$ and $\mathrm{g}'' \in B$. Clearly, $\mathrm{g}' = \mathrm{g}'' * \mathrm{s}$ for some $\mathrm{s} \in \mathrm{G}$ and

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}') \geq \min\{\alpha_{\mathrm{K}}^{p}(\mathrm{g}''), \alpha_{\mathrm{K}}^{p}(\mathrm{s})\}.$$

Since $\alpha_{\mathrm{K}}^{p}(\mathrm{g}')$ is the minimum membership degree, either $\alpha_{\mathrm{K}}^{p}(\mathrm{g}'') = \alpha_{\mathrm{K}}^{p}(\mathrm{g}')$ or $\alpha_{\mathrm{K}}^{p}(\mathrm{s}) = \alpha_{\mathrm{K}}^{p}(\mathrm{g}')$.

If $\alpha_{\mathrm{K}}^{p}(\mathrm{g}'') = \alpha_{\mathrm{K}}^{p}(\mathrm{g}')$, it follows that $\mathrm{g}'' \in A$, i.e., $A \cap B \neq \emptyset$ and $\mathrm{x} = \mathrm{g}''$.

If $\alpha_{\mathrm{K}}^{p}(\mathrm{s}) = \alpha_{\mathrm{K}}^{p}(\mathrm{g}')$, consider $\mathrm{g}'' = \mathrm{g}' * \mathrm{s}^{-1}$. Then

$$\beta_{\mathrm{K}}^{q}(\mathrm{g}'') \leq \max\{\beta_{\mathrm{K}}^{q}(\mathrm{g}'), \beta_{\mathrm{K}}^{q}(\mathrm{s}^{-1})\}.$$

Since $\beta_{\mathrm{K}}^{q}(\mathrm{g}'')$ is the maximum non-membership degree, either $\beta_{\mathrm{K}}^{q}(\mathrm{g}') = \beta_{\mathrm{K}}^{q}(\mathrm{g}'')$ or $\beta_{\mathrm{K}}^{q}(\mathrm{s}^{-1}) = \beta_{\mathrm{K}}^{q}(\mathrm{g}'')$.

If $\beta_{\mathrm{K}}^{q}(\mathrm{g}') = \beta_{\mathrm{K}}^{q}(\mathrm{g}'')$, it follows that $\mathrm{g}' \in B$, i.e., $A \cap B \neq \emptyset$ and $\mathrm{x} = \mathrm{g}'$.

If $\beta_{\mathrm{K}}^{q}(\mathrm{s}^{-1}) = \beta_{\mathrm{K}}^{q}(\mathrm{g}'')$, then it follows that $\alpha_{\mathrm{K}}^{p}(\mathrm{s}) = \alpha_{\mathrm{K}}^{p}(\mathrm{g}')$ and $\beta_{\mathrm{K}}^{q}(\mathrm{s}^{-1}) = \beta_{\mathrm{K}}^{q}(\mathrm{s}) = \beta_{\mathrm{K}}^{q}(\mathrm{g}'')$, which means $\mathrm{s} \in A \cap B$ and $\mathrm{x} = \mathrm{s}$.

□

Example 3.2. *In Example* 3.1*, for the* $(4,3)$*-fuzzy subgroup of* $(\mathbb{Z}_4, +_4)$*, the elements,* 1 *and* 3 *in* $\mathbb{Z}_4$ *possess the least membership degree and the greatest non-membership degree.*

Theorem 3.8. *Let* G *be a cyclic group and let* K *be a* (p,q)*-fuzzy subgroup of* G*, then the generators of* G *possess equal membership and non-membership degrees in* K.

Proof. Let g_1, g_2 be two generators of G. Since g_1 is a generator, $\mathrm{g}_2 = {\mathrm{g}_1}^{m}$ for some $m \in \mathbb{N}$.

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}_2) = \alpha_{\mathrm{K}}^{p}({\mathrm{g}_1}^{m}) \geq \alpha_{\mathrm{K}}^{p}(\mathrm{g}_1) \text{ and } \beta_{\mathrm{K}}^{q}(\mathrm{g}_2) = \beta_{\mathrm{K}}^{q}({\mathrm{g}_1}^{m}) \leq \beta_{\mathrm{K}}^{q}(\mathrm{g}_1) \quad (3)$$

Since g_2 is a generator, $\mathrm{g}_1 = {\mathrm{g}_2}^{n}$ for some $n \in \mathbb{N}$.

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1) = \alpha_{\mathrm{K}}^{p}({\mathrm{g}_2}^{n}) \geq \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2) \text{ and } \beta_{\mathrm{K}}^{q}(\mathrm{g}_1) = \beta_{\mathrm{K}}^{q}({\mathrm{g}_2}^{n}) \leq \beta_{\mathrm{K}}^{q}(\mathrm{g}_2) \quad (4)$$

From (3) and (4), $\alpha_K(g_1) = \alpha_K(g_2)$ and $\beta_K(g_1) = \beta_K(g_2)$. □

3.1. **Orders of (p, q)-fuzzy subgroups.**

Definition 3.2. *Let* K *be a (p, q)-fuzzy subgroup of group* G*. For* $g \in G$*, the least positive integer m such that $\alpha^p_K(g^m) = \alpha^p_K(e)$ and $\beta^q_K(g^m) = \beta^q_K(e)$ is called the order of* g *in* K*, denoted as $(p, q) - FO_K(g)$.*

Note that, the order of an element in a (p, q)-fuzzy subgroup K of G is always less than or equal to its order in G. Also, the order of an element and its inverse in K are equal.

