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AVAILABILITY AND RELIABILITY ANALYSIS OF A TWO-UNIT PARALLEL SYSTEM SUBJECT TO FAILURE AND REPAIR OPERATING IN A MULTI-LEVEL ENVIRONMENT

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Abstract. In this paper, a two-unit parallel system is considered subject to failure and repair in a multi-level environment. It is assumed that there are N levels of the environment. The failure time of any operating unit in the $k-$th level of the environment is exponentially distributed with parameter $\mu_k, k = 1, 2, \cdots, N$. There is a single repair facility. The system is in down-state when both units are in the repair facility. The down-state is designated as level 0 of the environment. The repairs are done under the 'first-come-first serve' policy. The repair time of a failed unit in the $k-$th level of the environment is exponentially distributed with parameter $\gamma_k, k = 0, 1, 2, \cdots, N$. When the system is in level 0 of the environment, one failed unit is undergoing repair and the other failed unit is waiting for repair. Upon completion of the repair of the unit, the system is immediately switched to operate in $r-$th level of the environment with positive probability $p_r, r = 1, 2, \cdots, N$ and the repair for the other failed unit immediately starts with rate γ_r. Using the techniques of renewal theory and Laplace transforms, transient state probability distribution, steady-state probability distribution, availability and reliability functions and mean down-time of the system are explicitly found. A numerical illustration is provided to highlight the system performance.

Keywords: Two-unit parallel system, multi-level environment, availability, reliability, mean down-time AMS Subject Classification: 60K10, 62N05, 62E15, 90B25

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1. Introduction

The quality of any stochastic system is characterised by its reliability up to any time and availability at any time. The treatise by Gnedenko etal [11] treats mathematical methods of reliability theory. Extensive research has been carried out for several decades in the past on this subject and an exhaustive bibliography is available in [14]. The monograph of Barlow and Prochan [5] describes the probabilistic methods and statistical theory of reliability and life testing. The recent updated edition by Birolini [7] presents the state-of-the-art of reliability, availability, maintainability, and safety of components, equipment and systems.

One of the several ways of improving the availability and reliability characteristics of a stochastic system is by providing parallel or redundant units. But providing a large number of standby units is not economically feasible. Furthermore, the components or units of a stochastic system are subject to deterioration or failure. Even if a system is provided with a few standby units and a repair facility to repair the failed units, there is always a possibility that all units will be queued in the repair facility for a non-trivial time interval, called a down-time of the system. A down-time ends immediately when the repair of the failed unit at the head of repair is completed and switched online for continuance of production. The study of statistical features of the total down-time of a stochastic system with parallel units over a period was carried out by several researchers in the past. Numerous research articles have been written about stochastic redundant systems. Srinivasan and Subramanian [23] have published a unified and comprehensive account of research work done on two-unit stochastic redundant systems. They exhibit in their book that very many different variations of two-unit stochastic redundant repairable systems can be identified. They also indicated some open problems that could be pursued in the coming years. Ravichandran [19] attempted to present a systematic treatment of redundant repairable systems. He also discussed the relevance and interconnections of reliability systems with other areas like inventory and queueing models.

The performance characteristics such as availability and reliability of systems are decided by parameters such as failure rate and repair rate of systems. In the afore-mentioned literature, it is generally assumed that the system parameters remain constant over time. The assumption of fixed values for the system parameters is unrealistic. For example, the life-time of an electrical equipment may be affected by voltage fluctuations which depend on the environment. As another example, consider an electrical power station where several transformers are set up for continuous power supply. If one online transformer fails, a standby transformer will be switched online and the failed transformer will go for repair. The life-time of a transformer depends on the load of the need that in turn depends on environmental factors. Further more, when a failed unit is repaired, it may not be statistically same as the unit before failure. Even the repair time distribution will be affected by down-time of the system. The reason may be attributed to the fact that their analysis becomes complex since the operating characteristics of such systems change their values as and when the environment changes its level randomly over time. These factors necessitated a new line of thinking to study the behaviour of stochastic redundant repairable systems operating in random environments. Much progress has been made in the past in the analysis of redundant repairable systems with fixed parameters (see Srinivasan and Subramanian [23]). Although a considerable amount of research has been done on the stochastic analysis of inventory systems and queueing systems operating in random environments (see for example, Feldman [10], Kalpakam and Arivarignan [12], Song and Zipkin [22], Özekici [15], Özekici and Parlar [16], Yadavalli and van Schoor [26], Paz and Yechiali [18], Udayabaskaran and Dora Pravina [24], Ammar etal [3], Akshaya Ramesh and Udayabaskaran [1, 2]), only a few number of articles have appeared on redundant repairable systems operating in random environments (see Çinlar and Özekici [8], Shaked and Shanthikumar [21], Sengupta [20], Baxter [6], Özekici [15], Özekici and Parlar [16], Assimakopoulos [4]). Lovas and Rásonyi [13] have made a theoretical study of Markov chains on a general state space in a random environment and proved the existence of limiting distributions with applications in queuing theory and machine learning. Pang et al. [17] have analysed birth and death processes in interactive random environments wherein the birth and death rates and the dynamics of the state of the environment are dependent on each other; and they have also proved that the joint invariant measure of the interactive impact leads to a product form solution supported by some applications of these processes in queueing and population growth models. Wei and Liu [25] have investigated reliability optimization problems for series and parallel systems under random shock environment, where the random shock environment for each subsystem is modeled by the nonhomogeneous Poisson process. Dong and Bai [9] have considered a $k-$out-of$-n : F$ system operating in a shock environment, where the external shocks induce state transitions. They have obtained component group failure rates by using Markov chain imbedding and Phase-type distribution. These articles indicate that much work is necessitated to explore the availability and reliability of repairable systems operating in a multi-level random environment. Accordingly, we make an attempt in the present paper to enrich the literature on availability and reliability of repairable systems by studying a stochastic model of a repairable two-unit redundant system operating in a multi-level random environment.

The rest of the paper is organized as follows:

In Section 2, the model is formulated. The governing equations of the model are derived in Section 3. Time-dependent state probabilities of the system are obtained in Section 4. Deduction of steady-state probabilities is done in Section 5. Availability analysis is carried out in Section 6. Section 7 is devoted to finding the mean down time of the system. In Section 8, reliability of the system is examined. Section 9 presents a numerical study of the model. A conclusion of the paper is provided in Section 10.

2. Model Description

Consider a two-unit parallel system in which both units are subject to failure and repair in a multi-level random environment. There are N levels of the environment. The failure time of any operating unit in the $k-$th level of the environment is exponentially distributed with parameter $\mu_k, k = 1, 2, \cdots, N$. There is a single repair facility whose service is level dependent. To be specific, if one of the two operating units in the $k-$level of the environment fails, it is immediately taken for repair and the repair of it commences immediately with rate γ_k. If the other operating unit fails before the completion of the repair of the failed unit, then the repair is pre-emptied and both units are in the repair facility. The system is in down-state when both units are in the repair facility. The down-state is designated as level 0 of the environment. The repair time of a failed unit in the $k-$th level of the environment is exponentially distributed with parameter $\gamma_k, k = 0, 1, 2, \cdots, N$. When the system is in level 0 of the environment, one failed unit is undergoing repair with rate γ_0 and the other failed unit is waiting for repair. Upon completion of the repair of the unit, the system is immediately switched to operate in $r-$th level of the environment with positive probability $p_r, r = 1, 2, \cdots, N$ such that $\sum_{r=1}^{N} p_r = 1$ and the repair for the other failed unit immediately starts with rate γ_r.

