TWMS J. App. and Eng. Math. V.15, N.9, 2025, pp. 2406-2420

AN EFFICIENT EIGHTH ORDER FAMILY OF ITERATIVE METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

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Abstract. In this paper, we analyse the underlying computational cost as well as applicability of the new iterative methods for real-life problems and illustrate that the new schemes produce approximations of greater numerical accuracy for solving nonlinear systems. Basins of attraction are also given for some test problems to study the convergence regions.

Keywords: Iterative methods, system of nonlinear equations, optimal eighth order of convergence, computational cost, convergence regions.

AMS Subject Classification: 65H10,65Y20

1. Introduction

Let us consider the system of equations $G(\vec{s}) = \mathbf{0}$ for solving nonlinear problems, where $G : \mathbb{U} \subseteq \mathbb{R}^t \to \mathbb{R}^t$ is a multivariate vector-valued function also $t \in N$. Numerous techniques have been developed to solve this problem. They are available to the scientific community, making iterative calculations fast and accurate, but there is always a margin of improvement to develop more computationally efficient iterative methods. Almost all physical phenomena exhibit nonlinear behaviour and mathematical modeling can be used to formulate many problems that lead to nonlinear systems of equations in the computational sciences. These, researchers presented applications for solving nonlinear systems of equations as economics modeling problems [12], combustion problems [8], kinematic problems [11], chemical equilibrium problems [13], neurophysiology problems [22], nonlinear cardiac mechanics problems [15], the kinematic synthesis problems for steering and Van der Pol equation problems [4]. The most frequent iterative strategy for solving nonlinear equations and systems of equations is Newton's method which possesses second-order convergence. In the past few decades, optimal fourth-order Jarratt [19] and Steffensen [18] type schemes have been developed but researchers have not remained successful in giving an optimal eighth-order method for solving systems of nonlinear equations. In fact, there

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§ Manuscript received: July 26, 2024; accepted: December 31, 2024.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.9;

are only a few seventh-order iterative schemes for solving nonlinear systems. In order to solve nonlinear equations and nonlinear systems, Wang and Zhang [23] introduced two seventh-order three-step Steffensen-type iterative algorithms.
Sharma and Arora [17] introduced a seventh order iterative approach for nonlinear systems using four functions, two matrix inversions and five divided differences in each iteration. An effective family of three-step iterative methods with seventh-order convergence was proposed by Abad et al. [1]. The weight functions are used to get the proposed approaches. Another seventh-order derivative-free iterative technique for solving nonlinear systems was presented by Wang et al. [24]. The new technique employs one matrix inversion at every iteration as:A three-step, two-parametric family of derivative-free algorithms having seventh-order of convergence was presented by Narang et al. [14] for nonlinear systems. Most recently, a novel three-point iterative method for solving a nonlinear system with seventh-order convergence is introduced by Behl and Arora [3]. Being motivated by ongoing research in this direction, we develop an eighth-order scheme for nonlinear systems. Due to the fact that each iterative strategy deal with nonlinear equations differently, we take into account the convergence order, number of iterations, number of function evaluations and the precision of the desired roots when analyzing the performance of an iterative approach. To verify the effectiveness of the new schemes, we use some numerical problems that enable us to validate the findings from the computational as well as dynamical point of view to evaluate our new methods against the well-known non-optimal eighth-order method given by Cordero et al. [6].

2. Development of the Iterative Scheme

We take the multivariate vector-valued function $G : \mathbb{U} \subseteq \mathbb{R}^t \to \mathbb{R}^t$ for which we can define the divided difference as:

$$[s, r; G]_{mk} \quad = (G_m[s_1, ..., s_{k-1}, s_k, r_{k+1}, ..., r_n] - G_m[s_1, ..., s_{k-1}, r_k, r_{k+1}, ..., r_n])/(s_k - r_k),$$

$$1 \le m, k \le n,$$

where the index m represents the m^{th} function and the index k denotes the nodes. In the procedure of developing our scheme, we employed the weight function technique involving divided differences. Our scheme comprises of three steps which are listed below:

$$\begin{aligned} v^{(t)} &= s^{(t)} - \left(G'\left(s^{(t)}\right)\right)^{-1} G\left(s^{(t)}\right), \\ z^{(t)} &= v^{(t)} - P\left(h^{(t)}\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(v^{(t)}\right), \\ s^{(t+1)} &= z^{(t)} - \left(R\left(h^{(t)}\right) + K(h^{(t)}\right) Q\left(u^{(t)}\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(z^{(t)}\right), \end{aligned} \tag{1}$$

where

$$h^{(t)} = I - \left(G'\left(s^{(t)}\right)\right)^{-1} [s^{(t)}, v^{(t)}; G],$$

and

$$u^{(t)} = I - \left(G'\left(s^{(t)}\right)\right)^{-1} [v^{(t)}, z^{(t)}; G] P\left(h^{(t)}\right).$$

where $P, R, K, Q : S_{n\times n}(\mathbb{R}) \to \Gamma(\mathbb{R}^t)$ with $S_{n\times n}$ be the set of $n \times n$ matrices and $\Gamma(\mathbb{R}^t)$, the set of linear operators from $\mathbb{R}^t$ to $\mathbb{R}^t$. The order of convergence of scheme (1) turns out to be eighth under the conditions on the weight function that can be described in the following theorem.

Theorem 2.1. *Let's assume that $G : \mathbb{U} \subseteq \mathbb{R}^t \to \mathbb{R}^t$ be a sufficiently differentiable function in a closed neighborhood $\mathbb{U}$ that contains the simple root Υ. We take into account the fact that $G^{'}(s)$ is continuous and non-singular at Υ. Additionally, if we use the initial guess $s^{(0)}$ close to the root and the conditions listed below are met then, convergence is assured.*

$$\begin{aligned} P(\mathbf{0}) &= I, P'(\mathbf{0}) = 2, P''(\mathbf{0}) = \mathbf{0}, P'''(\mathbf{0}) = \mathbf{0}, \left|P^{iv}(\mathbf{0})\right| < \infty, \\ R(\mathbf{0}) &= I, R'(\mathbf{0}) = 2, R''(\mathbf{0}) = 2, R'''(\mathbf{0}) = -24, \left|R^{iv}(\mathbf{0})\right| < \infty, \\ K(\mathbf{0}) &= I, K'(\mathbf{0}) = 4, \left|K''(\mathbf{0})\right| < \infty, \\ Q(\mathbf{0}) &= \mathbf{0}, Q'(\mathbf{0}) = I, \left|Q''(\mathbf{0})\right| < \infty. \end{aligned}$$