Example 3.3. *Let* K *be a $(4, 3)$-fuzzy subgroup of $(\mathbb{Z}_6, +_6)$,*

$$K = \{(0, 0.9, 0.2), (1, 0.9, 0.3), (2, 0.9, 0.2), (3, 0.9, 0.3), (4, 0.9, 0.2), (5, 0.9, 0.3)\}.$$

Here, $(p, q) - FO_K(1) = (p, q) - FO_K(5) = (p, q) - FO_K(3) = 2$ and $(p, q) - FO_K(2) = (p, q) - FO_K(4) = 1$.

Theorem 3.9. *Let* K *be a (p, q)-fuzzy subgroup of* G *and* e *be the identity element in* G*. Let $g_1 \in G$, and let t be a positive integer satisfying $\alpha^p_K({g_1}^t) = \alpha^p_K(e)$ and $\beta^q_K({g_1}^t) = \beta^q_K(e)$. Then, $(p, q) - FO_K(g_1)$ divides t.*

Proof. Let $g_1 \in G$, and let $(p, q) - FO_K(g_1) = m$. For $t \in \mathbb{N}$, by the division algorithm, there exists $s, r \in \mathbb{Z}$ such that

$$t = ms + r,\ 0 \leq r < m$$

$$r = t - ms$$

$$\begin{aligned}
\alpha^p_K({g_1}^r) &= \alpha^p_K({g_1}^{t-ms}) \\
&= \alpha^p_K({g_1}^t {g_1}^{-ms}) \\
&\geq \min\{\alpha^p_K({g_1}^t), \alpha^p_K({g_1}^{-ms})\} \\
&= \min\{\alpha^p_K({g_1}^t), \alpha^p_K({g_1}^{ms})\} \\
&\geq \min\{\alpha^p_K({g_1}^t), \alpha^p_K({g_1}^m)\} \text{ (by Theorem 3.1)} \\
&= \min\{\alpha^p_K(e), \alpha^p_K(e)\} \\
&= \alpha^p_K(e)
\end{aligned}$$

By Theorem 3.2, it follows that $\alpha^p_K(e) \geq \alpha^p_K({g_1}^r)$. Hence $\alpha^p_K({g_1}^r) = \alpha^p_K(e)$. Analogously, it can be shown that $\beta^q_K({g_1}^r) = \beta^q_K(e)$. However, $(p, q) - FO_K(g_1) = m$ implies that m is the least positive integer such that $\alpha^p_K({g_1}^m) = \alpha^p_K(e)$ and $\beta^q_K({g_1}^m) = \beta^q_K(e)$. Hence $r = 0$, which implies that t is a multiple of m or m divides t. □

Corollary 3.2. *Let* K *be a (p, q)-fuzzy subgroup of* G*. Then for each* $g \in G$*, the $(p, q) - FO_K(g)$ divides the order of* G*.*

Proof. By Theorem 3.9, for $g \in G$, its order in K divides its order in G, i.e., $(p, q) - FO_K(g)$ divides $o(g)$. For $g \in G$, as $o(g)$ divides the order of G (denoted as $|G|$), it follows that $(p, q) - FO_K(g)$ divides $|G|$. □

Definition 3.3. *Let* K *be a (p, q)-fuzzy subgroup of* G*. The order of* K *(denoted as $(p, q) - FO(K)$) is defined as the least common multiple of the orders of elements of* G *in* K*, that is $(p, q) - FO(K) = lcm\{(p, q) - FO_K(g) : g \in G\}$.*

Theorem 3.10. *Let* K *be a (p, q)-fuzzy subgroup of* G*. Then the (p, q)-$FO(K)$ divides the order of* G*.*

Proof. Since for each $\mathrm{g} \in \mathrm{G}$, by Corollary 3.2, $(p,q) - FO_{\mathrm{K}}(\mathrm{g})$ divides $|\mathrm{G}|$, it follows that $lcm\{(p,q) - FO_{\mathrm{K}}(\mathrm{g}) : \mathrm{g} \in \mathrm{G}\}$ divides $|\mathrm{G}|$, that is, $(p,q) - FO(\mathrm{K})$ divides $|\mathrm{G}|$. □

As in group theory, where the order of a subgroup of G divides the order of G, the order of a (p,q)-fuzzy subgroup of G also divides the order of G.

Theorem 3.11. *Let* K *be a* (p,q)*-fuzzy subgroup of* G *and* e *be the identity element in* G. *For* $\mathrm{g} \in \mathrm{G}$, $(p,q) - FO_{\mathrm{K}}(\mathrm{g}) = o(\mathrm{g})$ *if and only if* $\{\mathrm{g} \in \mathrm{G} : \alpha^p_{\mathrm{K}}(\mathrm{g}) = \alpha^p_{\mathrm{K}}(\mathrm{e})$ *and* $\beta^q_{\mathrm{K}}(\mathrm{g}) = \beta^q_{\mathrm{K}}(\mathrm{e})\} = \{\mathrm{e}\}$.