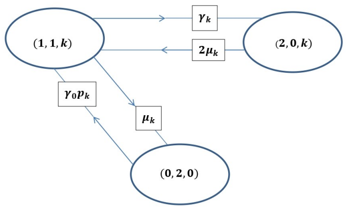
2.1. **Notations.**

$X_1(t)$	: The number of operable units at time t.
$X_2(t)$	: The number of units in the repair facility at time t.
$L(t)$	: The level of the environment at time t.
$\gamma_k e^{-\gamma_k t}$	: Probability density function of the repair-time of any unit in the repair facility while the environment is in level k.
N	: The number of levels of the multi-level environment.
$L(t) = 0$	: Both units are in the repair facility at time t.
$\mu_k e^{-\mu_k t}$	: Probability density function of the life-time of any unit in level k of the environment.
p_k	: The probability that the system is switched online to function in level k of the environment from level 0.
$Z(t)$	: $(X_1(t), X_2(t), L(t))$.
Ω	: The state space $\{(0, 2, 0)\} \cup \{(1, 1, k), (2, 0, k)\|k = 1, 2, \cdots, N\}$.
$P(i, j, k, t)$	: Probability that the system is in the state (i, j, k) at time t given that the system is in the state (0,2,0) at time $t = 0$.
$P^*(i, j, k, s)$	: Laplace transform of $P(i, j, k, t)$.

3. Governing Equations of the model

The state transition diagram of the stochastic model is given below:

Figure 1. Transition Diagram



We assume that the system is in the state $(0, 2, 0)$ at time $t = 0$. Using renewal-theoretic arguments, we obtain the following integral equations:

$$P(0,2,0,t) = e^{-\gamma_0 t} + \sum_{l=1}^{N} \mu_l \int_0^t P(1,1,l,u) e^{-\gamma_0 (t-u)} du, \tag{3.1}$$

$$P(1,1,k,t) = \int_0^t [2\mu_k P(2,0,k,u) + \gamma_0 p_k P(0,2,0,u)] e^{-(\mu_k+\gamma_k)(t-u)} du, \tag{3.2}$$

$$P(2,0,k,t) = \gamma_k \int_0^t P(1,1,k,u) e^{-2\mu_k (t-u)} du. \tag{3.3}$$

4. Transient Solution

Taking Laplace transform of both sides of (3.1) – (3.3), we obtain

$$(s + \gamma_0) P^*(0,2,0,s) = 1 + \sum_{l=1}^{N} \mu_l P^*(1,1,l,s), \tag{4.1}$$

$$(s+\mu_k+\gamma_k)P^*(1,1,k,s) = 2\mu_k P^*(2,0,k,s) + \gamma_0 p_k P^*(0,2,0,s), \tag{4.2}$$

$$(s+2\mu_k)P^*(2,0,k,s) = \gamma_k P^*(1,1,k,s). \tag{4.3}$$

By (4.2) and (4.3), we get

$$P^*(1,1,k,s) = \frac{\gamma_0 p_k(s+2\mu_k)}{s^2+s(3\mu_k+\gamma_k)+2\mu_k^2}P^*(0,2,0,s). \tag{4.4}$$

Substituting (4.4) in (4.3), we get

$$P^*(2,0,k,s) = \frac{\gamma_0\gamma_k p_k}{s^2+s(3\mu_k+\gamma_k)+2\mu_k^2}P^*(0,2,0,s). \tag{4.5}$$

By total probability axiom, we get

$$P^*(0,2,0,s) + \sum_{k=1}^{N} P^*(2,0,k,s) + \sum_{k=1}^{N} P^*(1,1,k,s) = \frac{1}{s}. \tag{4.6}$$

Substituting (4.4) and (4.5) in (4.6) and solving for $P^*(0,2,0,s)$, we get

$$P^*(0,2,0,s) = \frac{1}{(s+\gamma_0)[1-F^*(s)]}, \tag{4.7}$$

where

$$F^*(s) = \frac{\gamma_0}{(s+\gamma_0)}\sum_{j=1}^{N}\frac{p_j\mu_j(s+2\mu_j)}{s^2+s(3\mu_j+\gamma_j)+2\mu_j^2}. \tag{4.8}$$

The two zeros α_j and β_j of the quadratic polynomial $s^2+s(3\mu_j+\gamma_j)+2\mu_j^2$ are real, negative and distinct, and are given by

$$\alpha_j = \frac{-(3\mu_j+\gamma_j)-\sqrt{(3\mu_j+\gamma_j)^2-8\mu_j^2}}{2},$$

$$\beta_j = \frac{-(3\mu_j+\gamma_j)+\sqrt{(3\mu_j+\gamma_j)^2-8\mu_j^2}}{2}.$$

They have the following properties:

$$|\alpha_j| = \frac{(3\mu_j+\gamma_j)+\sqrt{(3\mu_j+\gamma_j)^2-8\mu_k^2}}{2}, \tag{4.9}$$

$$|\beta_j| = \frac{(3\mu_j+\gamma_j)-\sqrt{(3\mu_j+\gamma_j)^2-8\mu_j^2}}{2}, \tag{4.10}$$

$$|\alpha_j| > |\beta_j|, \alpha_j\beta_j = 2\mu_j^2, \tag{4.11}$$

$$s^2+s(3\mu_j+\gamma_j)+2\mu_j^2 \equiv (s+|\alpha_j|)(s+|\beta_j|). \tag{4.12}$$

The function $F^*(s)$ is analytic in the region Real$(s) > \gamma_0$. This condition ensures that $|F^*(s)| < 1$ in the region Real$(s) > \gamma_0$. By using (4.12) in (4.8) and splitting into partial fractions, we get

$$F^*(s) = \gamma_0\sum_{j=1}^{N} p_j\mu_j\left[\frac{2\mu_j-\gamma_0}{(\gamma_0-|\alpha_j|)(\gamma_0-|\beta_j|)}\frac{1}{(s+\gamma_0)}\right.$$

$$\left.+\frac{1}{(|\alpha_j|-|\beta_|)}\left\{\frac{(2\mu_j-|\beta_j|)}{(\gamma_0-|\beta_j|)}\frac{1}{(s+|\beta_j|)}-\frac{(2\mu_j-|\alpha_j|)}{(\gamma_0-|\alpha_j|)}\frac{1}{(s+|\alpha_j|)}\right\}\right]. \tag{4.13}$$

Taking inverse Laplace transform of both sides of (4.13), we get

$$F(t) = \gamma_0 \sum_{j=1}^{N} p_j \mu_j \left[\frac{2\mu_j - \gamma_0}{(\gamma_0 - |\alpha_j|)(\gamma_0 - |\beta_j|)} e^{-\gamma_0 t} \right.$$

$$\left. + \frac{1}{(|\alpha_j| - |\beta_|)} \left\{ \frac{(2\mu_j - |\beta_j|)}{(\gamma_0 - |\beta_j|)} e^{-|\beta_j| t} - \frac{(2\mu_j - |\alpha_j|)}{(\gamma_0 - |\alpha_j|)} e^{-|\alpha_j| t} \right\} \right]. \quad (4.14)$$