Proof. Let us consider that $e^{(t)} = s^{(t)} - \Upsilon$ is the error in the t^{th} iteration. The Taylor's series expansion of the function $G(s^{(t)})$and its first order derivative $G'(s^{(t)})$ with the assumption $|G'(\Upsilon)| \neq \mathbf{0}$ leads us to

$$G(s^{(t)}) = G'(\Upsilon)(e^{(t)} + c_2(e^{(t)})^2 + c_3(e^{(t)})^3 + c_4(e^{(t)})^4 + ... + O((e^{(t)})^9)), \tag{2}$$

where,

$$c_i = \frac{1}{i!}\left[G'(\Upsilon)\right]^{-1} G^{(i)}(\Upsilon), i = 2, 3, ...$$

and

$$G'(s^{(t)}) = G'(\Upsilon)(I + 2c_2e^{(t)} + 3c_3(e^{(t)})^2 + 4c_4(e^{(t)})^3 + ... + O((e^{(t)})^8)). \tag{3}$$

$$G''(s^{(t)}) = G'(\Upsilon)(2c_2 + 6c_3e^{(t)} + 12c_4(e^{(t)})^2 + 20c_5(e^{(t)})^3 + ... + O((e^{(t)})^6)). \tag{4}$$

$$G'''(s^{(t)}) = G'(\Upsilon)(6c_3 + 24c_4e^{(t)} + 60c_5(e^{(t)})^2 + ... + O((e^{(t)})^5)). \tag{5}$$

$$G^{(4)}(s^{(t)}) = G'(\Upsilon)(24c_4 + 120c_5e^{(t)} + 360c_6(e^{(t)})^2 + 840c_7(e^{(t)})^3 + O((e^{(t)})^4)). \tag{6}$$

Inversion of $G'(s^{(t)})$ gives

$$(G'(s^{(t)}))^{-1} = I - 2c_2e^{(t)} + (4c_2^2 - 3c_3)(e^{(t)})^2 + ... + O((e^{(t)})^5). \tag{7}$$

Subsequently,

$$v^{(t)} = s^{(t)} - (G'(s^{(t)}))^{-1}G\left(s^{(t)}\right). \tag{8}$$

By using (2) and (7) in (8), we get

$$v^{(t)} = c_2(e^{(t)})^2 + (2c_3 - 2c_2^2)(e^{(t)})^3 + (3c_4 - 7c_2c_3 + 4c_2^3)(e^{(t)})^4 + \sum_{i=5}^{8} A_i(e^{(t)})^i + O((e^{(t)})^9),$$

such that,

$$A_i = A_i(c_2, c_3, ...c_6, c_7), 5 \leqslant i \leqslant 8.$$

As

$$G(v^{(t)}) = G(s^{(t)}) \mid_{e^{(t)} \longrightarrow v^{(t)} - \Upsilon}$$

implies that

$$\begin{aligned} G(v^{(t)}) = {} & G'(\Upsilon)(c_2(e^{(t)})^2 + (2c_3 - 2c_2^2)(e^{(t)})^3 + (3c_4 - 7c_2c_3 + 4c_2^3)(e^{(t)})^4 \\ & + \sum_{i=5}^{8} A_i(e^{(t)})^i + O((e^{(t)})^9)), \end{aligned}$$

From (3) – (6) we obtain

$$\begin{aligned}[s^{(t)}, v^{(t)}; G] &= \frac{G(v^{(t)}) - G(s^{(t)})}{v^{(t)} - s^{(t)}} = G'(s^{(t)}) + \frac{G'(s^{(t)})}{2!}(v^{(t)} - s^{(t)}) \\ &\quad + \frac{G''(s^{(t)})}{3!}(v^{(t)} - s^{(t)})^2 + O(e^{(t)})^3 \\ &= G'(\Upsilon)(I + c_2 e^{(t)} + (c_3 + c_2^2)(e^{(t)})^2 + (-2c_2^3 + c_4 + 3c_3c_2)(e^{(t)})^3 + O(e^{(t)})^4).\end{aligned}$$

Also, we expand $h^{(t)} = I - \left(G'\left(s^{(t)}\right)\right)^{-1}[s^{(t)}, v^{(t)}; G]$ using Taylor's series expansion

$$h^{(t)} = c_2 e^{(t)} + (2c_3 - 3c_2^2)(e^{(t)})^2 + (3c_4 - 10c_2c_3 + 8c_2^3)(e^{(t)})^3 + \sum_{i=4}^{7} D_i (e^{(t)})^i + O((e^{(t)})^8), \quad (9)$$

for

$$D_i = D_i(c_2, c_3, ...c_6, c_7), 4 \leqslant i \leqslant 7.$$

Using, (9) the Taylor's expansion of function P about zero matrix is given by:

$$\begin{aligned}P(h^{(t)}) &= P(\mathbf{0}) + P'(\mathbf{0})c_2 e^{(t)} + (2P'(\mathbf{0})c_3 - 3P'(\mathbf{0})c_2^2 + \frac{1}{2}P''(\mathbf{0})c_2^2)(e^{(t)})^2 \\ &\quad + \sum_{i=3}^{7} E_i (e^{(t)})^i + O((e^{(t)})^8),\end{aligned}$$

where,

$$\begin{aligned}E_i &= E_i(c_2, c_3, ...c_6, c_7, P(\mathbf{0}), P'(\mathbf{0}), P''(\mathbf{0}), P'''(\mathbf{0}), P^{iv}(\mathbf{0}), P^{v}(\mathbf{0}), P^{vi}(\mathbf{0}), P^{vii}(\mathbf{0})), \\ &\quad 3 \leqslant i \leqslant 7.\end{aligned}$$

Consequently, the second substep

$$z^{(t)} = v^{(t)} - P(h^{(t)})G'(s^{(t)})^{-1}G(v^{(t)}),$$

becomes,

$$\begin{aligned}z^{(t)} &= (c_2 - P(\mathbf{0})c_2)(e^{(t)})^2 + (2c_3 - 2c_2^2 - 2P(\mathbf{0})c_3 + 4P(\mathbf{0})c_2^2 - P'(\mathbf{0})c_2^2)(e^{(t)})^3 \\ &\quad + (14P(\mathbf{0})c_2c_3 - 4P'(\mathbf{0})c_2c_3 + 7P'(\mathbf{0})c_2^3 - 3P(\mathbf{0})c_4 - 13P(\mathbf{0})c_2^3 - \frac{1}{2}P''(\mathbf{0})c_2^3 \\ &\quad + 3c_4 - 7c_2c_3 + 4c_2^3)(e^{(t)})^4 + \sum_{i=5}^{7} H_i (e^{(t)})^i + O((e^{(t)})^8), \quad (10)\end{aligned}$$

where,

$$\begin{aligned}H_i &= H_i(c_2, c_3, ...c_7, c_8, P(\mathbf{0}), P'(\mathbf{0}), P''(\mathbf{0}), P'''(\mathbf{0}), P^{iv}(\mathbf{0}), P^{v}(\mathbf{0}), P^{vi}(\mathbf{0})), \\ &\quad 5 \leqslant i \leqslant 8.\end{aligned}$$

