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ON VARIOUS ALGEBRAIC STRUCTURES IN MULTIDIMENSIONAL FUZZY SETS

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ABSTRACT. One recent generalization of Zadeh's fuzzy sets is the concept of multidimensional fuzzy sets. This work extends this concept further by introducing a generalized form of multidimensional fuzzy algebra. By focusing on multidimensional t-norms and t-conorms, we develop a comprehensive theory. This includes the notion of strong multidimensional fuzzy algebras and explores their properties, such as multidimensional groupoids, monoids, and groups. Additionally, we introduce equivalence relations on the collection of all multidimensional fuzzy sets and present an example of specific monoids within this collection. Finally, we demonstrate how a group structure can be imposed on a subcollection of orderless multidimensional fuzzy sets.

Keywords: Multidimensional fuzzy algebra, Multidimensional t-norm, Multidimensional t-conorm, Strong multidimensional fuzzy algebra, Equivalence classes, Multidimensional groups, Multidimensional fuzzy groupoid.

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1. Introduction

The most essential strategy for describing events with uncertainty and ambiguity is the development of fuzzy sets. Even though fuzzy sets are capable of handling many real-life situations, they fail to present some complex situations that have fluctuating ambiguity and vagueness. As a result, many mathematical fuzzy extension models have been suggested for expressing diverse occurrences. Among these models, interval-valued fuzzy sets [29, 10], hesitant fuzzy sets[28, 12], intuitionistic [2, 27] fuzzy sets, picture fuzzy sets [5], m-polar fuzzy sets [4] and n-dimensional fuzzy sets [24] are a few notable mathematical models that aid in data ambiguity resolution. They are used in a variety of industries, including medicine, engineering, finance, and data analysis. Despite the fact that these mathematical models do not provide any option to assign an adequate number of possible membership values to each element in order to avoid ambiguity in assigning a single membership or nonmembership value, or that these models do not provide a facility to mark the membership values of each element if the values are assigned by groups with different numbers.

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The n-dimensional fuzzy sets and m-polar fuzzy sets [11] are interesting fuzzy models in which the dimension of each set is specified as a positive integer, which specifies the maximum number of member values that an element can have. Nevertheless, in spite of this adaptability in choosing dimensions, these models do not provide the flexibility to customize the measurements of specific components in order to suit particular needs. This constraint emphasizes the importance of fuzzy sets with multiple dimensions. A refined version of n-dimensional fuzzy sets, Multidimensional Fuzzy Sets (MDFS) (Lima et al. [8]) allows elements to have different quantities of membership values without affecting others. For example, when an interviewer evaluates the skills of candidates, the application of an MDFS permits nuanced assessments, as illustrated by (0.30, 0.35, 0.55)/(0.30, 0.45, 0.50, 0.65)/(0.60, 0.65), which signifies the perceived level of ambiguity. On the contrary, when 2-dimensional fuzzy sets are utilized to capture the assessments, the interviewer's evaluation is limited to specific values, such as $(0.50, 0.55)$, $(0.45, 0.50)$, and $(0.55, 0.60)$. The implementation of this restrictive representation may introduce bias into the evaluation process as a consequence of the restricted number of members or cause data inflation by imposing a greater dimension on the model. The n-dimensional fuzzy sets are an extension of numerous existing structures, such as interval-valued fuzzy sets, and offer additional membership value options. However, it has several drawbacks, particularly when dealing with real situations involving a variable number of characteristics with reference to members of the universal set. In n-dimensional fuzzy sets, each element has membership value from the set $\mathcal{J}_n([0,1])$ which is a subset of $[0,1]^n$ with coordinates arranged in ascending order.[24]. In this mathematical framework, each element receives the same number of membership values regardless of the ambiguity that exists in each element. However, this is not the case in real life because each element in the universe has its own properties and ambiguity. This problem motivated mathematicians to introduce the concept of multidimensional fuzzy sets [19] with a structure in which each element may have a varied number of membership values. In the case of MDFS, depending on the level of uncertainty associated with each element, varying numbers of values in the range [0, 1] may be given to individual elements. As a result, MDFS offers a great deal of flexibility in terms of the quantity of values provided to each element based on demand.

Annaxsuel and Palmeira introduced and discussed the notion of MDFS [19] with the support of a partially ordered set $\mathcal{J}_\infty([0,1])$. Unlike n-dimensional, where the dimension of membership values is fixed, MDFS allows us to provide any number of membership values based on the degree of ambiguity in each element on its own. As a result, we no longer need to limit our comprehension of the components. MDFS gives a solid foundation for addressing mathematical difficulties in a more practical approach. When compared to other types of generalized fuzzy structures, such as type-2 fuzzy sets, [18, 3], genuine sets [21, 7], and so on, MDFS is more efficient in dealing with practical problems due to its simplicity and the freedom it gives to each element independent of other elements. More studies on MDFS such as Multidimensional complements, multidimensional t-norms, multidimensional t-conorms, etc. are made by [13]. Important results such as De Morgan's law and the bounds of norms can also be seen in the same. The study of various fuzzy measures in MDFS can be seen in [14]. Multidimensional entropy, multidimensional similarity measures, and multidimensional distance measures are all presented in the same paper, along with some key examples and findings. Studies regarding the graphical structure of multidimensional fuzzy sets can be seen in [17, 16]. In [15], a rough approximation of multidimensional fuzzy sets is shown utilizing distance measures and other aggregation operators that are important in application fields.