By Taylor's series expansion, (4.7) yields,

$$P^*(0,2,0,s) = \frac{1}{(s+\gamma_0)} \sum_{j=0}^{\infty} \{F^*(s)\}^j. \quad (4.15)$$

Taking inverse Laplace transform of both sides of (4.15), we obtain

$$P(0,2,0,t) = e^{-\gamma_0 t} + \int_0^t e^{-\gamma_0 u} F(t-u) du + \sum_{j=2}^{\infty} \int_0^t e^{-\gamma_0 u} F^{(j)}(t-u) du, \quad (4.16)$$

where $F^{(j)}(t)$ is the $j-$fold convolution of $F(t)$ defined by

$$F^{(1)}(t) = F(t), F^{(j)}(t) = \int_0^t F(u) F^{(j-1)}(t-u) du, j \geq 2.$$

Taking inverse transform of both sides of (4.4) and (4.5), we get

$$P(1,1,j,t) = \frac{\gamma_0 p_j}{(|\alpha_j| - |\beta_j|)} \left[(2\mu_j - |\beta_j|) e^{-|\beta_j| t} - (2\mu_j - |\alpha_j|) e^{-|\alpha_j| t} \right] \copyright P(0,2,0,t), \quad (4.17)$$

$$P(2,0,j,t) = \frac{\gamma_0 \gamma_j p_j}{(|\alpha_j| - |\beta_j|)} \left[e^{-|\beta_j| t} - e^{-|\alpha_j| t} \right] \copyright P(0,2,0,t). \quad (4.18)$$

5. Steady-state Solution

The steady-state probabilities are defined by

$$\pi(i,j,k) = \lim_{t \to \infty} P(i,j,k,t), (i,j,k) \in \Omega.$$

By the final value theorem of Laplace transform, we get

$$\pi(i,j,k) = \lim_{s \to 0} s P^*(i,j,k,s), (i,j,k) \in \Omega.$$

Multiplying both sides of (4.4) by s and taking $s \to 0$, we get

$$\pi(1,1,l) = \frac{\gamma_0 p_l}{\mu_l} \pi(0,2,0). \quad (5.1)$$

Multiplying both sides of (4.5) by s and taking $s \to 0$, we get

$$\pi(2,0,l) = \frac{\gamma_0 \gamma_l p_l}{2\mu_l^2} \pi(0,2,0). \quad (5.2)$$

By total probability axiom, we get

$$\pi(0,2,0) + \sum_{l=1}^{N} \pi(2,0,l) + \sum_{l=1}^{N} \pi(1,1,l) = 1. \quad (5.3)$$

Substituting (5.1) and (5.2) in (5.3) and solving for $\pi(0,2,0)$, we get

$$\pi(0,2,0) = \frac{1}{1 + \gamma_0 \sum_{m=1}^{N} \frac{p_m(2\mu_m + \gamma_m)}{2\mu_m^2}}. \quad (5.4)$$

Substituting (5.4) into (5.1) and (5.2), we get

$$\pi(1,1,l) = \frac{\frac{\gamma_0 p_l}{\mu_l}}{1+\gamma_0 \sum_{m=1}^{N} \frac{p_m(2\mu_m+\gamma_m)}{2\mu_m^2}}, l=1,2,\cdots,N, \quad (5.5)$$

$$\pi(2,0,l) = \frac{\frac{\gamma_0 \gamma_l p_l}{2\mu_l^2}}{1+\gamma_0 \sum_{m=1}^{N} \frac{p_m(2\mu_m+\gamma_m)}{2\mu_m^2}}, l=1,2,\cdots,N. \quad (5.6)$$

6. Availability Analysis

We assume that, at time $t=0$, both units are operable and the level of the environment is $k, k=1,2,\cdots,N$. Then, $Z(0)=(2,0,k), k=1,2,\cdots,N$. Let $A_k(t)$ be the probability that the system is available (that is, at least one unit is operable) at time t given that $Z(0)=(2,0,k), k=1,2,\cdots,N$. For the system to be operable at time t, the system is either in the state $(2,0,l), l=1,2,\cdots,N$ or in the state $(1,1,l), l=1,2,\cdots,N$ at time t. Let $\rho_0(t;k,l)$ and $\rho_1(t;k,l)$ denote, respectively, the probabilities that the system is available in states $(2,0,l)$ and $(1,1,l)$ at time t given that the system has been put in the state $(2,0,k)$ at the instant $t=0$. To derive expressions for the availability functions $\rho_r(t;k,l), r=0,1$, we need the following conditional probabilities:

$P_{0,0}(t;k,l)$: $\Pr[Z(t)=(2,0,l)|Z(0)=(2,0,k)], l,k=1,2,\cdots,N$
$P_{0,1}(t;k,l)$: $\Pr[Z(t)=(1,1,l)|Z(0)=(2,0,k)], l,k=1,2,\cdots,N$
$P_{0,2}(t;k,0)$: $\Pr[Z(t)=(0,2,0)|Z(0)=(2,0,k)], k=1,2,\cdots,N$
$P_{1,0}(t;k,l)$: $\Pr[Z(t)=(2,0,l)|Z(0)=(1,1,k)], l,k=1,2,\cdots,N$
$P_{1,1}(t;k,l)$: $\Pr[Z(t)=(1,1,l)|Z(0)=(1,1,k)], l=,k=1,2,\cdots,N$
$P_{1,2}(t;k,0)$: $\Pr[Z(t)=(0,2,0)|Z(0)=(1,1,k)], k=1,2,\cdots,N$
$P_{2,0}(t;0,l)$: $\Pr[Z(t)=(2,0,l)|Z(0)=(0,2,0)], l=1,2,\cdots,N$
$P_{2,1}(t;0,l)$: $\Pr[Z(t)=(1,1,l)|Z(0)=(0,2,0)], l=1,2,\cdots,N$
$P_{2,2}(t;0,0)$: $\Pr[Z(t)=(0,2,0)|Z(0)=(0,2,0)]$.

It is clear that the notation $P_{i,j}(t;k,l)$ stands for the conditional probability that the number of units under repair is j and the environment is in level l at time t given that the number of units under repair is i and the environment is in level k at time $t=0$. Using regeneration point technique, we derive the following system of integral equations for the above 9 probabilities:

$$P_{0,0}(t;k,l) = e^{-2\mu_k t}\delta_{k,l} + 2\mu_k \int_0^t e^{-2\mu_k u} P_{1,0}(t-u;k,l)du, \quad (6.1)$$

$$P_{0,1}(t;k,l) = 2\mu_k \int_0^t e^{-2\mu_k u} P_{1,1}(t-u;k,l)du, \quad (6.2)$$

$$P_{0,2}(t;k,0) = 2\mu_k \int_0^t e^{-2\mu_k u} P_{1,2}(t-u;k,0)du, \quad (6.3)$$

$$P_{1,0}(t;k,l) = \int_0^t e^{-(\mu_k+\gamma_k)u}[\mu_k P_{2,0}(t-u;0,l) + \gamma_k P_{0,0}(t-u;k,l)]du, \quad (6.4)$$