The conditions on P and its derivatives are chosen as:

$$P(\mathbf{0}) = I, P'(\mathbf{0}) = 2,$$

that transforms (10) to fourth order expression as:

$$z^{(t)} = (-c_2c_3 + 5c_2^3 - \frac{1}{2}P''(\mathbf{0})c_2^3)(e^{(t)})^4 + \sum_{i=5}^{7} J_i (e^{(t)})^i + O((e^{(t)})^8).$$

where

$$J_i = J_i(c_2, c_3, ...c_7, c_8, P''(\mathbf{0}), P'''(\mathbf{0}), P^{iv}(\mathbf{0}), P^{v}(\mathbf{0}), P^{vi}(\mathbf{0})), 5 \leqslant i \leqslant 8.$$

As

$$G(z^{(t)}) = G(s^{(t)}) \mid_{e^{(t)} \longrightarrow z^{(t)} - \Upsilon} .$$

By expanding Taylor's series of the function $G(z^{(t)})$, we also get the following expression

$$G(z^{(t)}) = G'(\Upsilon)((-c_2c_3 + 5c_2^3 - \frac{1}{2}P''(\mathbf{0})c_2^3)(e^{(t)})^4 + \sum_{i=5}^{8} J_i(e^{(t)})^i + O((e^{(t)})^9)),$$

We obtain following expression of the operator $[v^{(t)}, z^{(t)}; G]$

$$[v^{(t)}, z^{(t)}; G] = G'(\Upsilon)(I + c_2^2(e^{(t)})^2 + (2c_2c_3 - 2c_2^3)(e^{(t)})^3 + \sum_{i=4}^{6} M_i(e^{(t)})^i + O((e^{(t)})^7)),$$

where

$$M_i = M_i(c_2, c_3, ...c_6, c_7, P''(\mathbf{0}), P'''(\mathbf{0}), P^{(4)}(\mathbf{0})), 3 \leqslant i \leqslant 6.$$

Applying Taylor's series to $u^{(t)} = I - \left(G'\left(s^{(t)}\right)\right)^{-1}[v^{(t)}, z^{(t)}; G]P\left(h^{(t)}\right)$, we get

$$u^{(t)} = (-c_3 + 5c_2^2 - \frac{1}{2}P''(\mathbf{0})c_2^2)(e^{(t)})^2 + \sum_{i=3}^{6} M_i(e^{(t)})^i + O((e^{(t)})^7).$$

Then, we apply Taylor's series expansion to $R(h^{(t)})$, $K(h^{(t)}$ and $Q(u^{(t)})$ as follows

$$\begin{aligned} R(h^{(t)}) &= R(\mathbf{0}) + R'(\mathbf{0})c_2e^{(t)} + (2c_3R'(\mathbf{0}) - 3c_2^2R'(\mathbf{0}) + \frac{1}{2}c_2^2R''(\mathbf{0}))(e^{(t)})^2 \\ &\quad + \sum_{i=3}^{7} N_i(e^{(t)})^i + O((e^{(t)})^8), \end{aligned} \tag{11}$$

where,

$$\begin{aligned} N_i &= N_i(c_2, c_3, ...c_6, c_7, R'(\mathbf{0}), R''(\mathbf{0}), R'''(\mathbf{0}), R^{(4)}(\mathbf{0})), R^{(5)}(\mathbf{0}), R^{(6)}(\mathbf{0}), \\ &\quad R^{(7)}(\mathbf{0}), 3 \leqslant i \leqslant 7. \end{aligned}$$

and

$$\begin{aligned} K(h^{(t)} &= K(\mathbf{0}) + K'(\mathbf{0})c_2e^{(t)} + (2c_3K'(\mathbf{0}) - 3c_2^2K'(\mathbf{0}) + \frac{1}{2}c_2^2K''(\mathbf{0}))(e^{(t)})^2 \\ &\quad + \sum_{i=3}^{7} U_i(e^{(t)})^i + O((e^{(t)})^8), \end{aligned} \tag{12}$$

where,

$$\begin{aligned} U_i = U_i(c_2, c_3, ...c_6, c_7, K'(\mathbf{0}), K''(\mathbf{0}), K'''(\mathbf{0}), K^{(4)}(\mathbf{0}), \\ K^{(5)}(\mathbf{0}), K^{(6)}(\mathbf{0}), K^{(7)}(\mathbf{0})), 3 \leqslant i \leqslant 7. \end{aligned}$$

Also

$$\begin{aligned} Q(u^{(t)}) &= Q(\mathbf{0}) + Q'(\mathbf{0})(-c_3 + 5c_2^2 - \frac{1}{2}P''(\mathbf{0})c_2^2)(e^{(t)})^2 + Q'(\mathbf{0})(-2c_4 + 20c_2c_3 - 26c_2^3 \\ &\quad -2P''(\mathbf{0})c_2c_3 + 4P''(\mathbf{0})c_2^3 - P'''(\mathbf{0})c_2^3)(e^{(t)})^3 + \sum_{i=4}^{6} T_i(e^{(t)})^i + O((e^{(t)})^7), \end{aligned} \tag{13}$$

where

$$\begin{aligned} T_i &= T_i(c_2, c_3, ...c_6, c_7, Q'(\mathbf{0}), Q''(\mathbf{0}), Q'''(\mathbf{0}), P''(\mathbf{0}), P'''(\mathbf{0}), P^{(4)}(\mathbf{0}), P^{(5)}(\mathbf{0}) \\ &\quad , P^{(6)}(\mathbf{0})), 4 \leqslant i \leqslant 6. \end{aligned}$$