Rosenfeld developed the idea of fuzzy groups, which has a close relation with ordinary group theories [23]. In [6] Das et al. studied the fuzzy groups and their level subgroups. In [25] we can see the algebraic structure of intuitionistic fuzzy sets and their various properties. Divakaran et al. studied hesitant fuzzy group properties in [8]. In [1] Agboola et al. studied the neutrosophic fuzzy groups and discussed their applications. The algebraic properties of spherical fuzzy sets were studied by Perveen et al in 2023 [22]. Recently in 2023, Dogra et al studied the algebraic properties of the picture fuzzy sets [9]. More studies about fuzzy algebra can be seen in [20, 26]. Because multidimensional fuzzy sets play a vital role in dealing with problems that have complications due to vagueness, the notions of fuzzy group structure and group structure must be introduced to it. In this study, we examine multidimensional fuzzy algebraic characteristics using multidimensional norms to get highly broad findings. Following the preliminaries in Section 2, we introduce multidimensional fuzzy algebraic structures in Section 3 as extensions of groupoids, monoids, and groups of conventional fuzzy algebra. In Section 4, we will establish some equivalence relations on the collection of all multidimensional fuzzy sets and use them to construct certain monoids. Finally, in Section 5, we will construct a binary operation on a sub-collection of orderless multidimensional fuzzy sets, convert it to a group, and analyze some of the binary operation's features.

2. Preliminaries

2.1. **Multidimensional fuzzy sets.** Before proceeding to the notion of a multidimensional fuzzy set, it is crucial to establish fundamental terminology and notations. This segment serves to establish coherence and clarity in subsequent discourse.

Definition 2.1. [19]
Let $\mathcal{J}_n\big([0,1]\big) = \big\{ (z_1, \dots, z_n) \in [0,1]^n \mid z_1 \le z_2 \le \cdots \le z_n \big\}$ *where* $n \in \mathbb{N}$ *and let*

$$\mathcal{J}_\infty\big([0,1]\big) = \bigcup_{n=1}^{\infty} \mathcal{J}_n\big([0,1]\big)$$

Then a multidimensional fuzzy set on a crisp set U *is defined as a function* $\nu : U \to \mathcal{J}_\infty\big([0,1]\big)$

Let $\mathcal{X} \in \mathcal{J}_\infty\big([0,1]\big)$ then $|\mathcal{X}|$ denote the $n \in \mathbb{N}$ such that $\mathcal{X} \in \mathcal{J}_n\big([0,1]\big)$ called cardinality of $\mathcal{X}$ and for a multidimensional fuzzy set ν on Z and for $z \in Z$, $\nu_i(z)$ denote i^{th} component of $\nu(z)$. The benefit of MDFS over other fuzzy extensions, such $m-$polar fuzzy sets or n-dimensional fuzzy sets, etc., is that, according to its definition, $\nu(z)$ can have any cardinality regardless of the cardinality of other elements, given that $z \in Z$.

Let $\overline{1} = \{/1/m, m \in \mathbb{N}\}, \overline{0} = \{/0/m, m \in \mathbb{N}\}$,where $/k/ = (k, k..., k) \in \mathcal{J}_n\big([0,1]\big)$, $\dot{1}$ and $\dot{0}$ denote arbitrary element of $\overline{1}$ and $\overline{0}$ respectively.

We use a natural partial order on $\mathcal{J}_\infty([0,1])$ given by,

$\mathcal{X} \le_\infty \mathcal{Y} \Leftrightarrow |\mathcal{X}| = |\mathcal{Y}| = n$ and $\mathcal{X} \le_n^p \mathcal{Y}$ where $\le_n^p$ is the product order on $\mathcal{J}_n([0,1])$, $n \in \mathbb{N}$. Also, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two MDFS we say that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ if $\mathcal{M}_1(y) \le_\infty \mathcal{M}_2(y)$ for every y.

2.2. **Operators in Multidimensional Fuzzy Sets.** After a strong framework for storing and presenting data has been established, the next critical step is to incorporate aggregation operators that can link disparate data structures in order to facilitate exhaustive analysis. The $t-norms$ and $t-conorms$ for multidimensional fuzzy sets were introduced using the following two important functions H_1 and H_2 [13].

$H_1 : \mathcal{J}_\infty([0,1]) \times \mathcal{J}_\infty([0,1]) \to \mathbb{N}$ by

$$H_1(\mathcal{X}, \mathcal{Y}) = \begin{cases} |\mathcal{X}| \ if \ \exists l \in \mathbb{N} : x_i = y_i \\ for \ all \ i = 1, 2...l-1 \ and \ x_l < y_l. \\ |\mathcal{Y}| \ if \ \exists l \in \mathbb{N} : x_i = y_i \\ for \ all \ i = 1, 2...l-1 \ and \ x_l > y_l. \\ min\{|\mathcal{X}|, |\mathcal{Y}|\} \ if \ there \ is \ no \ such \ l \ exists. \end{cases}$$