$$P_{1,1}(t;k,l) = e^{-(\mu_k+\gamma_k)t}\delta_{k,l} + \int_0^t e^{-(\mu_k+\gamma_k)u}[\mu_k P_{2,1}(t-u;0,l) + \gamma_k P_{0,1}(t-u;k,l)]du, \quad (6.5)$$

$$P_{1,2}(t;k,0) = \int_0^t e^{-(\mu_k+\gamma_k)u}[\mu_k P_{2,2}(t-u;0,0) + \gamma_k P_{0,2}(t-u;k,0)]du, \quad (6.6)$$

$$P_{2,0}(t;0,l) = \gamma_0 \sum_{k=1}^{N} p_k \int_0^t e^{-\gamma_0 u} P_{1,0}(t-u;k,l)du, \tag{6.7}$$

$$P_{2,1}(t;0,l) = \gamma_0 \sum_{k=1}^{N} p_k \int_0^t e^{-\gamma_0 u} P_{1,1}(t-u;k,l)du, \tag{6.8}$$

$$P_{2,2}(t;0,0) = e^{-\gamma_0 t} + \gamma_0 \sum_{k=1}^{N} p_k \int_0^t e^{-\gamma_0 u} P_{1,2}(t-u;k,0)du. \tag{6.9}$$

Taking Laplace transforms of both sides of (6.1) – (6.9), we get

$$(s+2\mu_k)P^*_{0,0}(s;k,l) = \delta_{k,l} + 2\mu_k P^*_{1,0}(s;k,l), \tag{6.10}$$

$$(s+2\mu_k)P^*_{0,1}(s;k,l) = 2\mu_k P^*_{1,1}(s;k,l), \tag{6.11}$$

$$(s+2\mu_k)P^*_{0,2}(s;k,0) = 2\mu_k P^*_{1,2}(s;k,0), \tag{6.12}$$

$$(s+\mu_k+\gamma_k)P^*_{1,0}(s;k,l) = \mu_k P^*_{2,0}(s;0,l) + \gamma_k P^*_{0,0}(s;k,l), \tag{6.13}$$

$$(s+\mu_k+\gamma_k)P^*_{1,1}(s;k,l) = \delta_{k,l} + \mu_k P^*_{2,1}(s;0,l) + \gamma_k P^*_{0,1}(s;k,l), \tag{6.14}$$

$$(s+\mu_k+\gamma_k)P^*_{1,2}(s;k,0) = \mu_k P^*_{2,2}(s;0,0) + \gamma_k P^*_{0,2}(s;k,0), \tag{6.15}$$

$$(s+\gamma_0)P^*_{2,0}(s;0,l) = \gamma_0 \sum_{k=1}^{N} p_k P^*_{1,0}(s;k,l), \tag{6.16}$$

$$(s+\gamma_0)P^*_{2,1}(s;0,l) = \gamma_0 \sum_{k=1}^{N} p_k P^*_{1,1}(s;k,l), \tag{6.17}$$

$$(s+\gamma_0)P^*_{2,2}(s;0,0) = 1 + \gamma_0 \sum_{k=1}^{N} p_k P^*_{1,2}(s;k,0). \tag{6.18}$$

Using (6.13) in (6.10), we get

$$P^*_{0,0}(s;k,l) = \frac{(s+\mu_k+\gamma_k)}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2}\delta_{k,l} + \frac{2\mu_k^2}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2}P^*_{2,0}(s;0,l). \tag{6.19}$$

Using (6.14) in (6.11), we get

$$P^*_{0,1}(s;k,l) = \frac{2\mu_k\delta_{k,l}}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2} + \frac{2\mu_k^2}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2}P^*_{2,1}(s;0,l). \tag{6.20}$$

Using (6.15) in (6.12), we get

$$P^*_{0,2}(s;k,0) = \frac{2\mu_k^2}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2}P^*_{2,2}(s;0,0), \tag{6.21}$$

Using (6.10) in (6.13), we get

$$P^*_{1,0}(s;k,l) = \frac{\gamma_k\delta_{k,l}}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2} + \frac{\mu_k(s+2\mu_k)}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2}P^*_{2,0}(s;0,l). \tag{6.22}$$

Using (6.11) in (6.14), we get

$$P^*_{1,1}(s;k,l) = \frac{(s+2\mu_k)\delta_{k,l}}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2} + \frac{\mu_k(s+2\mu_k)}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2}P^*_{2,1}(s;0,l). \tag{6.23}$$

Using (6.12) in (6.15), we get

$$P^*_{1,2}(s;k,0) = \frac{\mu_k(s+2\mu_k)}{s^2+(3\mu_k+\gamma_k)s+2\mu_k^2}P^*_{2,2}(s;0,0). \tag{6.24}$$

Multiplying both sides of (6.22) by $\gamma_0 p_k$ and summing from $k = 1$ to N, we get

$$\gamma_0 \sum_{k=1}^{N} p_k P^*_{1,0}(s; k, l) = \frac{\gamma_0 \gamma_l p_l}{s^2 + (3\mu_l + \gamma_l)s + 2\mu_l^2} + \gamma_0 \sum_{k=1}^{N} \frac{\mu_k p_k (s + 2\mu_k)}{s^2 + (3\mu_k + \gamma_k)s + 2\mu_k^2} P^*_{2,0}(s; 0, l). \tag{6.25}$$

Using (6.16) in (6.25) and solving for $P^*_{2,0}(s; 0, l)$, we get

$$P^*_{2,0}(s; 0, l) = \frac{\frac{\gamma_0 \gamma_l p_l}{s^2 + (3\mu_l + \gamma)s + 2\mu_l^2}}{(s + \gamma_0) - \gamma_0 \sum_{k=1}^{N} \frac{\mu_k p_k (s+2\mu_k)}{s^2 + (3\mu_k + \gamma_k)s + 2\mu_k^2}}. \tag{6.26}$$

Multiplying both sides of (6.23) by $\gamma_0 p_k$ and summing from $k = 1$ to N, we get

$$\gamma_0 \sum_{k=1}^{N} p_k P^*_{1,1}(s; k, l) = \frac{\gamma_0 p_l (s + 2\mu_l)}{s^2 + (3\mu_l + \gamma_l)s + 2\mu_l^2} + \gamma_0 \sum_{k=1}^{N} \frac{p_k \mu_k (s + 2\mu_k)}{s^2 + (3\mu_k + \gamma_k)s + 2\mu_k^2} P^*_{2,1}(s; 0, l). \tag{6.27}$$

Using (6.17) in (6.27) and solving for $P^*_{2,1}(s; 0, l)$, we get

$$P^*_{2,1}(s; 0, l) = \frac{\frac{\gamma_0 p_l (s+2\mu_l)}{s^2 + (3\mu_l + \gamma_l)s + 2\mu_l^2}}{(s + \gamma_0) - \gamma_0 \sum_{k=1}^{N} \frac{p_k \mu_k (s+2\mu_k)}{s^2 + (3\mu_k + \gamma_k)s + 2\mu_k^2}}. \tag{6.28}$$

Multiplying both sides of (6.24) by $\gamma_0 p_k$ and summing from $k = 1$ to N, we get

$$\gamma_0 \sum_{k=1}^{N} p_k P^*_{1,2}(s; k, 0) = \sum_{k=1}^{N} \frac{\gamma_0 p_k \mu_k (s + 2\mu_k)}{s^2 + (3\mu_k + \gamma_k)s + 2\mu_k^2} P^*_{2,2}(s; 0, 0). \tag{6.29}$$