Consequently, in the last step

$$s^{(t+1)} = z^{(t)} - \left(R\left(h^{(t)}\right) + K(h^{(t)}\right) Q\left(u^{(t)}\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(z^{(t)}\right), \tag{14}$$

by using (11), (12), (13) in (14), we obtain

$$\begin{aligned} s^{(t+1)} = {} & (c_2c_3 - 5c_2^3 + \frac{1}{2}P''(\mathbf{0})c_2^3)(-I + R(\mathbf{0}) + K(\mathbf{0})Q(\mathbf{0}))(e^{(t)})^4 + (2R(\mathbf{0})c_2c_4 - 2c_2c_4 \\ & -2c_3^2 + 32c_3c_2^2 - 36c_2^4 - 3P''(\mathbf{0})c_3c_2^2 + 5P''(\mathbf{0})c_2^4 - \frac{1}{6}P'''(\mathbf{0})c_2^4 + R'(\mathbf{0})c_3c_2^2 - 5R'(\mathbf{0})c_2^4 \\ & -34R(\mathbf{0})c_3c_2^2 - 6R(\mathbf{0})P''(\mathbf{0})c_2^4 + \frac{1}{6}R(\mathbf{0})P'''(\mathbf{0})c_2^4 + 46K(\mathbf{0})Q(\mathbf{0})c_2^4 + 2K(\mathbf{0})Q(\mathbf{0})c_3^2 \\ & +4c_2^2Q(\mathbf{0})c_3 + 2c_2^4Q(\mathbf{0})P''(\mathbf{0}) + \frac{1}{2}R'(\mathbf{0})c_2^4P''(\mathbf{0}) + 3R(\mathbf{0})P''(\mathbf{0})c_3c_2^2 + 2K(\mathbf{0})Q(\mathbf{0})c_2c_4 \\ & -34K(\mathbf{0})Q(\mathbf{0})c_3c_2^2 - 6K(\mathbf{0})Q(\mathbf{0})P''(\mathbf{0})c_2^4 + \frac{1}{6}K(\mathbf{0})Q(\mathbf{0})P'''(\mathbf{0})c_2^4 + 46R(\mathbf{0})c_2^4 + 2R(\mathbf{0})c_3^2 \\ & -20c_2^4Q(\mathbf{0}) + 3K(\mathbf{0})Q(\mathbf{0})P''(\mathbf{0})c_3c_2^2)(e^{(t)})^5 + \sum_{i=5}^{8} W_i(e^{(t)})^i + O((e^{(t)})^9), \end{aligned} \tag{15}$$

where,

$$W_i = W_i(c_2, c_3, ...c_6, c_7, c_8, K(\mathbf{0}), K'(\mathbf{0}), K''(\mathbf{0}), K'''(\mathbf{0}), K^{iv}(\mathbf{0}), R(\mathbf{0}),$$
$$R'(\mathbf{0}), R''(\mathbf{0}), R'''(\mathbf{0}), P''(\mathbf{0}), P'''(\mathbf{0}), P^{iv}(\mathbf{0}), P^{v}(\mathbf{0}), Q(\mathbf{0}), Q'(\mathbf{0}), Q''(\mathbf{0})), 5 \leqslant i \leqslant 8.$$

Applying

$$P''(\mathbf{0}) = \mathbf{0}, P'''(\mathbf{0}) = \mathbf{0}, \left|P^{iv}(\mathbf{0})\right| < \infty,$$

and

$$\begin{aligned} R(\mathbf{0}) &= I, R'(\mathbf{0}) = 2, R''(\mathbf{0}) = 2, R'''(\mathbf{0}) = -24, \left|R^{iv}(\mathbf{0})\right| < \infty, \\ K(\mathbf{0}) &= I, K'(\mathbf{0}) = 4, \left|K''(\mathbf{0})\right| < \infty, \\ Q(\mathbf{0}) &= \mathbf{0}, Q'(\mathbf{0}) = I, \left|Q''(\mathbf{0})\right| < \infty, \end{aligned}$$

in (15), we finally have

$$\begin{aligned} s^{(t+1)} = {} & -\frac{1}{24}(-c_2c_3 + 5c_2^3)(-1080c_2^4 - P^{iv}(\mathbf{0})c_2^4 + R^{iv}(\mathbf{0})c_2^4 + 60K''(\mathbf{0})c_2^4 + 300Q''(\mathbf{0})c_2^4 \\ & +456c_3c_2^2 - 120Q''(\mathbf{0})c_3c_2^2 - 12K''(\mathbf{0})c_3c_2^2 - 24c_2c_4 + 12Q''(\mathbf{0})c_3^2 - 24c_3^2)(e^{(t)})^8 \\ & +O((e^{(t)})^9). \end{aligned}$$

□

This error analysis exhibits that the given scheme (1) has attained the eighth-order of convergence.

Remark: It can be observed that the scheme (1) attains eighth-order convergence with four evaluations of G and its derivatives. So, it is optimal in the sense of Kung-Traub conjecture [10] from a univariate point of view and efficient from a multivariate point of view. Next, we present a few particular cases of our new scheme (1) as:

Case 1: If we consider the weight functions $P(h^{(t)}), R(h^{(t)}), K(h^{(t)})$ and $Q(u^{(t)})$ in the following forms:

$$P(h^{(t)}) = a_0 + a_1h^{(t)} + a_2(h^{(t)})^2,$$
$$R(h^{(t)}) = b_0 + b_1h^{(t)} + b_2(h^{(t)})^2 + b_3(h^{(t)})^3,$$
$$K(h^{(t)}) = c_0 + c_1h^{(t)} + c_2(h^{(t)})^2,$$
$$Q(u^{(t)}) = d_0 + d_1u^{(t)} + d_2(u^{(t)})^2,$$

with

$$a_0 = I, a_1 = 2, a_2 = \mathbf{0},$$
$$b_0 = I, b_1 = 2, b_2 = I, b_3 = -4,$$
$$c_0 = I, c_1 = 4, c_2 = c_2,$$
$$d_0 = \mathbf{0}, d_1 = I, d_2 = d_2,$$

then we have an eighth-order scheme specified as FS_1 by taking $c_2 = -2$ and $d_2 = 4$, which is given below:

$$\begin{aligned} v^{(t)} &= s^{(t)} - (G'\left(s^{(t)}\right))^{-1} G\left(s^{(t)}\right), \\ z^{(t)} &= v^{(t)} - \left(I + 2h^{(t)}\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(v^{(t)}\right), \\ s^{(t+1)} &= z^{(t)} - \left(I + 2h^{(t)} + (h^{(t)})^2 - 4(h^{(t)})^3\right) + \left(I + 4h^{(t)} - 2(h^{(t)})^2\right) \\ &\quad \left(u^{(t)} + 4(u^{(t)})^2\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(z^{(t)}\right). \end{aligned}$$