$H_2 : \mathcal{J}_\infty([0,1]) \times \mathcal{J}_\infty([0,1]) \to \mathbb{N}$ be defined by

$$H_2(\mathcal{X},\mathcal{Y}) = \begin{cases} |\mathcal{X}| \; if \; \exists l \in \mathbb{N} : x_{n+1-i} = y_{m+1-i} \\ for \; all \; i = 1, 2 ... l-1 \\ and \; x_{n+1-l} > y_{m+1-l}. \\ |\mathcal{Y}| \; if \; \exists l \in \mathbb{N} : x_{n+1-i} = y_{m+1-i} \; for \\ \; all \; i = 1, 2 ... l-1 \\ and \; x_{n+1-l} < y_{m+1-l}. \\ min\{|\mathcal{X}|, |\mathcal{Y}|\} \; if \; there \; is \; no \; such \; l \; exists. \end{cases}$$

where $\mathcal{X} = (x_1 \ldots x_n)$ *and* $\mathcal{Y} = (y_1 \ldots y_m)$ $n, m \in \mathbb{N}$.

The functions H_1 and H_2 help to find the cardinality of the vector that gives the minimum and maximum membership values, respectively. Then Multidimensional t-norm and Multidimensional t-conorm for MDFS are given, respectively, by;

Definition 2.2. *Multidimensional t-norm is a function* $\mathbb{P} : \mathcal{J}_\infty([0,1]) \times \mathcal{J}_\infty([0,1]) \to \mathcal{J}_\infty([0,1])$ *satisfying following axioms,*

- $(N1)$ $\left|\mathbb{P}(\mathcal{W},\mathcal{Y})\right| = H_1(\mathcal{W},\mathcal{Y})$
- $(N2)$ $\mathbb{P}(\mathcal{W},\mathcal{Y}) = \mathbb{P}(\mathcal{Y},\mathcal{W})$
- $(N3)$ *If* $\mathcal{Y} \leq_\infty \mathcal{Z}$ *then* $\mathbb{P}(\mathcal{W},\mathcal{Y}) \leq_\infty \mathbb{P}(\mathcal{W},\mathcal{Z})$, *given* $|\mathcal{W}| = |\mathcal{Y}| = |\mathcal{Z}|$
- $(N4)$ $\mathbb{P}(\mathcal{W}, \dot{1}) = \mathcal{W}, \mathcal{W} \notin \bar{1} \setminus \{\dot{1}\}$
- $(N5)$ $\mathbb{P}\big(\mathcal{W}, \mathbb{P}(\mathcal{Y},\mathcal{Z})\big) = \mathbb{P}\big(\mathbb{P}(\mathcal{W},\mathcal{Y}),\mathcal{Z}\big)$, *whenever* $|\mathcal{W}| = |\mathcal{Y}| = |\mathcal{Z}|$

where $\mathcal{W},\mathcal{Y},\mathcal{Z} \in \mathcal{J}_\infty([0,1])$.

Definition 2.3. *Multidimensional t-conorm is a function* $\mathbb{Q} : \mathcal{J}_\infty([0,1]) \times \mathcal{J}_\infty([0,1]) \to \mathcal{J}_\infty([0,1])$ *satisfying following axioms,*

- $(C1)$ $|\mathbb{Q}(\mathcal{W},\mathcal{Y})| = H_2(\mathcal{W},\mathcal{Y})$
- $(C2)$ $\mathbb{Q}(\mathcal{W},\mathcal{Y}) = \mathbb{Q}(\mathcal{Y},\mathcal{W})$
- $(C3)$ *If* $\mathcal{Y} \leq_\infty \mathcal{Z}$ *then* $\mathbb{Q}(\mathcal{W},\mathcal{Y}) \leq_\infty \mathbb{Q}(\mathcal{W},\mathcal{Z})$, *given* $|\mathcal{W}| = |\mathcal{Y}| = |\mathcal{Z}|$
- $(C4)$ $\mathbb{Q}(\mathcal{W}, \dot{0}) = \mathcal{W}, \mathcal{W} \notin \bar{0} \setminus \{\dot{0}\}$
- $(C5)$ $\mathbb{Q}(\mathcal{W}, \mathbb{Q}(\mathcal{Y},\mathcal{Z})) = \mathbb{Q}(\mathbb{Q}(\mathcal{W},\mathcal{Y}),\mathcal{Z})$, *whenever* $|\mathcal{W}| = |\mathcal{Y}| = |\mathcal{Z}|$ *where* $\mathcal{W},\mathcal{Y},\mathcal{Z} \in \mathcal{J}_\infty\big([0,1]\big)$

Then the standard Multidimensional t-norm and standard Multidimensional t-conorm for MDFS are given respectively by;

If $H_1(\mathcal{V},\mathcal{W}) = k$ then

$$min(\mathcal{V},\mathcal{W}) = (v_1 \wedge w_1 \ldots v_k \wedge w_k)$$

where $v_i = w_i = 1 \; for \; all \; i > k$ and
if $H_2(\mathcal{V},\mathcal{W}) = l$ then

$$max(\mathcal{V},\mathcal{W}) = (v_{n-(l-1)} \vee w_{m-(l-1)}, \ldots, v_n \vee w_m),$$

where $v_{n-i} = w_{m-j} = 0 \; for \; all \; i \geq n \; and \; j \geq m.$

The standard multidimensional compliment is given by

$C_s(a_1, a_2, \cdots, a_n) = (1 - a_n, 1 - a_{n-1}, \cdots, 1 - a_1)$.