Using (6.18) in (6.29), we get

$$P^*_{2,2}(s; 0, 0) = \frac{1}{(s + \gamma_0) - \gamma_0 \sum_{k=1}^{N} \frac{p_k \mu_k (s+2\mu_k)}{s^2 + (3\mu_k + \gamma_k)s + 2\mu_k^2}}. \tag{6.30}$$

Let $\psi(t)$ represent the inverse Laplace transform defined by

$$\psi(t) = L^{-1} \sum_{k=1}^{N} \left[\frac{p_k \mu_k (s + 2\mu_k)}{s^2 + (3\mu_k + \gamma_k)s + 2\mu_k^2} \right]. \tag{6.31}$$

Using (4.12) in (6.31) and taking inverse transform, we have

$$\psi(t) = \sum_{k=1}^{N} \frac{p_k \mu_k}{(|\alpha_k| - |\beta_k|)} \left[(2\mu_k - |\beta_k|) e^{-|\beta_k| t} - (2\mu_k - |\alpha_k|) e^{-|\alpha_k| t} \right]. \tag{6.32}$$

Taking inverse transform of (6.30) and using (6.32), we get

$$P_{2,2}(t; 0, 0) = \sum_{n=0}^{\infty} \frac{e^{-\gamma_0 t} (\gamma_0 t)^n}{n!} \copyright \psi^{(n)}(t). \tag{6.33}$$

Plugging (6.30) into (6.28) and taking inverse Laplace transform, we have

$$P_{2,1}(t; 0, l) = \frac{\gamma_0 p_l}{(|\alpha_l| - |\beta_l|)} \left[((2\mu_l - |\beta_l|) e^{-|\beta_l| t} - (2\mu_l - |\alpha_l|) e^{-|\alpha_l| t} \right] \copyright P_{2,2}(t; 0, 0). \tag{6.34}$$

Plugging (6.30) into (6.26) and taking inverse Laplace transform, we have

$$P_{2,0}(t; 0, l) = \frac{\gamma_0 \gamma_l p_l}{(|\alpha_l| - |\beta_l|)} \left[((2\mu_l - |\beta_l|) e^{-|\beta_l| t} - (2\mu_l - |\alpha_l|) e^{-|\alpha_l| t} \right] \copyright P_{2,2}(t; 0, 0). \tag{6.35}$$

Taking inverse Laplace transform of (6.19), (6.20) and (6.21), we get

$$P_{0,0}(t;k,l) = \frac{\delta_{k,l}}{(|\alpha_k| - |\beta_k|)} \left[(\mu_k + \gamma_k - |\beta_k|)e^{-|\beta_k|t} - (\mu_k + \gamma_k - |\alpha_k|)e^{-|\alpha_k|t}\right] + \frac{2\mu_k^2}{(|\alpha_k| - |\beta_k|)} \left(e^{-|\beta_k|t} - e^{-|\alpha_k|t}\right) ©P_{2,0}(t;0,l), \tag{6.36}$$

$$P_{0,1}(t;k,l) = \frac{2\mu_k\delta_{k,l}}{(|\alpha_k| - |\beta_k|)} \left(e^{-|\beta_k|t} - e^{-|\alpha_k|t}\right) + \frac{2\mu_k^2}{(|\alpha_k| - |\beta_k|)} \left(e^{-|\beta_k|t} - e^{-|\alpha_k|t}\right) ©P_{2,1}(t;0,l), \tag{6.37}$$

$$P_{0,2}(t;k,0) = \frac{2\mu_k^2}{(|\alpha_k| - |\beta_k|)} \left(e^{-|\beta_k|t} - e^{-|\alpha_k|t}\right) ©P_{2,2}(t;0,0). \tag{6.38}$$

Taking inverse Laplace transform of (6.22), (6.23) and (6.24), we get

$$P_{1,0}(t;k,l) = \frac{\gamma_k\delta_{k,l}}{(|\alpha_k| - |\beta_k|)} \left(e^{-|\beta_k|t} - e^{-|\alpha_k|t}\right) + \frac{\mu_k}{(|\alpha_k| - |\beta_k|)} \left[(2\mu_k - \beta_k)e^{-|\beta_k|t} - (2\mu_k - \alpha_k)e^{-|\alpha_k|t}\right] ©P_{2,0}(t;0,l), \tag{6.39}$$

$$P_{1,1}(t;k,l) = \frac{\delta_{k,l}}{(|\alpha_k| - |\beta_k|)} \left[(2\mu_k - |\beta_k|)\, e^{-|\beta_k|t} - (2\mu_k - |\alpha_k|)\, e^{-|\alpha_k|t}\right] + \frac{\mu_k}{(|\alpha_k| - |\beta_k|)} \left[(2\mu_k - |\beta_k|)\, e^{-|\beta_k|t} - (2\mu_k - |\alpha_k|)\, e^{-|\alpha_k|t}\right] ©P_{2,1}(t;0,0), \tag{6.40}$$

$$P_{1,2}(t;k,0) = \frac{\mu_k}{(|\alpha_k| - |\beta_k|)} \left[(2\mu_k - |\beta_k|)\, e^{-|\beta_k|t} - (2\mu_k - |\alpha_k|)\, e^{-|\alpha_k|t}\right] ©P_{2,2}(t;0,0). \tag{6.41}$$

It can be shown that

$$P_{2,0}(t;0,l) = P(2,0,l,t), l = 1,2,\cdots,N,$$
$$P_{2,1}(t;0,l) = P(1,1,l,t), l = 1,2,\cdots,N.$$

Now, we can easily obtain the availability function $A_k(t)$ as given by

$$A_k(t) = \sum_{l=1}^{N}[\rho_0(t;k,l) + \rho_1(t;k,l)] = \sum_{l=1}^{N}[P_{0,0}(t;k,l) + P_{0,1}(t;k,l)]. \tag{6.42}$$

7. Mean down time of the system

We consider a Bernoulli random variable defined by

$$K(t) = \begin{cases} 0, & \text{if } Z(t) \in \{(2,0,k),(1,1,k)|k = 1,2,\cdots,N\} \text{ and} \\ 1, & \text{otherwise.} \end{cases}$$

If $D(t)$ represents the down time of the system in the interval $[0,t]$, then we have

$$D(t) = \int_0^t K(w)dw$$

and hence the mean down time of the system in $[0,t]$ is given by

$$E[D(t)] = E\left[\int_0^t K(w)dw\right] = \int_0^t E[K(w)]dw = \int_0^t Pr[Z(w) = (0,2,0)]dw = \int_0^t P(0,2,0,w)dw. \tag{7.1}$$

Using (4.16) in (7.1), we get

$$E[D(t)] = \sum_{j=0}^{\infty} (-1)^j \int_0^t \int_0^w F^{(j)}(u)dudw. \tag{7.2}$$

8. Reliability of the system

We consider auxiliary functions $P_{0,k}(t)$ and $P_{1,k}(t)$ defined by

$$P_{0,k}(t) = Pr\{Z(t) = (0,2,0), Z(u) \neq (0,2,0) \forall u \in (0,t) | Z(0) = (2,0,k)\},$$
$$P_{1,k}(t) = Pr\{Z(t) = (0,2,0), Z(u) \neq (0,2,0) \forall u \in (0,t) | Z(0) = (1,1,k)\}.$$

Then, $P_{0,k}(t)dt$ represents the conditional probability that the system has not visited the down state $(0,2,0)$ throughout the interval $(0,t)$ and it enters into the down state $(0,2,0)$ between t and $t+dt$ given that both units are operable at $t=0$, and the environment is in level k. In a similar manner, $P_{1,k}(t)dt$ represents the conditional probability that the system has not visited the down state $(0,2,0)$ throughout the interval $(0,t)$ and it enters into the down state $(0,2,0)$ between t and $t+dt$ given that one unit is operable at $t=0$, the other unit is under repair and the environment is in level k.