Case 2: If the weight functions $P(h^{(t)})$, $R(h^{(t)})$, $K(h^{(t)})$ and $Q(u^{(t)})$ are of the following forms:

$$P(h^{(t)}) = a_0 + a_1 h^{(t)} + a_2 (h^{(t)})^2,$$
$$R(h^{(t)}) = b_0 + b_1 h^{(t)} + b_2 (h^{(t)})^2 + b_3 (h^{(t)})^3,$$
$$K(h^{(t)}) = (I + c_2 (h^{(t)})^2)^{-1} (I + c_1 h^{(t)}),$$
$$Q(u^{(t)}) = d_0 + d_1 u^{(t)} + d_2 (u^{(t)})^2,$$

along with

$$a_0 = I, a_1 = 2, a_2 = \mathbf{0},$$
$$b_0 = I, b_1 = 2, b_2 = I, b_3 = -4I,$$
$$c_1 = 4, c_2 = c_2,$$
$$d_0 = \mathbf{0}, d_1 = I, d_2 = d_2,$$

then $c_2 = -I$ and $d_2 = 4$, we obtain the following eighth-order scheme namely FS_2

$$\begin{aligned} v^{(t)} &= s^{(t)} - (G'\left(s^{(t)}\right))^{-1} G\left(s^{(t)}\right), \\ z^{(t)} &= v^{(t)} - \left(I + 2h^{(t)}\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(v^{(t)}\right), \\ s^{(t+1)} &= z^{(t)} - \left(I + 2h^{(t)} + (h^{(t)})^2 - 4(h^{(t)})^3\right) + \left(I - (h^{(t)})^2\right)^{-1}\left(I + 4h^{(t)}\right) \\ &\quad \left(u^{(t)} + 4(u^{(t)})^2\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(z^{(t)}\right). \end{aligned}$$

Case 3: If the weight functions $P(h^{(t)})$, $R(h^{(t)})$ $K(h^{(t)})$ and $Q(u^{(t)})$ are of the following forms:

$$P(h^{(t)}) = a_0 + a_1 h^{(t)} + a_2 (h^{(t)})^2,$$
$$R(h^{(t)}) = (b_0 + b_1 h^{(t)} + b_2 (h^{(t)})^2)^{-1},$$
$$K(h^{(t)}) = (I + c_2 (h^{(t)})^2)^{-1} (I + c_1 h^{(t)}),$$

and

$$Q(u^{(t)}) = d_0 + d_1 u^{(t)} + d_2 (u^{(t)})^2,$$

with

$$a_0 = I, a_1 = 2, a_2 = \mathbf{0},$$
$$b_0 = I, b_1 = -2, b_2 = 3,$$

$$c_1 = 4, c_2 = c_2,$$
$$d_0 = \mathbf{0}, d_1 = I, d_2 = d_2,$$

then for $c_2 = -I$ *and* $d_2 = 4$, we have the following eighth-order scheme named as FS_3

$$\begin{aligned} v^{(t)} &= s^{(t)} - \left(G'\left(s^{(t)}\right)\right)^{-1} G\left(s^{(t)}\right), \\ z^{(t)} &= v^{(t)} - \left(I + 2h^{(t)}\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(v^{(t)}\right), \\ s^{(t+1)} &= z^{(t)} - \left(I - 2h^{(t)} + 3(h^{(t)})^2\right)^{-1} + \left(I - (h^{(t)})^2\right)^{-1} \\ &\quad \left(I + 4h^{(t)}\right)\left(u^{(t)} + 4(u^{(t)})^2\right)\left(G'\left(s^{(t)}\right)\right)^{-1} G\left(z^{(t)}\right). \end{aligned}$$

3. Computational cost

In order to find the computational efficiency index (CEI) of the new multistep iterative methods for solving nonlinear systems, we use the order of convergence (OC), number of iterations (IT), number of function evaluations (FE), total numbers of Jacobian matrix evaluations (JME), total number of steps of the iterative method (NOS), number of LU-factors (LU.F) and total number of linear operations evaluations (LOE) to utilize the criteria given in [2] as follows:

$$CEI = \frac{OC}{IT + LU.F + LOE + FE + JME + NOS} \tag{16}$$

In table 1, the computational efficiency indices of our methods are compared with the eighth-order method presented by Cordero et al. [6] named as AC is given as:

Table 1: Computational Efficiency Index

$Methods$	OC	IT	NOS	JME	FE	CEI
AC	8	3	3	2	3	$\frac{8}{6+\frac{1}{3}n^3+5n^2+\frac{8}{3}n}$
FS	8	3	3	1	3	$\frac{8}{6+\frac{1}{3}n^3+4n^2+\frac{8}{3}n}$

The number of functional evaluations and its derivatives are taken as n and n^2 respectively, as in [5]. The $\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n$ products-quotient is essentially used in the solution of LU decomposition of m linear systems with the same coefficient matrix. From Table 1, we see that the underlying cost analysing parameters seem to be equivalent, but the impact of having one less Jacobian matrix evaluation is considerable which further results into greater efficiency of our new scheme.

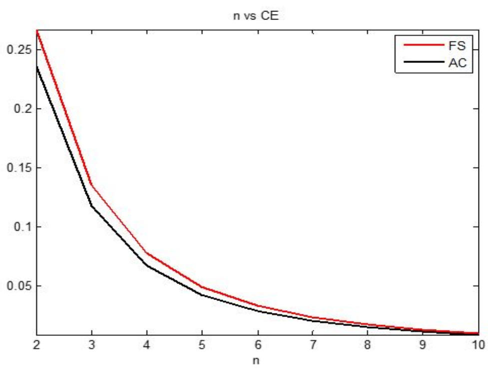


FIGURE 1. Computational Efficiency Index

4. Numerical and Dynamical Analysis

Next, we compare the results of our schemes FS_1, FS_2 and FS_3 by considering some problems from different engineering fields and fluid mechanics with respect to the number of iterations t, absolute residual error of the function $\left\|G\left(s^{(t)}\right)\right\|$ and absolute error in two successive iterations $\left\|s^{(t)}-s^{(t-1)}\right\|$ and computational order of convergence [20] expressed as $\delta \approx \dfrac{\ln\left(\frac{\left\|G\left(s^{(t+1)}\right)\right\|}{\left\|G\left(s^{(t)}\right)\right\|}\right)}{\ln\left(\frac{\left\|G\left(s^{(t)}\right)\right\|}{\left\|G\left(s^{(t-1)}\right)\right\|}\right)}$. Our methods are compared with the non-optimal eighth-order method presented by Cordero et al. [6]. The numerical results are given in Tables 2-4.