Definition 2.4. *Let $\mathbb{A}_i, i \in I$ be an indexed family of multidimensional fuzzy sets of $\mathbf{X}$. Then this family is said to Equi cardinal family if $|\mathbb{A}_i(x)| = |\mathbb{A}_j(x)|$ for all $x \in \mathbf{X}$ and for all $i, j \in I$*

Definition 2.5. *Let $\mathbf{U}$ be a nonempty set and given $r : \mathbf{U} \to \mathbb{N}$. Then an orderless multidimensional fuzzy set over $\mathbf{U}$ is a set of the form*

$$\mathcal{N} = \left\{ \left(y, \mu^1_{\mathcal{N}}(y), \ldots, \mu^{r(y)}_{\mathcal{N}}(y) \right) \mid y \in \mathbf{U} \right\}$$

where $\mu^i_{\mathcal{N}} : \mathbf{U} \to [0, 1]$ for every i.

Note : Clearly, every multidimensional fuzzy set is an orderless multidimensional fuzzy set.

2.3. Fuzzy Algebra.

Definition 2.6. [23]*Let $(G, *)$ be a groupoid(semigroup) then we recall a fuzzy set in G say g a fuzzy subgroupoid of G if, for all x, y in G,*

$$g(x * y) \geq \min(g(x), g(y))$$

A fuzzy groupid g is said to fuzzy subgroup if g satisfies $g(x^{-1}) = g(x)$ for every $x \in G$.

3. Multidimensional Fuzzy Algebra

When we want to study various algebraic structures of MDFS, we need to choose MDFS that give membership values with the same cardinality as the functions H_1 after the group operation(or equivalently H_2).

Definition 3.1. *Let $(G, *)$ be a groupoid and let $\mathcal{N}$ be a MDFS on G then $\mathcal{N}$ said to coincides with multidimensional t-norms with respect to $*$ if*

$$|\mathcal{N}(x * y)| = H_1(\mathcal{N}(x), \mathcal{N}(y))$$

for every $x, y \in G$. $\mathcal{N}$ said to coincides with multidimensional t-conorms with respect to $*$, if $|\mathcal{N}(x * y)| = H_2(\mathcal{N}(x), \mathcal{N}(y))$ for every $x, y \in G$.

3.1. Multidimensional Fuzzy Groupoid and Strong Multidimensional Fuzzy Groupoid.

Groupid is the basic structure of algebra and next, we study the basic properties of multidimensional fuzzy groupoids and strong multidimensional fuzzy groupoids.

Definition 3.2. *Let $\mathbb{I}$ be a Multidimensional t-norm on G then an MDFS $\mathcal{N}$ on $(G, *)$ which coincides with multidimensional t-norms is said to be a Multidimensional fuzzy groupoid (MDFGP) with respect to $\mathbb{I}$ if $\mathbb{I}(\mathcal{N}(x), \mathcal{N}(y)) \leq_\infty \mathcal{N}(x * y)$ for every $x, y \in G$.*

Definition 3.3. *Let $\mathbb{U}$ be a Multidimensional t-conorm on G then an MDFS $\mathcal{N}$ on $(G, *)$ which coincides with multidimensional t-conorms is said to be a Strong multidimensional fuzzy groupoid (SMDFGP) with respect to $\mathbb{U}$ if $\mathbb{U}(\mathcal{N}(x), \mathcal{N}(y)) \leq_\infty \mathcal{N}(x * y)$ for every $x, y \in G$.*

Result 3.1. *Let $\mathbb{I}$ be the standard Multidimensional t-norm and $\mathcal{N}$ be a MDFGP on G with respect to $\mathbb{I}$ then for each $\alpha \in \mathcal{J}_\infty([0, 1])$ the set $A_\alpha = \{x \in G, \alpha \leq_\infty \mathcal{N}(x)\}$ is a groupoid.*

Proof. The proof follows directly from the definition of MDFGP with respect to Standard Multidimensional t-norm and the fact that if $\alpha \leq_\infty X$ and $\alpha \leq_\infty Y$ then $\alpha \leq_\infty min(X, Y)$. □

But A_α need not to be a groupoid always. For example, if we take the drastic min operator [13] then the corresponding A_α is not a groupoid for many MDFGP. But this is not the case with Strong Multidimensional fuzzy groupoids. In that case, we have the following theorem, and this is one of the reasons why the strong multidimensional fuzzy groupoids are called strong.

Theorem 3.1. *Let $\mathcal{N}$ be an SMDFGP on G with respect to a Multidimensional t-conorm $\mathbb{U}$ then $A_\alpha = \{x \in G, \alpha \leq_\infty \mathcal{N}(x)\}$ is a groupoid.*

Proof. Let $x, y \in A_\alpha$ then we have $\alpha \leq_\infty \mathcal{N}(x) = \mathbb{U}(\mathcal{N}(x), \dot{0}) \leq_\infty \mathbb{U}(\mathcal{N}(x), \mathcal{N}(y)) \leq_\infty \mathcal{N}(x * y)$ where $\dot{0}$ is from $\overline{0}$ with $|\dot{0}| = |\mathcal{N}(y)|$. Hence $\alpha \leq_\infty \mathcal{N}(x * y)$ and thus $x * y \in A_\alpha$. □

Next, we define multidimensional characteristic functions to explore multidimensional algebraic structures more deeply.