Proof. Let $T = \{\mathrm{g} \in \mathrm{G} : \alpha^p_{\mathrm{K}}(\mathrm{g}) = \alpha^p_{\mathrm{K}}(\mathrm{e}) \text{ and } \beta^q_{\mathrm{K}}(\mathrm{g}) = \beta^q_{\mathrm{K}}(\mathrm{e})\}$.
Suppose there exists a $\mathrm{g}_1 \in \mathrm{G}$ such that $(p,q) - FO_{\mathrm{K}}(\mathrm{g}_1) \neq o(\mathrm{g}_1)$, that is,
let $(p,q) - FO_{\mathrm{K}}(\mathrm{g}_1) = m$ and $o(\mathrm{g}_1) = n$, with $m \neq n$.
$o(\mathrm{g}_1) = n \implies {\mathrm{g}_1}^n = \mathrm{e} \implies \alpha^p_{\mathrm{K}}({\mathrm{g}_1}^n) = \alpha^p_{\mathrm{K}}(\mathrm{e})$ and $\beta^q_{\mathrm{K}}({\mathrm{g}_1}^n) = \beta^q_{\mathrm{K}}(\mathrm{e})$.
$(p,q) - FO_{\mathrm{K}}(\mathrm{g}_1) = m$ gives that m is the least positive integer such that $\alpha^p_{\mathrm{K}}({\mathrm{g}_1}^m) = \alpha^p_{\mathrm{K}}(\mathrm{e})$ and $\beta^q_{\mathrm{K}}({\mathrm{g}_1}^m) = \beta^q_{\mathrm{K}}(\mathrm{e})$, hence, it follows that $m < n$. Therefore ${\mathrm{g}_1}^m \in T$, which means $T \neq \{\mathrm{e}\}$. Thus, it is proved that if $T = \{\mathrm{e}\}$, then $(p,q) - FO_{\mathrm{K}}(\mathrm{g}) = o(\mathrm{g})$ for $\mathrm{g} \in \mathrm{G}$.
The converse part follows directly. □

3.2. (p,q)**-fuzzy cosets.** In group theory, it is proven as a consequence of Lagrange's theorem that the number of left (right) cosets of a subgroup divides the order of the group. This article attempts to derive an analogous result for a (p,q)-fuzzy subgroup of G, for which the notion of (p,q)-fuzzy cosets of a (p,q)-fuzzy subgroup is introduced.

Definition 3.4. *Let* $\mathrm{K} = \{(\mathrm{g}, \alpha^p_{\mathrm{K}}(\mathrm{g}), \beta^q_{\mathrm{K}}(\mathrm{g})) : \mathrm{g} \in \mathrm{G}\}$ *be a* (p,q)*-fuzzy subgroup of* G. *Then, for* $\mathrm{x} \in \mathrm{G}$,

(1) *the* (p,q)*-fuzzy set* $\mathrm{xK} = \{(\mathrm{g}, \alpha^p_{\mathrm{xK}}(\mathrm{g}), \beta^q_{\mathrm{xK}}(\mathrm{g})) : \mathrm{g} \in \mathrm{G}\}$ *on* G, *where* $\alpha^p_{\mathrm{xK}}(\mathrm{g}) = \alpha^p_{\mathrm{K}}(\mathrm{x}^{-1}\mathrm{g})$ *and* $\beta^q_{\mathrm{xK}}(\mathrm{g}) = \beta^q_{\mathrm{K}}(\mathrm{x}^{-1}\mathrm{g})$, *is called the* (p,q)*-fuzzy left coset of* K *determined by* x.
(2) *the* (p,q)*-fuzzy set* $\mathrm{Kx} = \{(\mathrm{g}, \alpha^p_{\mathrm{Kx}}(\mathrm{g}), \beta^q_{\mathrm{Kx}}(\mathrm{g})) : \mathrm{g} \in \mathrm{G}\}$ *on* G, *where* $\alpha^p_{\mathrm{Kx}}(\mathrm{g}) = \alpha^p_{\mathrm{K}}(\mathrm{gx}^{-1})$ *and* $\beta^q_{\mathrm{Kx}}(\mathrm{g}) = \beta^q_{\mathrm{K}}(\mathrm{gx}^{-1})$, *is called the* (p,q)*-fuzzy right coset of* K *determined by* x.

Definition 3.5. *Let* $\mathrm{K} = \{(\mathrm{g}, \alpha^p_{\mathrm{K}}(\mathrm{g}), \beta^q_{\mathrm{K}}(\mathrm{g})) : \mathrm{g} \in \mathrm{G}\}$ *be a* (p,q)*-fuzzy subgroup of* G. *Then* K *is called a* (p,q)*-fuzzy normal subgroup of* G *if* $\mathrm{xK} = \mathrm{Kx}$ *for all* $\mathrm{x} \in \mathrm{G}$.

Theorem 3.12. *Let* $\mathrm{K} = \{(\mathrm{g}, \alpha^p_{\mathrm{K}}(\mathrm{g}), \beta^q_{\mathrm{K}}(\mathrm{g}) : \mathrm{g} \in \mathrm{G}\}$ *be a* (p,q)*-fuzzy subgroup of* G. *Then* K *is* (p,q)*-fuzzy normal subgroup of* G *if and only if* $\alpha^p_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2) = \alpha^p_{\mathrm{K}}(\mathrm{g}_2\mathrm{g}_1)$ *and* $\beta^q_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2) = \beta^q_{\mathrm{K}}(\mathrm{g}_2\mathrm{g}_1)$ *for* $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$.