Using regeneration point technique, we obtain the following integral equations:

$$P_{0,k}(t) = \int_0^t e^{-2\mu_k u} 2\mu_k P_{1,k}(t-u)du. \tag{8.1}$$

$$P_{1,k}(t) = e^{-(\mu_k+\gamma_k)t}\mu_k + \int_0^t e^{-(\mu_k+\gamma_k)u}\gamma_k P_{0,k}(t-u)du. \tag{8.2}$$

Taking Laplace transform of both sides of (8.1) and (8.2), we get

$$(s+2\mu_k)P^*_{0,k}(s) = 2\mu_k P^*_{1,k}(s), \tag{8.3}$$

$$(s+\mu_k+\gamma_k)P^*_{1,k}(s) = \mu_k + \gamma_k P^*_{0,k}(s). \tag{8.4}$$

Solving for $P^*_{0,k}(s)$ and $P^*_{1,k}(s)$, we get

$$P^*_{0,k}(s) = \frac{2\mu_k^2}{s^2 + (3\mu_k+\gamma_k)s + 2\mu_k^2}, \tag{8.5}$$

$$P^*_{1,k}(s) = \frac{\mu_k\{s^2 + (3\mu_k+\gamma_k)s + 2\mu_k^2\} + 2\gamma_k\mu_k^2}{(s+\mu_k+\gamma_k)\{s^2 + (3\mu_k+\gamma_k)s + 2\mu_k^2\}}. \tag{8.6}$$

The reliability function $R_k(t)$ of the system is the probability that the system enters into down state only after time t given that the system started in the state $(2,0,k)$ at time $t=0$. Then, we have

$$R_k(t) = \int_t^\infty P_{0,k}(u)du. \tag{8.7}$$

Taking Laplace transform of both sides of (8.7), we get

$$R^*_k(s) = \frac{1 - P^*_{0,k}(s)}{s}. \tag{8.8}$$

Substituting (8.5) into (8.8), we get

$$R^*_k(s) = \frac{s + (3\mu_k+\gamma_k)}{s^2 + (3\mu_k+\gamma_k)s + 2\mu_k^2}. \tag{8.9}$$

Using (4.12) in (8.9), we get

$$R^*_k(s) = \frac{s + (3\mu_k+\gamma_k)}{(s+|\alpha_k|)(s+|\beta_k|)}. \tag{8.10}$$

Splitting into partial fractions and then taking inverse Laplace transform, (8.10) leads to

$$R_k(t) = \frac{1}{(|\beta_k| - |\alpha_k|)}\left[\{(3\mu_k + \gamma_k) - |\alpha_k|\}e^{-|\alpha_k|t} - \{(3\mu_k + \gamma_k) - |\beta_k|\}e^{-|\beta_k|t}\right]. \quad (8.11)$$

The mean lifetime of the system is given by

$$T_{0,k} = \lim_{s\to 0} R_k^*(s) = \frac{(3\mu_k + \gamma_k)}{\alpha_k\beta_k} = \frac{(3\mu_k + \gamma_k)}{2\mu_k^2}. \quad (8.12)$$

The result (8.12) coincides with the result obtained by Gnedenko et al. (1969).

9. **A Numerical Illustration**

For the purpose of illustration, we assume that there are 5 levels for the environment; that is, $N = 5$. By assumption of the model, the environment changes its level at the instant of switching from down-state to up-state. We fix the probabilities for selecting the environments as follows: $p_1 = 0.10, p_2 = 0.15, p_3 = 0.20, p_4 = 0.35, p_5 = 0.20$. The failure rates and repair rates in the environments are assumed as follows:

$$\begin{aligned} \mu_1 &= 0.1, \quad \gamma_1 = 2.5;\\ \mu_2 &= 0.2, \quad \gamma_2 = 3.5;\\ \mu_3 &= 0.3, \quad \gamma_3 = 4.5;\\ \mu_4 &= 0.4, \quad \gamma_4 = 5.5;\\ \mu_5 &= 0.5, \quad \gamma_5 = 6.5. \end{aligned}$$

The down-state has been considered as level 0. Repair alone is done in level 0 and the repair rate in down state is $\gamma_0 = 8.0$. For achieving a reliable system, we have assumed that the repair rate is greater than the failure rate in each active level of the environment. By using (5.4), (5.5) and (5.6), the steady-state probabilities for the present illustration are given by

$$\begin{array}{ll} \pi(0,2,0) = 0.0034 & \\ \pi(1,1,1) = 0.0274 & \pi(2,0,1) = 0.3425\\ \pi(1,1,2) = 0.0206 & \pi(2,0,2) = 0.1798\\ \pi(1,1,3) = 0.0183 & \pi(2,0,3) = 0.1370\\ \pi(1,1,4) = 0.0240 & \pi(2,0,4) = 0.1648\\ \pi(1,1,5) = 0.0110 & \pi(2,0,5) = 0.0712. \end{array}$$

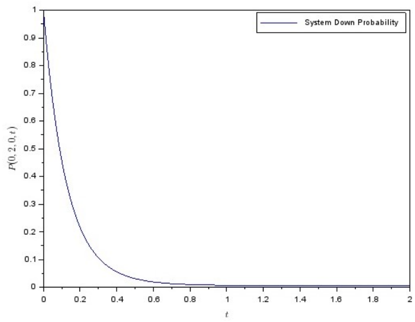
Next, we proceed to obtain time-dependent state probabilities $P(i,j,k,t), (i,j,k) \in \Omega$ and $t > 0$. We assume that the system is in the state $(0,2,0)$ at time $t = 0$ so that $P(0,2,0,0) = 1$. Using (4.16), (4.17) and (4.18), we computed the system probabilities $P(i,j,k,t)$ for values of $t \in (0,1)$. First we computed numerical values of $P(0,2,0,t)$ for values of t in the interval $0 < t \leq 1$. In table 1, we provide system down transient probabilities at time points in $0 < t \leq 1$. We observe that the system-down probability decreases as t increases. We depict the above behaviour in Figure 2. This behaviour is quite natural, since the repair rate is chosen much higher than the failure rate.