Definition 3.4. *Let $S \subseteq G$ then a multidimensional characteristics function of S is an MDFS χ_S on G with $\chi_S(x) \in \overline{1}$ for $x \in S$ and $\chi_S(x) \in \overline{0}$ for $x \notin S$*

Theorem 3.2. *Let G be a group and $S \subseteq G$ then S is a groupoid if and only if χ_S is an MDFGP (where χ_S is assumed to coincide with the multidimensional t-norm).*

Proof. Let S be a groupoid and let $x, y \in S$ then $\mathbb{I}(\chi_S(x), \chi_S(y)) \leq_\infty \chi_S(x * y) = \dot{1}$. Now if at least one of x, y is not in S then

$$\dot{0} = \mathbb{I}(\chi_S(x), \chi_S(y)) \leq_\infty \chi_S(x * y)$$

Hence χ_S is an MDFGP.
Now, conversely, assume that χ_S is an MDFGP, then for $x, y \in S$ we have by $(I4)$ $\dot{1} = \mathbb{I}(\chi_S(x), \chi_S(y)) \leq_\infty \chi_S(x * y) \Rightarrow \chi_S(x * y) = \dot{1}$ and hence $x * y \in S$. Thus, S is a groupoid. □

Result 3.2. *Let $S \subseteq G$ and if χ_S is an SMDFGP then S is a groupoid, and the proof follows similar to the above one. But the converse is not true. This can be seen when we use the drastic max operator as the t-conorm* [13].

The following theorem gives the case where χ_S becomes an SMDFGP.

Theorem 3.3. *Let $(R, +, *)$ be a ring and $S \subseteq R$ be an ideal, then χ_S is an SMDFGP with respect to the operation $*$.*

Proof. Let at least one of x, y be in S then $\mathbb{U}(\chi_S(x), \chi_S(y)) \leq_\infty \chi_S(x * y) = \dot{1}$ and if both x, y is not in S then we have $\dot{0} = \mathbb{U}(\chi_S(x), \chi_S(y)) \leq_\infty \chi_S(x * y)$. Thus, χ_S is an SMDFGP. □

The following theorems give the relations corresponding to the infimum of the equi-cardinal family of MDFGPs and SMDFGPs.

Theorem 3.4. *Let $\mathcal{N}_i, i \in I$ be an equi-cardinal family of MDFGPs on G with respect to the Multidimensional t-norm $\mathbb{I}$ then $\inf_{i \in I} \mathcal{N}_i$ is again an MDFGP on G.*

Proof. For $x, y \in G$ we have $\mathbb{I}(\mathcal{N}_i(x), \mathcal{N}_i(y)) \leq_\infty \mathcal{N}_i(x * y)$ for every i. Hence

$$\inf_{i \in I} \mathbb{I}(\mathcal{N}_i(x), \mathcal{N}_i(y)) \leq_\infty \mathcal{N}_i(x * y) \tag{1}$$

.
Now, $\inf_{i \in I} \mathcal{N}_i(x) \leq_\infty \mathcal{N}_i(x)$ and $\inf_{i \in I} \mathcal{N}_i(y) \leq_\infty \mathcal{N}_i(y)$ for every i. Thus we have

$$\mathbb{I}(\inf_{i \in I} \mathcal{N}_i(x), \inf_{i \in I} \mathcal{N}_i(y)) \leq_\infty \mathbb{I}(\mathcal{N}_i(x), \mathcal{N}_i(y))$$

for every i. Hence taking infimum we get

$$\mathbb{I}(\inf_{i \in I} \mathcal{N}_i(x), \inf_{i \in I} \mathcal{N}_i(y)) \leq_\infty \inf_{i \in I} \mathbb{I}(\mathcal{N}_i(x), \mathcal{N}_i(y)) \tag{2}$$

Hence, from (1) and (2) we get
$\mathbb{I}(\inf_{i \in I} \mathcal{N}_i(x), \inf_{i \in I} \mathcal{N}_i(y)) \leq_\infty \mathcal{N}_i(x * y)$ for every i.
Now taking infimum on the RHS, we get $\mathbb{I}(\inf_{i \in I} \mathcal{N}_i(x), \inf_{i \in I} \mathcal{N}_i(y)) \leq_\infty \inf_{i \in I} \mathcal{N}_i(x * y)$. Thus $\inf_{i \in I} \mathcal{N}_i$ is an MDFGP. □

Theorem 3.5. *Let $\mathcal{N}_i, i \in I$ be a family of SMDFGPs on G with respect to the multidimensional t-conorm $\mathbb{U}$ then $\inf_{i\in I} \mathcal{N}_i$ is again an SMDFGP on G.*

Proof. The proof follows similar to the above one. □

Theorem 3.6. *Let $S_i, i \in I$ be a family of groupoids of a group G. Let χ_{S_i} be a multidimensional characteristic function of S_i then $\inf \chi_{S_i}$ and $\chi_{\bigcap_{i\in I} S_i}$ both are multidimensional characteristic functions of $\bigcap_{i\in I} S_i$*

Proof. Let $x \in \bigcap_{i\in I} S_i \Rightarrow \chi_{S_i}(x) \in \bar{1}$ for every i and hence $\inf \chi_{S_i}(x) \in \bar{1}$. Now if $x \notin \bigcap_{i\in I} S_i \Rightarrow \chi_{S_i}(x) \in \bar{0}$ for some i and hence $\inf \chi_{S_i}(x) \in \bar{0}$. Thus $\inf \chi_{S_i}$ is multidimensional characteristic functions of $\bigcap_{i\in I} S_i$ and the result follows.