Proof. Let K be a (p,q)-fuzzy normal subgroup of G, which means $\mathrm{xK} = \mathrm{Kx}$ for all $\mathrm{x} \in \mathrm{G}$, that is

$$\alpha^p_{\mathrm{xK}}(\mathrm{g}) = \alpha^p_{\mathrm{Kx}}(\mathrm{g}) \text{ and } \beta^q_{\mathrm{xK}}(\mathrm{g}) = \beta^q_{\mathrm{Kx}}(\mathrm{g}) \text{ for all } \mathrm{x} \in \mathrm{G}.$$

$$\alpha^p_{\mathrm{K}}(\mathrm{x}^{-1}\mathrm{g}) = \alpha^p_{\mathrm{K}}(\mathrm{gx}^{-1}) \text{ and } \beta^q_{\mathrm{K}}(\mathrm{x}^{-1}\mathrm{g}) = \beta^q_{\mathrm{K}}(\mathrm{gx}^{-1}) \text{ for all } \mathrm{x} \in \mathrm{G}.$$

For two elements $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$:

$$\alpha^p_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2)) = \alpha^p_{\mathrm{K}}(\mathrm{g}_1({\mathrm{g}_2}^{-1})^{-1}) = \alpha^p_{\mathrm{K}}(({\mathrm{g}_2}^{-1})^{-1}\mathrm{g}_1) = \alpha^p_{\mathrm{K}}(\mathrm{g}_2\mathrm{g}_1) \text{ and}$$

$$\beta^q_{\mathrm{K}}(\mathrm{g}_1\mathrm{g}_2)) = \beta^q_{\mathrm{K}}(\mathrm{g}_1({\mathrm{g}_2}^{-1})^{-1}) = \beta^q_{\mathrm{K}}(({\mathrm{g}_2}^{-1})^{-1}\mathrm{g}_1) = \beta^q_{\mathrm{K}}(\mathrm{g}_2\mathrm{g}_1).$$

To prove the converse, suppose $\alpha_K^p(g_1g_2) = \alpha_K^p(g_2g_1)$ and $\beta_K^q(g_1g_2) = \beta_K^q(g_2g_1)$ for g_1, $g_2 \in G$. This implies

$$\alpha_K^p\left(x^{-1}g\right) = \alpha_K^p\left(gx^{-1}\right) \text{ and } \beta_K^q(x^{-1}g) = \beta_K^q(gx^{-1})$$

for $x, g \in G$.

Thus, $xK = Kx$ for all $x \in G$.

□

Theorem 3.13 states that a (p, q)-fuzzy subgroup of G is a (p, q)-fuzzy normal subgroup of G if and only if the membership and non-membership functions are constant within the conjugacy classes of G.

Theorem 3.13. *Let* K *be a* (p, q)*-fuzzy subgroup of* G*. Then* K *is* (p, q)*-fuzzy normal subgroup of* G *if and only if* $\alpha_K^p(gxg^{-1}) = \alpha_K^p(x)$ *and* $\beta_K^q(gxg^{-1}) = \beta_K^q(x)$ *for* $x, g \in G$.

Proof. Let K be a (p, q)-fuzzy normal subgroup of G. For x, g $\in$ G,

$$\alpha_K^p(gxg^{-1}) = \alpha_K^p((gx)g^{-1}) = \alpha_K^p(g^{-1}(gx)) = \alpha_K^p(x) \text{ (by Theorem 3.12).}$$

Similarly, it can be proved that $\beta_K^q(gxg^{-1}) = \beta_K^q(x)$.
Conversely, suppose $\alpha_K^p(gxg^{-1}) = \alpha_K^p(x)$ and $\beta_K^q(gxg^{-1}) = \beta_K^q(x)$ for $x, g \in G$.
For $g_1, g_2 \in G$:

$$\begin{aligned}\alpha_K^p(g_1g_2) &= \alpha_K^p((g_2^{-1}g_2)g_1g_2)\\ &= \alpha_K^p((g_2^{-1}g_2)g_1(g_2^{-1})^{-1})\\ &= \alpha_K^p(g_2^{-1}(g_2g_1)(g_2^{-1})^{-1})\\ &= \alpha_K^p(g_2g_1).\end{aligned}$$

Similarly, it can be shown that $\beta_K^q(g_1g_2) = \beta_K^q(g_2g_1)$. Therefore, by Theorem 3.12, K is a (p, q)-fuzzy normal subgroup of G. □

Example 3.4. *Consider the symmetric group* S_3*.* S_3 *has three conjugacy classes: cycle type* 3*,* $2+1$ *and,* $1+1+1$*. Any* (p, q)*-fuzzy subgroup of* S_3 *with the same membership and non-membership values for each cycle type in* S_3 *will be a* (p, q)*-fuzzy normal subgroup of* S_3*. For instance,*

$$S_3 = \{\rho_0 = (1), \rho_1 = (1\,2\,3), \rho_2 = (1\,3\,2), \mu_1 = (2\,3), \mu_2 = (1\,3), \mu_3 = (1\,2)\}$$

$K' = \{(\rho_0, 0.9, 0.1), (\rho_1, 0.75, 0.8), (\rho_2, 0.75, 0.8), (\mu_1, 0.75, 0.7), (\mu_2, 0.75, 0.7), (\mu_3, 0.75, 0.7)\}$ *is a* $(4, 3)$*-fuzzy normal subgroup of* S_3*.*

Theorem 3.14. *Let* K *be a* (p, q)*-fuzzy normal subgroup of* G*. Then* $H = \{g \in G : \alpha_K^p(g) = \alpha_K^p(e), \beta_K^q(g) = \beta_K^q(e)\}$ *is a normal subgroup of* G*.*

Proof. By Theorem 3.6, H is a subgroup of G. To prove that H is a normal subgroup of G. Since K is a (p, q)-fuzzy normal subgroup of G, for $x \in H$ and $g \in G$, by Theorem 3.13

$$\alpha_K^p(gxg^{-1}) = \alpha_K^p(x) = \alpha_K^p(e) \text{ and } \beta_K^q(gxg^{-1}) = \beta_K^q(x) = \beta_K^q(e),$$

which implies $gxg^{-1} \in H$. Hence, H is a normal subgroup of G. □

Theorem 3.15. *Let* K *be a* (p, q)*-fuzzy normal subgroup of* G*. Let* F *denote the collection of all* (p, q)*-fuzzy left cosets of* K*. Then* F *forms a group under the operation* $*$ *defined as* $xK * x'K = xx'K$ *for* x*,* $x' \in G$.