TABLE 1. System down probability $P(0,2,0,t)$ as time t increases

t	$P(0,2,0,t)$	t	$P(0,2,0,t)$	t	$P(0,2,0,t)$	t	$P(0,2,0,t)$	t	$P(0,2,0,t)$
0.01	0.9232	0.11	0.4223	0.21	0.2001	0.31	0.0991	0.41	0.0520
0.02	0.8526	0.12	0.3912	0.22	0.1861	0.32	0.0926	0.42	0.0489
0.03	0.7876	0.13	0.3626	0.23	0.1732	0.33	0.0867	0.43	0.0460
0.04	0.7278	0.14	0.3361	0.24	0.1612	0.34	0.0811	0.44	0.0434
0.05	0.6727	0.15	0.3117	0.25	0.1502	0.35	0.0760	0.45	0.0409
0.06	0.6220	0.16	0.2892	0.26	0.1399	0.36	0.0712	0.46	0.0386
0.07	0.5752	0.17	0.2685	0.27	0.1305	0.37	0.0668	0.47	0.0365
0.08	0.5322	0.18	0.2493	0.28	0.1217	0.38	0.0626	0.48	0.0345
0.09	0.4925	0.19	0.2315	0.29	0.1136	0.39	0.0588	0.49	0.0326
0.10	0.4560	0.20	0.2152	0.30	0.1061	0.40	0.0553	0.50	0.0309

t	$P(0,2,0,t)$	t	$P(0,2,0,t)$	t	$P(0,2,0,t)$	t	$P(0,2,0,t)$	t	$P(0,2,0,t)$
0.51	0.0293	0.61	0.0180	0.71	0.0122	0.81	0.0092	0.91	0.0075
0.52	0.0278	0.62	0.0173	0.72	0.0118	0.82	0.0090	0.92	0.0074
0.53	0.0264	0.63	0.0166	0.73	0.0115	0.83	0.0088	0.93	0.0073
0.54	0.0251	0.64	0.0159	0.74	0.0111	0.84	0.0086	0.94	0.0072
0.55	0.0238	0.65	0.0153	0.75	0.0108	0.85	0.0084	0.95	0.0071
0.56	0.0227	0.66	0.0147	0.76	0.0105	0.86	0.0082	0.96	0.0070
0.57	0.0216	0.67	0.0141	0.77	0.0102	0.87	0.0081	0.97	0.0069
0.58	0.0206	0.68	0.0136	0.78	0.0099	0.88	0.0079	0.98	0.0068
0.59	0.0197	0.69	0.0131	0.79	0.0097	0.89	0.0078	0.99	0.0067
0.60	0.0188	0.70	0.0127	0.80	0.0094	0.90	0.0076	1.00	0.0066

FIGURE 2. System-Down Probability versus Time



Next we proceed to obtain the availability of the system at various time points. For the purpose of illustration, we assume $Z(0) = (0,2,0)$ and find the numerical values of the availability function $A(t)$ at time points in the interval $(0,1)$. To achieve this task, we need the probabilities $P_{2,0}(t;0,l)$ and $P_{2,1}(t;0,l)$ at $t = 0.1, 0.2, \cdots, 2.0$. Using these probabilities, we obtain $A(t) = \sum_{l=1}^{5}[P_{2,0}(t;0,l) + P_{2,1}(t;0,l)]$. We furnish the availability probabilities $P_{2,0}(t;0,l)$ and $P_{2,1}(t;0,l)$ in table 2 and table 3.

TABLE 2. $P_{2,0}(t;0,l)$ with respect to environment level l and time t

$t \downarrow l \rightarrow$	1	2	3	4	5
0.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.0070	0.0141	0.0232	0.0476	0.0309
0.20	0.0205	0.0394	0.0622	0.1230	0.0770
0.30	0.0344	0.0636	0.0968	0.1851	0.1125
0.40	0.0468	0.0832	0.1226	0.2280	0.1354
0.50	0.0570	0.0980	0.1403	0.2553	0.1490
0.60	0.0652	0.1087	0.1520	0.2718	0.1565
0.70	0.0717	0.1162	0.1595	0.2813	0.1605
0.80	0.0767	0.1215	0.1642	0.2867	0.1624
0.90	0.0806	0.1251	0.1670	0.2894	0.1632
1.00	0.0836	0.1276	0.1687	0.2907	0.1633
1.10	0.0859	0.1293	0.1697	0.2911	0.1631
1.20	0.0877	0.1305	0.1702	0.2910	0.1626
1.30	0.0891	0.1312	0.1704	0.2906	0.1621
1.40	0.0902	0.1318	0.1704	0.2900	0.1615
1.50	0.0910	0.1321	0.1703	0.2893	0.1609
1.60	0.0917	0.1323	0.1702	0.2886	0.1602
1.70	0.0922	0.1325	0.1700	0.2878	0.1596
1.80	0.0926	0.1326	0.1698	0.2870	0.1589
1.90	0.0930	0.1326	0.1696	0.2863	0.1583
2.00	0.0933	0.1327	0.1694	0.2855	0.1576

TABLE 3. $P_{2,1}(t;0,l)$ with respect to environment level l and time t

$t \downarrow l \rightarrow$	1	2	3	4	5
0.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.0476	0.0674	0.0850	0.1409	0.0764
0.20	0.0589	0.0786	0.0938	0.1482	0.0770
0.30	0.0564	0.0709	0.0808	0.1229	0.0621
0.40	0.0495	0.0590	0.0648	0.0965	0.0484
0.50	0.0419	0.0478	0.0515	0.0767	0.0389
0.60	0.0351	0.0388	0.0418	0.0634	0.0330
0.70	0.0293	0.0321	0.0352	0.0550	0.0295
0.80	0.0246	0.0272	0.0308	0.0498	0.0275
0.90	0.0209	0.0237	0.0280	0.0467	0.0263
1.00	0.0180	0.0213	0.0262	0.0448	0.0255
1.10	0.0158	0.0196	0.0250	0.0437	0.0251
1.20	0.0141	0.0184	0.0243	0.0429	0.0248
1.30	0.0128	0.0176	0.0238	0.0425	0.0246
1.40	0.0118	0.0171	0.0235	0.0421	0.0244
1.50	0.0111	0.0167	0.0233	0.0419	0.0243
1.60	0.0105	0.0164	0.0231	0.0417	0.0241
1.70	0.0100	0.0163	0.0230	0.0415	0.0240
1.80	0.0097	0.0162	0.0229	0.0414	0.0239
1.90	0.0095	0.0161	0.0229	0.0412	0.0238
2.00	0.0093	0.0160	0.0228	0.0411	0.0237

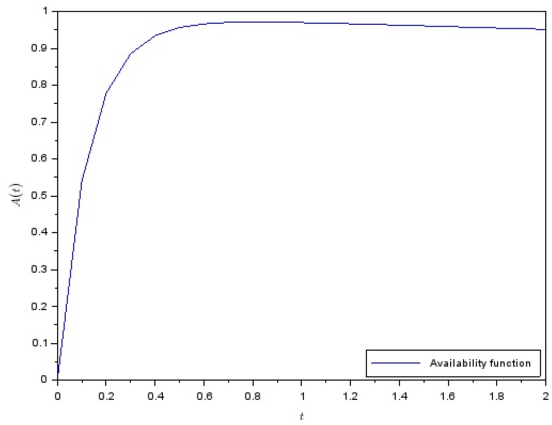
The values of availability function $A(t)$ for various values of t are listed in table 4. We

TABLE 4. Availability $A(t)$ with respect to time t

t	$A(t)$	t	$A(t)$	t	$A(t)$	t	$A(t)$
0.10	0.5401	0.60	0.9663	1.10	0.9683	1.60	0.9588
0.20	0.7786	0.70	0.9703	1.20	0.9665	1.70	0.9569
0.30	0.8855	0.80	0.9714	1.30	0.9647	1.80	0.9550
0.40	0.9342	0.90	0.9709	1.40	0.9628	1.90	0.9533
0.50	0.9564	1.00	0.9697	1.50	0.9609	2.00	0.9514

plot $A(t)$ in figure 3. We find that the probability $A(t)$ of availability increases very rapidly as t increases up to the time point 0.8 and then slowly decreases keeping above the level 0.9.