□

3.2. **Multidimensional Fuzzy Monoids and Strong Multidimensional Fuzzy Monoids.** In this section, the study focuses on multidimensional fuzzy monoids, which are the generalization of algebraic structure monoids and clearly an extension of multidimensional fuzzy groupoids.

Definition 3.5. *Let G be groupoid with identity e and $\mathbb{I}$ be a multidimensional t-norm on G then an MDFS $\mathcal{N}$ on G is said to be a Multidimensional Fuzzy Monoid (MDFM) with respect to $\mathbb{I}$ if $\mathcal{N}$ is an MDFGP with respect to $\mathbb{I}$ and satisfies $\mathcal{N}(x) \leq_\infty \mathcal{N}(e)$ for every x with $|\mathcal{N}(x)| = |\mathcal{N}(e)|$*

Definition 3.6. *Let G be groupoid with identity e and $\mathbb{U}$ be a multidimensional t-conorm on G then an MDFS $\mathcal{N}$ on G is said to be a Strong Multidimensional Fuzzy Monoid (SMDFM) with respect to $\mathbb{U}$ if $\mathcal{N}$ is an SMDFGP with respect to $\mathbb{U}$ and satisfies $\mathcal{N}(x) \leq_\infty \mathcal{N}(e)$ for every x with $|\mathcal{N}(x)| = |\mathcal{N}(e)|$*

Theorem 3.7. *Let $\mathcal{N}$ be an MDFM on $(G, *)$ and let $A = \{x \in G : \mathcal{N}(x) \leq_\infty \mathcal{N}(e)\}$ and $B = \{x \in G : \mathcal{N}(x) = \mathcal{N}(e)\}$ then A is always a monoid, and if the respective multidimensional t-norm $\mathbb{I}$ is idempotent, then B is also a monoid.*

Proof. First, we prove that A is monoid. Let $x, y \in A$ then $|\mathcal{N}(x * y)| = H_1(\mathcal{N}(x), \mathcal{N}(y)) = |\mathcal{N}(x)| = |\mathcal{N}(y)| = |\mathcal{N}(e)|$. Hence $\mathcal{N}(x * y) \leq_\infty \mathcal{N}(e) \Rightarrow x * y \in A$. Since, $e \in A$, we have A as a monoid.

Now consider the second part. Let $x, y \in B$ then, as above, we have

$$\mathcal{N}(x * y) \leq_\infty \mathcal{N}(e) \tag{3}$$

Now,

$$\mathcal{N}(e) = \mathbb{I}(\mathcal{N}(e), \mathcal{N}(e)) = \mathbb{I}(\mathcal{N}(x), \mathcal{N}(y)) \leq_\infty \mathcal{N}(x * y) \tag{4}$$

Thus from (3) and (4) we have $\mathcal{N}(x * y) = \mathcal{N}(e)$ and hence $x * y \in B$. Since $e \in B$, we have B as a monoid. □

The next theorem gives another reason why the strong multidimensional fuzzy monoids are called strong.

Theorem 3.8. *Let $\mathcal{N}$ be an SMDFM on $(G, *)$ and let $A = \{x \in G : \mathcal{N}(x) \leq_\infty \mathcal{N}(e)\}$, $A_\alpha = \{x \in G : \alpha \leq_\infty \mathcal{N}(x)\}$ where $\alpha \leq_\infty \mathcal{N}(e)$ and $B = \{x \in G : \mathcal{N}(x) = \mathcal{N}(e)\}$ then A, A_α and B are always monoids, irrespective of whether the corresponding multidimensional t-conorm $\mathbb{U}$ is idempotent or not.*

Proof. The proof that A and A_α are monoids follows similarly to the above proof. Now, clearly, $e \in B$ and for $x, y \in B$ we have $|\mathcal{N}(x * y)| = H_2(x, y) = |\mathcal{N}(x)| = |\mathcal{N}(y)| = |\mathcal{N}(e)|$. Hence

$$\mathcal{N}(x * y) \leq_\infty \mathcal{N}(e) \tag{5}$$

Now,

$$\begin{aligned} \mathcal{N}(e) = \mathbb{U}(\mathcal{N}(e), \dot{0}) &\leq_\infty \mathbb{U}(\mathcal{N}(e), \mathcal{N}(e)) \\ &= \mathbb{U}(\mathcal{N}(x), \mathcal{N}(y)) \leq_\infty \mathcal{N}(x * y) \end{aligned} \tag{6}$$

where $|\mathcal{N}(e)| = |\dot{0}|$ and hence from (5) and (6) we have $\mathcal{N}(x * y) = \mathcal{N}(e)$ and thus $x * y \in B$. Thus, B is a monoid. □

The theorem gives the characterization of MDFM using the characteristics function.