Proof. Initially, it is verified that the operation $*$ is well defined on F.
Let x_1, x_2, x_3, $x_4 \in G$ such that $x_1K = x_2K$ and $x_3K = x_4K$. It needs to be verified that $x_1K * x_3K = x_2K * x_4K$.
From $x_1K = x_2K$, it implies that $\alpha^p_{x_1K}(g) = \alpha^p_{x_2K}(g)$ and $\beta^q_{x_1K}(g) = \beta^q_{x_2K}(g)$ for $g \in G$.
Similarly, $x_3K = x_4K$ implies that $\alpha^p_{x_3K}(g) = \alpha^p_{x_4K}(g)$ and $\beta^q_{x_3K}(g) = \beta^q_{x_4K}(g)$ for $g \in G$.
Hence, for $g \in G$:

$$\alpha^p_K(x_1^{-1}g) = \alpha^p_K(x_2^{-1}g),\ \beta^q_K(x_1^{-1}g) = \beta^q_K(x_2^{-1}g) \tag{5}$$

$$\alpha^p_K(x_3^{-1}g) = \alpha^p_K(x_4^{-1}g),\ \beta^q_K(x_3^{-1}g) = \beta^q_K(x_4^{-1}g) \tag{6}$$

To verify $x_1x_3K = x_2x_4K$, it needs to be shown that, for $g \in G$,

$$\alpha^p_{x_1x_3K}(g) = \alpha^p_{x_2x_4K}(g) \text{ and } \beta^q_{x_1x_2K}(g) = \beta^q_{x_2x_4K}(g),$$

i.e.,

$$\alpha^p_K((x_1x_3)^{-1}g) = \alpha^p_K((x_2x_4)^{-1}g) \text{ and } \beta^q_K((x_1x_3)^{-1}g) = \beta^q_K((x_2x_4)^{-1}g).$$

Consider $\alpha^p_K((x_1x_3)^{-1}g)$,

$$\begin{aligned}\alpha^p_K((x_1x_3)^{-1}g) &= \alpha^p_K({x_3}^{-1}{x_1}^{-1}g)\\ &= \alpha^p_K({x_3}^{-1}{x_1}^{-1}x_2{x_2}^{-1}g)\\ &= \alpha^p_K({x_3}^{-1}{x_1}^{-1}x_2x_4{x_4}^{-1}{x_2}^{-1}g)\\ &\geq \min\{\alpha^p_K({x_3}^{-1}{x_1}^{-1}x_2x_4), \alpha^p_K({x_4}^{-1}{x_2}^{-1}g)\}\end{aligned}$$

Substituting ${x_1}^{-1}x_2x_4$ for g in (6),

$$\begin{aligned}\alpha^p_K({x_3}^{-1}{x_1}^{-1}x_2x_4) &= \alpha^p_K({x_4}^{-1}{x_1}^{-1}x_2x_4)\\ &= \alpha^p_K({x_1}^{-1}x_2) \quad \text{(since K is a } (p,q)\text{-fuzzy normal subgroup of G)}\\ &= \alpha^p_K(e) \quad \text{(by substituting g in (5) with } x_2)\end{aligned}$$

Thus, $\alpha^p_K({x_3}^{-1}{x_1}^{-1}g) \geq \alpha^p_K({x_4}^{-1}{x_2}^{-1}g)$.
Following similar steps, it can be shown that $\alpha^p_K({x_4}^{-1}{x_2}^{-1}g) \geq \alpha^p_K({x_3}^{-1}{x_1}^{-1}g)$.
Hence, for $g \in G$, $\alpha^p_K({x_3}^{-1}{x_1}^{-1}g) = \alpha^p_K({x_4}^{-1}{x_2}^{-1}g)$.
Similarly, it can also be shown that, $\beta^q_K({x_3}^{-1}{x_1}^{-1}g) = \beta^q_K({x_4}^{-1}{x_2}^{-1}g)$ for $g \in G$.
Thus, $x_1x_3K = x_2x_4K$, i.e., the operation $*$ on F is well defined.
The identity element in F is eK, and $xK * x^{-1}K = xx^{-1}K = eK$, $x^{-1}K{*}xK = x^{-1}xK = eK$, which gives the inverse of xK as $x^{-1}K$.
Hence, F is a group under the operation $*$. □

Corollary 3.3. *Let* K *be a* (p, q)*-fuzzy normal subgroup of of* G*. Let* F *denote the collection of all* (p, q)*-fuzzy left cosets of* K*, which forms a group under the operation* $*$ *defined in Theorem* 3.15*. Define a map* $f : G \to F$ *as* $f(x) = xK$*, then* f *is a group homomorphism with ker* $f = \{x \in G : \alpha^p_K(x) = \alpha^p_K(e), \beta^q_K(x) = \beta^q_K(e)\}$*, where* e *is the identity element in* G*.*

Proof. Let $x_1, x_2 \in G$, then $f(x_1x_2) = x_1x_2K = x_1K * x_2K = f(x_1) * f(x_2)$. Thus, f is a group homomorphism.