FIGURE 3. Availability $A(t)$



Lastly, we proceed to exhibit numerically and pictorially the behaviour of reliability function as a function of time t. The table 3 provides reliability values and the figure 4 depicts the reliability as a function of time. It is observed that, at any time t, the reliability satisfies the inequality $R_1(t) > R_2(t) > R_3(t) > R_4(t) > R_5(t)$. This is quite true, since $\mu_1 < \mu_2 < \mu_3 < \mu_4 < \mu_5$.

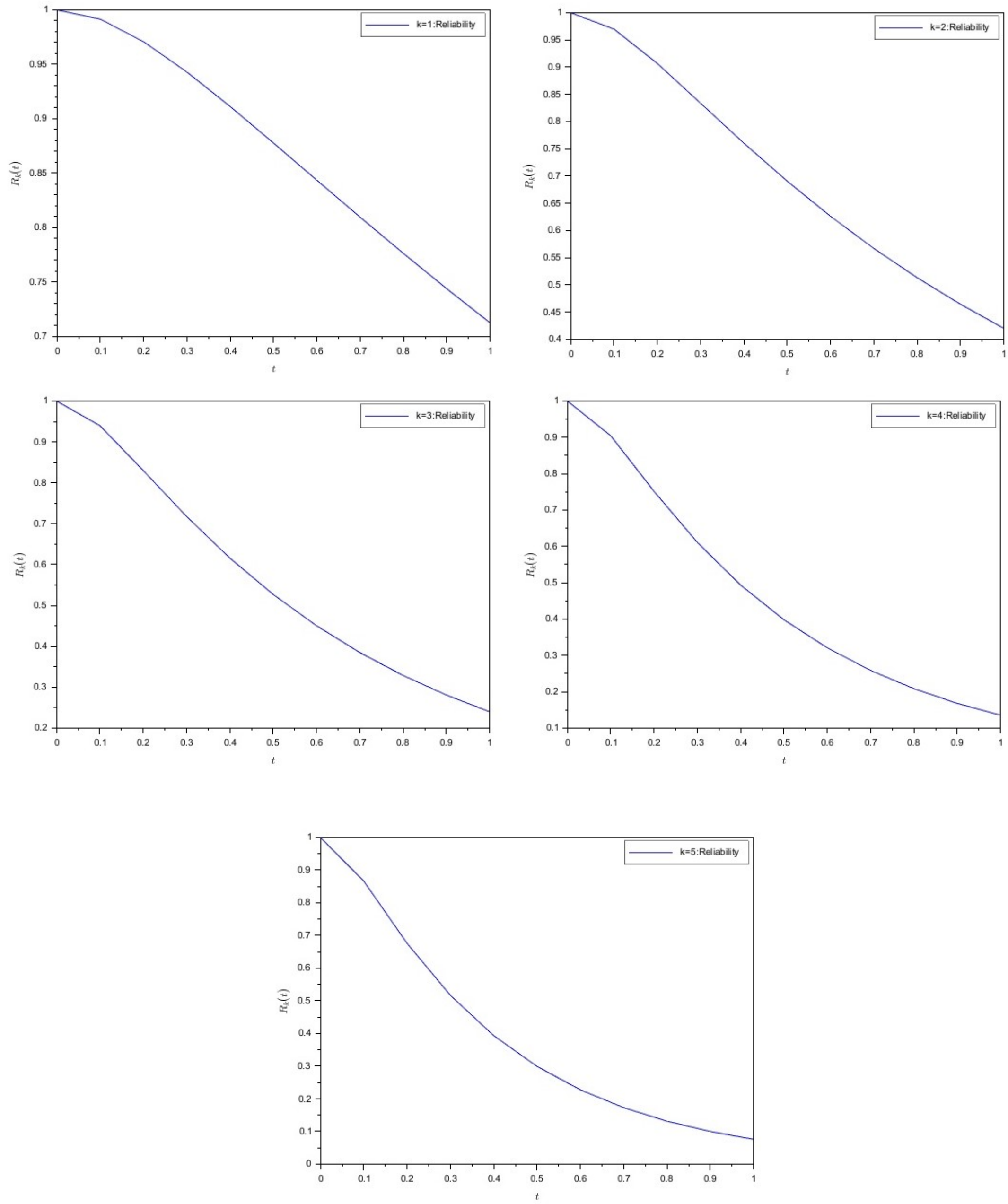
TABLE 5. Reliability in different environments

k	t	$R_k(t)$	k	t	$R_k(t)$	k	t	$R_k(t)$	k	t	$R_k(t)$	k	t	$R_k(t)$
1	0.0	1.0	2	0.0	1.0	3	0.0	1.0	4	0.0	1.0	5	0.0	1.0
	0.1	0.9914913		0.1	0.9699101		0.1	0.9398449		0.1	0.9045258		0.1	0.8662156
	0.2	0.9707515		0.2	0.9068499		0.2	0.8301570		0.2	0.7514317		0.2	0.6757731
	0.3	0.9429190		0.3	0.8336897		0.3	0.7178580		0.3	0.6107711		0.3	0.5166605
	0.4	0.9112221		0.4	0.7602569		0.4	0.6159947		0.4	0.4934769		0.4	0.3933570
	0.5	0.8776791		0.5	0.6905614		0.5	0.5270869		0.5	0.3980420		0.5	0.2992154
	0.6	0.8435436		0.6	0.6260376		0.6	0.4505354		0.6	0.3209129		0.6	0.2275620
	0.7	0.8095851		0.7	0.5669978		0.7	0.3849503		0.7	0.2586951		0.7	0.1730607
	0.8	0.7762675		0.8	0.5132815		0.8	0.3288641		0.8	0.2085322		0.8	0.1316114
	0.9	0.7438621		0.9	0.4645444		0.9	0.2809341		0.9	0.1680945		0.9	0.1000893
	1.0	0.7125191		1.0	0.4203857		1.0	0.2399847		1.0	0.1354980		1.0	0.0761170

10. **Conclusion**

We considered a two-unit parallel system which is subject to failure and repair in a multi-level random environment. We assumed that units have environment level dependent life-time and repair rates. We applied renewal-theoretic approach to obtain the governing equations of the model. We explicitly obtained the availability and reliability of the system. There is a limitation of the model in the sense that the model assumes that the failure and repair rates are not time-dependent and they are assigned with fixed values jointly and instantaneously from a finite set of levels of fixed rates randomly at those epochs e when the number of failed units switches from 2 to 1. That is, between any two successive e events, the system evolves like a two-unit parallel system subject to constant failure and repair rates. This restriction can be relaxed to trigger a new direction of viewing availability and reliability of repairable systems operating in random environment. As a future study of parallel systems, we may extend the present model by incorporating time-dependent rates of failure and repair that are modulated by a random environment. We can also attempt to study reliability and availability of repairable systems with three

FIGURE 4. Reliability as a function of time



or more units operating in random environment.

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