Example 1: In order to relate the temperatures and pressures on either side of a detonation wave that is traveling into an area of unburned gas, the following equations can be used (see[7], p. 331):

$$\begin{aligned}\frac{\alpha_2 m_2 t_1}{m_1 t_2}\left(\frac{p_2}{p_1}\right)^2-(\alpha_2+1)\frac{p_2}{p_1}+1 &= 0,\\ \frac{\Delta h_{r1}}{c_{p2}t_1}+\frac{t_2}{t_1}-1-\left(\frac{(\alpha_2-1)\,m_2}{2\alpha_2 m_1}\left(\frac{p_2}{p_1}-1\right)\left(1+\frac{m_1 t_2 p_1}{m_2 t_1 p_2}\right)\right) &= 0.\end{aligned}$$

Here t_i = absolute temperature, p_i = absolute pressure, α_2 = ratio of specific heat at constant pressure to that at constant volume, m = mean molecular weight, Δh_{r1} = heat of reaction, c_{p2} = specific heat and the subscripts 1 and 2 refer to the unburned and burned gas, respectively. Also m_1 = 12 g/g mol, m_2 = 18 g/g mol, t_1 = 300 K, α_2,= 1.31, Δh_{r1} = −58,300 cal/g mol , c_{p2} = 9.86 cal/(g mo1.K) and p_1 = 1 atm. The roots of this system are:

$$(5380.016949, 0.44217461)^t, (9071.7439017, 35.11000478)^t$$

So, the initial guess is taken as $(2, 2)^t$.

Table 2: Numerical performance for chemical engineering problem

Cases	t	$\Vert s^{(t)}-s^{(t-1)}\Vert_\infty$	$\Vert G\left(s^{(t)}\right)\Vert_\infty$	δ
FS_1	1	$1.07718e(3)$	$9.50226e(3)$	
	2	$1.43607e(4)$	$4.40190e(3)$	
	3	$2.57443e(4)$	$4.76972e(1)$	5.88038
FS_2	1	$9.73545e(3)$	$6.54039e(1)$	
	2	$6.64765e(2)$	$2.49181e(-2)$	
	3	$9.49266e(-1)$	$3.19759e(-19)$	4.94040
FS_3	1	$9.71278e(3)$	$6.40349e(1)$	
	2	$6.41061e(2)$	$5.37881e(-2)$	
	3	1.97681	$1.21939e(-18)$	5.41157
AC	1	$8.70827e(2)$	$1.11675e(4)$	
	2	$1.26913e(4)$	$4.24165e(3)$	
	3	$2.43446e(4)$	$7.33656e(1)$	4.19113

According to Table 2, our methods FS_2 and FS_3 perform better than the others. It displays faster convergence towards the root within three iterations, while others do not.

Example 2: Consider a problem taken from [9] where we need to figure out the steady-state levels of naphthalene in the liver and lungs. A 2×2 system of equations is obtained.

The roots of this system are:

$$(-9.486, 280.417)^t, (287.127, 361.311)^t, (-0.173, -2.104)^t, (-84.923, -2.105)^t.$$

The initial guess is taken as:

$$s^{(0)} = (S^{(0)}_{lung}, S^{(0)}_{liver}) = (1, 1)^t$$

Table 3: Numerical performance of iterative methods for bio-engineering problem

Cases	t	$\Vert s^{(t)} - s^{(t-1)} \Vert_\infty$	$\Vert G\left(s^{(t)}\right) \Vert_\infty$	δ
FS_1	1	$4.08920e(2)$	$4.38565e(1)$	
	2	$1.22792e(2)$	$5.63819e(-11)$	
	3	$1.53816e(-10)$	$2.53643e(-87)$	8.27077
FS_2	1	$5.03477e(2)$	$7.75805e(1)$	
	2	$2.17349e(2)$	$1.09930e(-9)$	
	3	$2.99873e(-9)$	$1.07113e(-79)$	8.11267
FS_3	1	$4.93644e(2)$	$7.40695e(1)$	
	2	$2.07516e(2)$	$8.84226e(-10)$	
	3	$2.41205e(-9)$	$2.94566e(-80)$	8.28314
AC	1	$2.94996e(2)$	$8.74069e(1)$	
	2	$2.33203e(2)$	$8.39016e(-4)$	
	3	$2.28793e(-3)$	$1.01261e(-49)$	9.15113

It is displayed in Table 3 that our method FS_1 performs better than the other methods. It shows faster convergence of FS_1, FS_2 and FS_3 than the AC method towards the root using three iterations.

Example 3: Consider a model based on atmospheric fluid dynamics to study intensity of atmospheric fluid motion given by Lorenz [16].The desired root for this system is,

$$(1.7320508075, 1.73205080, 1)^t.$$

where we take the initial guess as $s^{(0)} = (5, 5, 2)^t$.

Table 4: Numerical performance of iterative methods for fluid dynamics problem

Cases	t	$\Vert s^{(t)} - s^{(t-1)} \Vert_\infty$	$\Vert G\left(s^{(t)}\right) \Vert_\infty$	δ
FS_1	1	3.26497	$6.45504e(-3)$	
	2	$3.72042e(-3)$	$4.85933e(-15)$	
	3	$3.27123e(-15)$	$3.34367(-75)$	7.43709
FS_2	1	3.25219	$2.81378e(-2)$	
	2	$1.57497e(-2)$	$4.87885e(-13)$	
	3	$1.29975e(-13)$	$4.97629e(-66)$	7.34054
FS_3	1	3.24560	$3.78798e(-2)$	
	2	$2.23407e(-2)$	$3.11267e(-12)$	
	3	$1.12468e(-12)$	$3.80729e(-62)$	7.32875
AC	1	3.18700	$1.74916e(-1)$	
	2	$8.09451e(-2)$	$1.69780e(-10)$	
	3	$9.80229e(-11)$	$3.54417e(-63)$	8.06400

In Table 4, we can see that our method FS_1 performs better than the other methods. It shows that our methods FS_1 and FS_2 converge more quickly than the AC method towards the root using three iterations.
In conclusion, it is evident from Tables 2-4 that approximation solutions obtained from our methods possess greater or equal accuracy than the existing method. So, in that sense, our method is robust and computationally efficient, as its computational efficiency is greater than the existing method. Using just four functions makes it valuable.
Furthermore, we examine the dynamics of the new methods based on the graphical tool known as the attraction basins. In 2D space, the attraction basin is a region where iterations will always be iterated towards the attractor without choosing an initial guess. Vrscay and Gilbert [21] were the first to present this concept. The dynamical characteristics of FS's behavior are illustrated using the vector-valued function with visual basins of attraction and their related iteration colouring. The point is coloured according to the roots to which it converge. This enables us to determine whether the method converges within the given number of iterations. Each figure is accompanied by a colour map that indicates how many iterations it takes for the convergence to take place. The listed test problems, which are systems of polynomials in two variables, are taken into consideration.
Example 4: Let us take the following system of equations:

$$\begin{aligned} 3s_1^2s_2 - s_2^3 &= 0, \\ -1 + s_1^3 - 3s_1s_2^2 &= 0, \end{aligned}$$

with solutions

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)^t, \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^t, (1, 0)^t.$$