$$\begin{aligned} ker\ f &= \{x \in G : f(x) = eK\} \\ &= \{x \in G : xK = eK\} \\ &= \{x \in G : \alpha^p_{xK}(g) = \alpha^p_{eK}(g), \beta^q_{xK}(g) = \beta^q_{eK}(g)\ ,\ \text{ for all } g \in G\} \\ &= \{x \in G : \alpha^p_K(x^{-1}g) = \alpha^p_K(g), \beta^q_K(x^{-1}g) = \beta^q_K(g),\ \text{ for all } g \in G\} \\ &= \{x \in G : \alpha^p_K(x) = \alpha^p_K(e), \beta^q_K(x) = \beta^q_K(e)\}\ \text{(by Theorem 3.5)} \end{aligned}$$

Note that, by Theorem 3.6, $ker\ f$ is a subgroup of G. □

Theorem 3.16. *Let* K *be a (p,q)-fuzzy normal subgroup of* G*. Then the number of (p,q)-fuzzy left cosets of* K *divides the order of* G.

Proof. Let K be a (p,q)-fuzzy normal subgroup of G and let $F = \{xK : x \in G\}$ be the collection of all (p,q)-fuzzy left cosets of K. From Corollary 3.3, the function $f : G \to F$, defined by $f(x) = xK$ is a group homomorphism with

$$ker\ f = \{x \in G : \alpha^p_K(x) = \alpha^p_K(e), \beta^q_K(x) = \beta^q_K(e)\}.$$

Note that, $kerf$ forms a subgroup of G. Let H denote $ker\ f$, and express G as the disjoint union of cosets of H:

$$G = x_1H \cup x_2H \cup x_3H \ldots \cup x_kH.$$

To show that there exists a one-to-one correspondence between the elements of F and the cosets of H in G.
Consider the coset $x_iH \in F$ and $h \in H$,

$$\begin{aligned} f(x_ih) &= x_ihK \\ &= x_iK * hK \\ &= x_iK * eK\ \text{(by Theorem 3.5)} \\ &= x_iK \end{aligned}$$

i.e., f maps each element in the coset x_iH to the (p,q)-fuzzy coset x_iK. Hence, define a mapping f' from $\{x_iH : 1 \le i \le k\}$ to F as $f'(x_iH) = x_iK$, which is well defined.
To show that f' is one-to-one.
Suppose $f'(x_iH) = f'(x_jH)$,

$$\begin{aligned} f'(x_iH) = f'(x_jH) &\implies x_iK = x_jK \\ &\implies \alpha^p_K(x_i^{-1}g) = \alpha^p_K(x_j^{-1}g),\ \beta^q_K(x_i^{-1}g) = \beta^q_K(x_j^{-1}g),\ \text{ for all } g \in G \\ &\implies \alpha^p_K(x_i^{-1}x_j) = \alpha^p_K({x_j}^{-1}x_j),\ \beta^q_K(x_i^{-1}x_j) = \beta^q_K(x_j^{-1}x_j)\ \text{ for } g = x_j \\ &\implies \alpha^p_K(x_i^{-1}x_j) = \alpha^p_K(e),\ \beta^q_K(x_i^{-1}x_j) = \beta^q_K(e) \\ &\implies x_i^{-1}x_j \in H \\ &\implies x_iH = x_jH. \end{aligned}$$

Hence, f' is one-to-one, which means that the cardinality of $\{x_iH : 1 \le i \le k\}$ equals the cardinality of F. By Lagrange's theorem, the number of left cosets of H in G divides the order of G. That is, the cardinality of $\{x_iH : 1 \le i \le k\}$ divides the order of G, and hence the cardinality of F divides the order of G. □

The theorems proved for (p, q)-fuzzy left cosets can also be proved analogously for (p, q)-fuzzy right cosets.

3.3. (p, q)**-fuzzy level subgroup.** From a given (p, q)-fuzzy subgroup of G, some subgroups of G, called (p, q)-fuzzy level subgroups, can be identified. This section aims to clarify the concept of (p, q)-fuzzy level subgroups and highlight the key results associated with this idea.

Definition 3.6. *Let* $\mathrm{K} = \big\{\big(\mathrm{g}, \alpha_{\mathrm{K}}^{p}(\mathrm{g}), \beta_{\mathrm{K}}^{q}(\mathrm{g})\big) : \mathrm{g} \in \mathrm{G}\big\}$ *be a* (p, q)*-fuzzy set on* G. *Let* $\mu, \nu \in [0, 1]$. *Then the set* $\mathrm{K}_{\mu,\nu} = \{\mathrm{g} \in \mathrm{G} : \alpha_{\mathrm{K}}^{p}(\mathrm{g}) \geq \mu, \beta_{\mathrm{K}}^{q}(\mathrm{g}) \leq \nu\}$ *is called a* (p, q)*-fuzzy level subset of* K.

Theorem 3.17. *Let* K *be a* (p, q)*-fuzzy subgroup of* G*, then for each* $\mu, \nu \in [0, 1]$ *satisfying* $\mu \leq \alpha_{\mathrm{K}}^{p}(\mathrm{e})$ *and* $\nu \geq \beta_{\mathrm{K}}^{q}(\mathrm{e})$*, the* (p, q)*-fuzzy level subset* $\mathrm{K}_{\mu,\nu}$ *is a subgroup of* G.