Theorem 3.9. *Let $(G, *)$ be a group and $S \subseteq G$ then S is a monoid if and only if the corresponding multidimensional characteristic function χ_S which coincides with multidimensional t-norms is an MDFM(Given there is at least one $x \in S$ with $|\chi_S(x)| = |\chi_S(e)|$).*

Proof. Since every monoid is a groupoid, if $S \subseteq G$ is a monoid then we have χ_S as an MDFGP by a previous theorem. Now let $x \in G$ and $|\chi_S(x)| = |\chi_S(e)|$ then $\chi_S(x) \leq_\infty \dot{1} = \chi_S(e)$. Hence χ_S is an MDFM.
Now, conversely assume that χ_S is an MDFM then clearly S is a groupoid. Since there exist $x \in S$ with $|\chi_S(x)| = |\chi_S(e)|$ we have $\dot{1} = \chi_S(x) \leq_\infty \chi_S(e)$. Hence $\chi_S(e) = \dot{1}$ and thus $e \in S$. So S is a monoid. □

The following theorem gives a characterization of ideals,

Theorem 3.10. *Let $(R, +, *)$ be a ring and $S \subseteq R$ be a subring. Let χ_S be the multidimensional characteristic function of S which coincides with multidimensional t-conorms. Then S is an ideal if and only if χ_S is an SMDFM.*

Proof. If S is ideal then by Theorem (3.3) we have χ_S as an SMDFGP. Now if $x \in R$ and $|\chi_S(x)| = |\chi_S(e)|$ then $\chi_S(x) \leq_\infty \dot{1} = \chi_S(e)$. Thus χ_S is an SMDFM.
Now, conversely assume that χ_S is an SMDFM with respect to $\mathbb{U}$. Let $x \in S$ and $y \in R$ then $\dot{1} = \mathbb{U}(\dot{1}, \dot{0}) \leq_\infty \mathbb{U}(\chi_S(x), \chi_S(y)) \leq_\infty \chi_S(x * y)$. Hence $x * y \in S$ and S is an ideal.

□

Theorem 3.11. *Let $\mathcal{N}_i, i \in I$ be a collection of equicardinal MDFMs corresponding to a multidimensional t-norm $\mathbb{I}$ then $\inf_{i \in I} \mathcal{N}_i$ is again an MDFM with respect to $\mathbb{I}$.*

Proof. We have already proved that $\inf_{i \in I} \mathcal{N}_i$ is an MDFGP.
Now, if $|\inf_{i \in I} \mathcal{N}_i(x)| = |\inf_{i \in I} \mathcal{N}_i(e)|$ then $|\mathcal{N}_i(x)| = |\mathcal{N}_i(e)|$ for every i as $\{\mathcal{N}_i\}$ is equicardinal. Hence, $\mathcal{N}_i(x) \leq_\infty \mathcal{N}_i(e)$ for every i. Thus $\inf_{i \in I} \mathcal{N}_i(x) \leq_\infty \inf_{i \in I} \mathcal{N}_i(e)$. Hence, $\inf_{i \in I} \mathcal{N}_i$ is an MDFM.
□

Theorem 3.12. *Let $\mathcal{N}_i, i \in I$ be a collection of equicardinal SMDFMs corresponding to a multidimensional t-conorm $\mathbb{U}$ then $\inf_{i \in I} \mathcal{N}_i$ is again an SMDFM with respect to $\mathbb{U}$.*

Proof. The proof follows similar to the above one. □

Theorem 3.13. *Let $\mathcal{N}$ be an SMDFM, then for any $x \in G$ the sequence $\mathcal{N}(x^n), n = 1, 2, 3...$ is monotonically increasing sequence.*

Proof. We have $\mathcal{N}(x) \leq_\infty \mathbb{U}(\mathcal{N}(x), \mathcal{N}(x)) \leq_\infty \mathcal{N}(x^2)$. (Since if $|V| = |W|$ then $V \leq_\infty \mathbb{U}(V, W)$). Similarly, $\mathcal{N}(x^3) \leq_\infty \mathbb{U}(\mathcal{N}(x), \mathcal{N}(x^2)) \leq_\infty \mathcal{N}(x^2)$. (Since, $\mathcal{N}(x) \leq_\infty \mathcal{N}(x^2) \Rightarrow |\mathcal{N}(x)| = |\mathcal{N}(x^2)|$). Continuing like this we have $\{\mathcal{N}(x^n)\}$ is monotonically increasing. □

Theorem 3.14. *Let $\mathcal{N}$ be an MDFM, with an idempotent multidimensional t-norm then for any $x \in G$ the sequence $\{\mathcal{N}(x^{k^n})\}, n = 1, 2, 3...$ is monotonically increasing for every positive integer k.*

Proof. The proof follows directly.

□

3.3. **Multidimensional Fuzzy Groups and Strong Multidimensional Fuzzy Groups.** Next, we study the group structure of multidimensional fuzzy sets

Definition 3.7. *Let G be a group and $\mathcal{N}$ be an MDFM on G. Then $\mathcal{N}$ is said to be a Multidimensional Fuzzy Group (MDFG) if it satisfies $\mathcal{N}(x) = \mathcal{N}(x^{-1})$ for every $x \in G$.*

Definition 3.8. *Let G be a group and $\mathcal{N}$ be an SMDFM on G. Then $\mathcal{N}$ is said to be a Strong Multidimensional Fuzzy Group (SMDFG) if it satisfies $\mathcal{N}(x) = \mathcal{N}(x^{-1})$ for every $x \in G$.*

Theorem 3.15. *Let $G, *$ be a group and $S \subseteq G$ and let χ_S be the multidimensional characteristic function of S which coincides with multidimensional t-norms then S is a subgroup if and only if χ_S is an MDFG.*