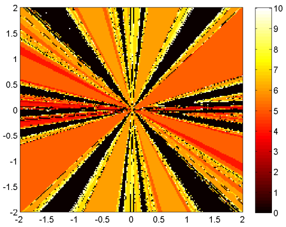
Example 5: The following system of equations is taken into consideration:

$$\begin{aligned} s_1^3 - s_2 = 0, \\ s_2^3 - s_1 = 0, \end{aligned}$$

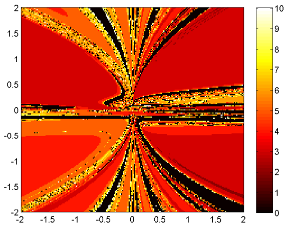
with solutions

$$(-1, -1)^t, (0, 0)^t, (1, 1)^t.$$

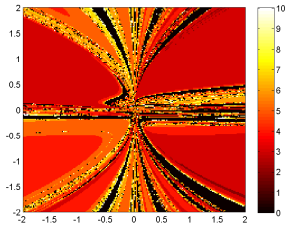
Graphs are created with $\| G(s)^{(t)})\|_\infty < 10^{-3}$ and t = 10, where t is the maximum number of iterations selected. Under the suggested iterative approach FS, the attraction basins will exhibit distinct dynamical traits.



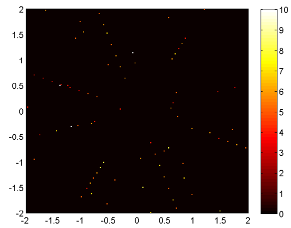
(A) FS_1



(B) FS_2



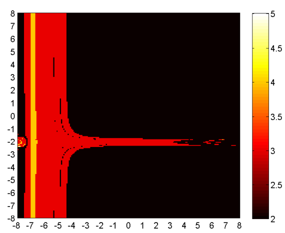
(C) FS_3



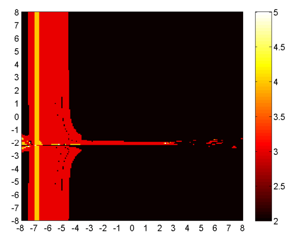
(D) AC

FIGURE 2. Attraction Basins of Example 1

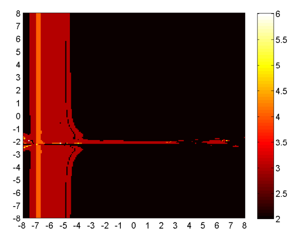
We choose a rectangular region $[-2, 2] \times [-2, 2]$ (fig. 2) under discretized into 200 by 200 grid points to fulfill the purpose. The black colour shows a divergence region, or when methods fail to converge within the maximum number of iterations. Here, FS_2 and FS_3 have a larger convergence region than FS_1, while AC does not converge to the required roots.



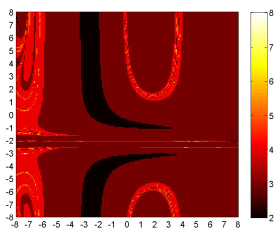
(A) FS_1



(B) FS_2

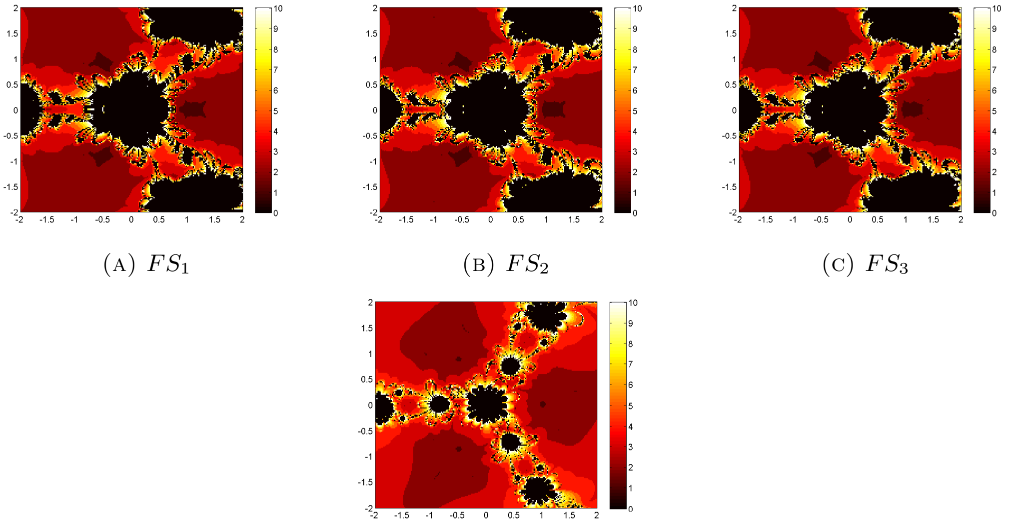


(C) FS_3



(D) AC

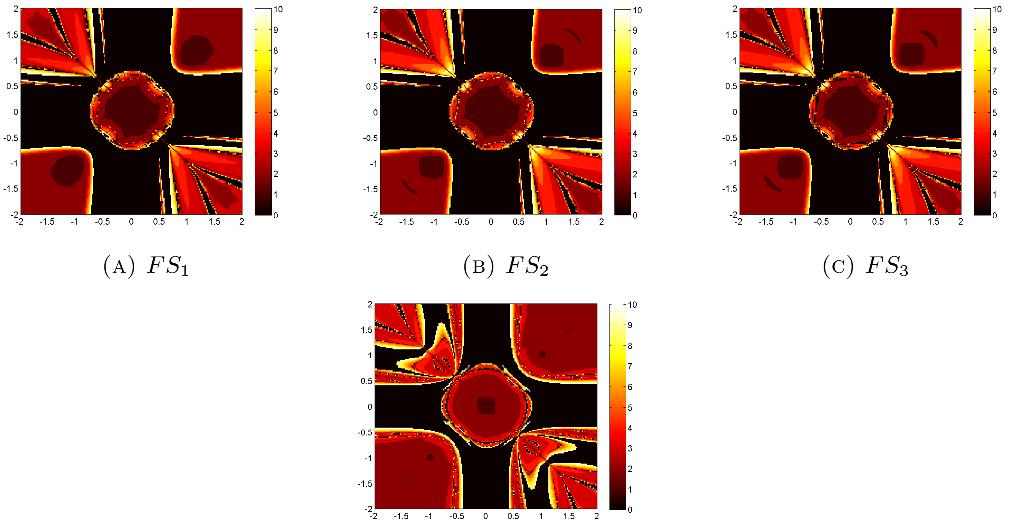
FIGURE 3. Attraction Basins of Example 2