Proof. Suppose K is a (p, q)-fuzzy subgroup of G. Let $\mu, \nu \in [0, 1]$ and satisfy $\alpha_{\mathrm{K}}^{p}(\mathrm{e}) \geq \mu$ and $\beta_{\mathrm{K}}^{q}(\mathrm{e}) \leq \nu$, which implies $\mathrm{e} \in \mathrm{K}_{\mu,\nu}$. Hence, $\mathrm{K}_{\mu,\nu}$ is non-empty.
Let $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{K}_{\mu,\nu}$, i.e.,

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1) \geq \mu,\ \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2) \geq \mu \text{ and } \beta_{\mathrm{K}}^{q}(\mathrm{g}_1) \leq \nu,\ \beta_{\mathrm{K}}^{q}(\mathrm{g}_2) \leq \nu.$$

Since K is a (p, q)-fuzzy subgroup of G,

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1{\mathrm{g}_2}^{-1}) \geq \min\{\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1), \alpha_{\mathrm{K}}^{p}({\mathrm{g}_2}^{-1})\} = \min\{\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1), \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2)\} \geq \min\{\mu, \mu\} = \mu.$$

Also,

$$\beta_{\mathrm{K}}^{q}(\mathrm{g}_1{\mathrm{g}_2}^{-1}) \leq \max\{\beta_{\mathrm{K}}^{q}(\mathrm{g}_1), \beta_{\mathrm{K}}^{q}({\mathrm{g}_2}^{-1})\} = \max\{\beta_{\mathrm{K}}^{q}(\mathrm{g}_1), \beta_{\mathrm{K}}^{q}(\mathrm{g}_2)\} \leq \max\{\nu, \nu\} = \nu.$$

Hence, $\mathrm{g}_1{\mathrm{g}_2}^{-1} \in \mathrm{K}_{\mu,\nu}$, i.e., $\mathrm{K}_{\mu,\nu}$ is a subgroup of G. □

The subgroup $\mathrm{K}_{\mu,\nu}$ of G in Theorem 3.17 is referred to as a (p, q)-fuzzy level subgroup of K. Notably, G itself is a (p, q)-fuzzy level subgroup of K, obtained by setting μ to a value less than or equal to the smallest membership degree of all elements in G, and ν to a value greater than or equal to the largest non-membership degree of all elements in G.

Theorem 3.18. *Let* K *be a* (p, q)*-fuzzy set on* G *with* $\alpha_{\mathrm{K}}^{p}(\mathrm{e}) \geq \alpha_{\mathrm{K}}^{p}(\mathrm{g})$ *and* $\beta_{\mathrm{K}}^{q}(\mathrm{e}) \leq \beta_{\mathrm{K}}^{q}(\mathrm{g})$ *for all* $\mathrm{g} \in \mathrm{G}$. *If for each* $\mu, \nu \in [0, 1]$ *satisfying* $\mu \leq \alpha_{\mathrm{K}}^{p}(\mathrm{e})$ *and* $\nu \geq \beta_{\mathrm{K}}^{q}(\mathrm{e})$*, the* (p, q)*-fuzzy level subset* $\mathrm{K}_{\mu,\nu}$ *is a subgroup of* G*, then* K *is a* (p, q)*-fuzzy subgroup of* G.

Proof. Let $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{G}$. Take $\mu = \min\{\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1), \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2)\}$ and $\nu = \max\{\beta_{\mathrm{K}}^{q}(\mathrm{g}_1), \beta_{\mathrm{K}}^{q}(\mathrm{g}_2)\}$.
Clearly, $\mu \leq \alpha_{\mathrm{K}}^{p}(\mathrm{e})$ and $\nu \geq \beta_{\mathrm{K}}^{q}(\mathrm{e})$. Also,

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1) \geq \mu,\ \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2) \geq \mu \text{ and } \beta_{\mathrm{K}}^{q}(\mathrm{g}_1) \leq \nu,\ \beta_{\mathrm{K}}^{q}(\mathrm{g}_2) \leq \nu,$$

which implies $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{K}_{\mu,\nu}$. Since $\mathrm{K}_{\mu,\nu}$ is a subgroup of G, it follows that $\mathrm{g}_1\mathrm{g}_2 \in \mathrm{K}_{\mu,\nu}$, which implies

$$\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1\mathrm{g}_2) \geq \mu = \min\{\alpha_{\mathrm{K}}^{p}(\mathrm{g}_1), \alpha_{\mathrm{K}}^{p}(\mathrm{g}_2)\} \text{ and } \beta_{\mathrm{K}}^{q}(\mathrm{g}_1\mathrm{g}_2) \leq \nu = \max\{\beta_{\mathrm{K}}^{q}(\mathrm{g}_1), \beta_{\mathrm{K}}^{q}(\mathrm{g}_2)\}.$$

Let $\mathrm{g}_1 \in \mathrm{G}$, choosing $\mu = \alpha_{\mathrm{K}}^{p}(\mathrm{g}_1)$ and $\nu = \beta_{\mathrm{K}}^{q}(\mathrm{g}_1)$, it follows that $\mathrm{g}_1 \in \mathrm{K}_{\mu,\nu}$. Since $\mathrm{K}_{\mu,\nu}$ is a subgroup of G, ${\mathrm{g}_1}^{-1} \in \mathrm{K}_{\mu,\nu}$, which implies

$$\alpha_{\mathrm{K}}^{p}({\mathrm{g}_1}^{-1}) \geq \mu \text{ and } \beta_{\mathrm{K}}(\mathrm{g}_1^{-1}) \leq \nu,$$

i.e.,

$$\alpha_{\mathrm{K}}^{p}({\mathrm{g}_1}^{-1}) \geq \alpha_{\mathrm{K}}^{p}(\mathrm{g}_1) \text{ and } \beta_{\mathrm{K}}^{q}({\mathrm{g}_1}^{-1}) \leq \beta_{\mathrm{K}}^{q}(\mathrm{g}_1).$$