(A) FS_1 (B) FS_2 (C) FS_3

(D) AC

FIGURE 4. Attraction Basins of Example 4



(A) FS_1 (B) FS_2 (C) FS_3

(D) AC

FIGURE 5. Attraction Basins of Example 5

Let us take a rectangular region $[-8, 8] \times [-8, 8]$ (fig. 3) under discretized into 200 by 200 grid points, taking into consideration all sets of roots. The black colour represents that methods converge to their desired roots using fewer iterations. Here, FS_1, FS_2 and FS_3 use a smaller number of iterations than the AC method. We choose a rectangular region $[-2, 2] \times [-2, 2]$ (fig. 4) under discretized into 200 by 200 grid points to fulfill the purpose. The black colour indicates the divergence region, or when methods fail to converge within the prescribed number of iterations. Here, AC method has a larger convergence region

than the FS_1, FS_2 and FS_3. Next we take a rectangular region $[-2,2] \times [-2,2]$ (fig. 5) under discretized into 200 by 200 grid points. The black colour is due to the divergence region, or when methods fail to converge within the maximum number of iterations. Here, all methods have approximately the same region of convergence.
In the above examples, we can easily see that our methods have larger convergence zones and quick convergence towards the roots.

5. Conclusion

We developed a new competitive eighth-order scheme for solving nonlinear systems of equations using weight functions. The proposed method's accuracy and reliability is adequate in terms of iteration counts, the number of function evaluations, and absolute error. We compare the numerical examples from the fields of chemical engineering, bio-engineering and fluid dynamics problems of our schemes with the existing non-optimal eighth-order iterative scheme [6] to show the overall performance and efficiency.

References

[1] Abad, M., Cordero, A. and Torregrosa, J. R., (2014), A family of seventh order schemes for solving nonlinear systems, Bull. Math. Soc. Sci. Math. Roumanie Tome., (105)57, pp. 133-145.

[2] AL-Obaidi, R. H. and Darvishi, M. T., (2022), A comparative study on qualification criteria of nonlinear solvers with introducing some new ones, J. Mathematics., 2022.

[3] Behl, R. and Arora, H., (2022), A novel scheme having seventh-order convergence for nonlinear systems, J. Comput. Appl. Math., 404.

[4] Behl, R., Argyros, I. K., Mallawi, F. O. and Alharbi, S. K., (2023), Extending the applicability of highly efficient iterative methods for nonlinear equations and their applications, Mathematics, 125.

[5] Capdevila, R. R., Cordero, A., and Torregrosa, J. R., (2019), A new three-step class of iterative methods for solving nonlinear systems, Mathematics., 7.

[6] Cordero, A., Gómez, E. and Torregrosa, J. R., (2017), Efficient high-order iterative methods for solving nonlinear systems and their application on heat conduction problems, Complexity., 2017.

[7] Crnahan, B., Luther, H. A. and Wilkes, J. O., (1969), Applied Numerical Methods, Wiley, New York.

[8] Hentenryck, P. V., McAllester, D. and Kapur, D., (1997), Solving polynomial systems using a branch and prune approach, SIAM J. Numer. Anal., 34, pp. 797-827.

[9] King, M. R. and Mody, N. A., (2010), Numerical and statistical methods for bioengineering: Applications in matlab, Cambridge University Press, New York.

[10] Kung, H. T. and Traub, J. F., (1974), Optimal order of one-point and multipoint iteration, J. ACM., 21, pp. 643-651.

[11] Morgan, A. P., (1987), Computing all solutions to polynomial systems using homotopy continuation, Appl. Math. Comput., 24, pp. 115-138.

[12] Morgan, A. P., (1987), Solving polynomial systems using continuation for scientific and engineering problems, Englewood Cliffs, NJ: Prentice-Hall.

[13] Meintjes, K. and Morgan, A. P., (1990), Chemical equilibrium systems as numerical test problems, ACM Trans. Math. Softw., 16, pp. 143-151.

[14] Narang, M., Bhatia, S. and Kanwar, V., (2017), New efficient derivative free family of seventh-order methods for solving systems of nonlinear equations, Numer. Algorithms., 76, pp. 283-307.

[15] Pavarino, L. F., Scacchi, S. and Zampini, S., (2015), Newton Krylov-BDDC solvers for nonlinear cardiac mechanics, Comput. Methods Appl. Mech. Engrg., 295, pp. 562-580.

[16] Qureshi, S., Ramos, H. and Soomro, A. K., (2021), A new nonlinear ninth-order root-finding method with error analysis and basins of attraction, Mathematics., 9.

[17] Sharma, J. R. and Arora, H., (2014), A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations, Numer. Algorithms., 67, pp. 917-933.

[18] Sharma, J. R., Argyros, I. K. and Singh, H., (2022), Semilocal convergence analys is of an efficient Steffensentype fourth order method, J. Anal., 31, pp. 1573-1586.

[19] Soleymani, F., Khattri, S. K. and Vanania, S. K., (2012), Two new classes of optimal Jarratt-type fourth-order methods, Appl. Math. Lett., 25, pp. 847-853.

[20] Soleymani, F., Sharifi, M., Shateyi, S. and Haghani, F. K., (2014), Iterative methods for nonlinear equations or systems and their applications J. Appl. Math., 2014.
[21] Vrscay, E. R. and Gilbert, W. J., (1998), Extraneous fixed points basin boundaries and chaotic dynamics for Schroder and Konig rational iteration functions, Numer. Math., pp. 52, 1-16.
[22] Verschelde, J., Verlinden, P. and Cools, R., (1994), Homotopies exploiting Newton polytopes for solving sparse polynomial systems, SIAM J. Numer. Anal., 31, pp. 915-930.
[23] Wang, X. and Zhang, T., (2013), A family of Steffensen type methods with seventh-order convergence, Numer. Algorithms., 62, pp. 429-444.
[24] Wang, X., Zhang, T., Qian, W. and Teng, M., (2015), Seventh-order derivative-free iterative method for solving nonlinear systems, Numer. Algorithms., 70, pp. 545-558.

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