Substituting ${g_1}^{-1}$ for g_1, it follows that

$$\alpha_K^p(g_1) \geq \alpha_K^p({g_1}^{-1}) \text{ and } \beta_K^q(g_1) \leq \beta_K^q({g_1}^{-1}).$$

Hence, $\alpha_K^p({g_1}^{-1}) = \alpha_K^p(g_1)$ and $\beta_K^q({g_1}^{-1}) = \beta_K^q(g_1)$. Thus, K is a (p, q)-fuzzy subgroup of G. □

Definition 3.7. *Let* K *be a* (p, q)*-fuzzy subgroup of* G*. An element* $g \in G$ *that has the smallest membership degree and the largest non-membership degree among all elements of* G *is called a generator element of* K*.*

Theorem 3.7 proves the existence of a generator element for a (p, q)-fuzzy subgroup of G.

Theorem 3.19. *Let* G *be a cyclic group, and let* K *be a* (p, q)*-fuzzy subgroup of* G*. Then the generators of* G *are generator elements of* K*.*

Proof. Let g be a generator of G. Then for any $x \in G$, $x = g^t$ for some $t \in \mathbb{N}$. Consider a (p, q)-fuzzy subgroup K of G, then

$$\alpha_K^p(x) \geq \alpha_K^p(g) \text{ and } \beta_K^q(x) \leq \beta_K^q(g),$$

which implies that g is a generator element of K. □

Note that, a generator element of K need not be a generator of the group G.

Theorem 3.20. *Let* K *be a* (p, q)*-fuzzy subgroup of* G*. Then* K *is a* (p, q)*-fuzzy normal subgroup of* G *if and only if for each* $\mu, \nu \in [0, 1]$ *satisfying* $\mu \leq \alpha_K^p(e)$ *and* $\nu \geq \beta_K^q(e)$*, the* (p, q)*-fuzzy level subset* $K_{\mu,\nu}$ *is a normal subgroup of* G*.*

Proof. Suppose K is a (p, q)-fuzzy normal subgroup of G. By Theorem 3.17, the (p, q)-fuzzy level subset $K_{\mu,\nu}$ for each $\mu, \nu \in [0, 1]$ satisfying $\mu \leq \alpha_K^p(e)$ and $\nu \geq \beta_K^q(e)$, is a subgroup of G.
Let $x \in K_{\mu,\nu}$ and $g \in G$. By Theorem 3.13,

$$\alpha^p(gxg^{-1}) = \alpha^p(x) \text{ and } \beta^q(gxg^{-1}) = \beta^q(x).$$

Since $x \in K_{\mu,\nu}$, it follows that $\alpha^p(gxg^{-1}) = \alpha^p(x) \geq \mu$ and $\beta^q(gxg^{-1}) = \beta^q(x) \leq \nu$, which implies $gxg^{-1} \in K_{\mu,\nu}$. Thus, $K_{\mu,\nu}$ is a normal subgroup of G.

Conversely, suppose K is a (p, q)-fuzzy subgroup of G and for each $\mu, \nu \in [0, 1]$ satisfying $\mu \leq \alpha_K^p(e)$ and $\nu \geq \beta_K^q(e)$, the (p, q)-fuzzy level subset $K_{\mu,\nu}$ is a normal subgroup of G. To prove that K is a (p, q)-fuzzy normal subgroup of G.
Let $x, g \in G$, assign $\alpha_K^p(x) = \mu$ and $\beta_K^q(x) = \nu$, then $x \in K_{\mu,\nu}$. Since $K_{\mu,\nu}$ is a normal subgroup of G, it follows that $gxg^{-1} \in K_{\mu,\nu}$, which implies

$$\alpha_K^p(gxg^{-1}) \geq \mu = \alpha_K^p(x) \text{ and } \beta_K^q(gxg^{-1}) \leq \nu = \beta_K^q(x) \tag{7}$$

For $gxg^{-1} \in K_{\mu,\nu}$ and $g^{-1} \in G$, by equation (7)

$$\begin{aligned} &\alpha_K^p(gxg^{-1}) \leq \alpha_K^p(g^{-1}gxg^{-1}(g^{-1})^{-1}) = \alpha_K^p(x) \text{ and} \\ &\beta_K^q(gxg^{-1}) \geq \beta_K^q(g^{-1}gxg^{-1}(g^{-1})^{-1}) = \beta_K^q(x) \end{aligned} \tag{8}$$

From (7) and (8), it can be concluded that $\alpha_K^p(gxg^{-1}) = \alpha_K^p(x)$ and $\beta_K^q(gxg^{-1}) = \beta_K^q(x)$. Hence, by Theorem 3.13, K is a (p, q)-fuzzy normal subgroup of G. □

4. Conclusions

The objective of this article is to introduce the concept of (p, q)-fuzzy subgroups of a finite group. It provides the definition of a (p, q)-fuzzy subgroup of a finite group G and examines various properties associated with it. Additionally, the article defines (p, q)-fuzzy cosets, (p, q)-fuzzy normal subgroups, and (p, q)-fuzzy level subgroups, while also providing several characteristics of these concepts, along with examples. Future research may explore concepts such as (p, q)-fuzzy homomorphisms and (p, q)-fuzzy isomorphisms.

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