Proof. We have $x \in S$ if and only if $x^{-1} \in S$ and $\chi_S(x) = \chi_S(x^{-1})$ are equivalent conditions. Now χ_S is an MDFM if and only if S is a monoid. Using these two results the proof follows directly. □

Theorem 3.16. *Let $(R, +, *)$ be a field and $S \subseteq R$ be a subfield. Let χ_S denote the multidimensional characteristic function of S which coincides with multidimensional t-conorms. Then χ_S is an SMDFG if and only if $S = R$.*

Proof. If $S = R$ then $\chi_S(x) = \dot{1}$ for every x and the result follows directly. Conversely assume that χ_S is SMDFG. If possible assume that $S \neq R$ then there is at least one $x \in R$ which is not in S. Thus we have, for $0 \neq y \in S$, $x * y \notin S$ (Otherwise $x = (x * y) * y^{-1} \in S$). Hence $\chi_S(x * y) = \dot{0}$ but $\mathbb{U}(\chi_S(x), \chi_S(y)) = \mathbb{U}(\dot{0}, \dot{1}) = \dot{1}$ which implies χ_S is not an SMDFG and hence a contradiction. Thus we have $S = R$.

□

Theorem 3.17. *Let $\mathcal{M}_\infty$ be an MDFG on $(G, *)$ and let $A = \{x \in G : \mathcal{M}_\infty(x) \leq_\infty \mathcal{M}_\infty(e)\}$ and $B = \{x \in G : \mathcal{M}_\infty(x) = \mathcal{M}_\infty(e)\}$ then A is always a subgroup of G, and if the respective multidimensional t-norm $\mathbb{I}$ is idempotent, then B is also a subgroup. Let $\mathcal{M}_\in$ be an SMDFG on $(G, *)$ and let $A = \{x \in G : \mathcal{M}_\in(x) \leq_\infty \mathcal{M}_\in(e)\}$, $A_\alpha = \{x \in G : \alpha \leq_\infty \mathcal{M}_\in(x)\}$ where $\alpha \leq_\infty \mathcal{M}_\in(e)$ and $B = \{x \in G : \mathcal{M}_\in(x) = \mathcal{M}_\in(e)\}$ then A, A_α and B are always subgroups.*

Proof. The proof is similar to that of MDFM and SMDFM. □

Theorem 3.18. *If $|\mathcal{N}(x)| = |\mathcal{N}(y)|$ and $\mathcal{N}$ is an SMDFG then $\mathcal{N}(x) = \mathcal{N}(y)$*

Proof. If $|\mathcal{N}(x)| = |\mathcal{N}(y)|$ then we have $|\mathcal{N}(y^{-1})| = H_2(x, y) = |\mathcal{N}(x * y)|$.
Thus

$$\begin{aligned} \mathcal{N}(y) \leq_\infty \mathcal{N}(y^{-1}) &\leq_\infty \mathbb{U}(\mathcal{N}(x * y), \mathcal{N}(y^{-1}) \\ &\leq_\infty \mathcal{N}(x * y * y^{-1}) = \mathcal{N}(x) \end{aligned} \tag{7}$$

Similarly, we have

$$\mathcal{N}(x) \leq_\infty \mathcal{N}(y) \tag{8}$$

Hence we have $\mathcal{N}(x) = \mathcal{N}(y)$ □

Theorem 3.19. *Let $\mathcal{N}$ be an SMDFG, then for any $x \in G$ the sequence $\{\mathcal{N}(x^n)\}, n = 1, 2, 3...$ is a constant sequence.*

Proof. We have already proved that the above sequence is monotonically increasing. Now $\mathcal{N}(x^n) \leq_\infty \mathcal{N}(x^{n+1}) \Rightarrow |\mathcal{N}(x^n)| = |\mathcal{N}(x^{n+1})|$. Then, by the previous theorem, $\{\mathcal{N}(x^n)\}$ is constant. □

Corollary 3.1. *If G is a cyclic group then any SMDFG $\mathcal{N}$ on G is a constant function.*

Next, the theorem gives a relation between the multidimensional fuzzy group and the normal subgroup generated by it.

Theorem 3.20. *Let $\mathcal{M}$ be an MDFG over G and $N = \{y \in G : \mathcal{M}(x*y*x^{-1}) = \mathcal{M}(y), \forall x \in G\}$ and let $\mathcal{M}(x*y) = \mathbb{I}(\mathcal{M}(x), \mathcal{M}(y))$ for every $x, y \in N$. Then N is a normal subgroup of G.*

Proof. Clearly, $e \in N$. Now let $y_1, y_2 \in N$ then for any $x \in G$ we have $\mathcal{M}(x*(y_1*y_2)*x^{-1}) = \mathcal{M}((x*y_1*x^{-1})*(x*y_2*x^{-1})) = \mathbb{I}(\mathcal{M}(x*y_1*x^{-1}), \mathcal{M}(x*y_2*x^{-1}))$ since $(x*y_1*x^{-1}), (x*y_2*x^{-1}) \in N$. Now we have,

$$\begin{aligned}&\mathbb{I}(\mathcal{M}(x*y_1*x^{-1}), \mathcal{M}(x*y_2*x^{-1}))\\&= \mathbb{I}(\mathcal{M}(y_1), \mathcal{M}(y_2)) = \mathcal{M}(y_1*y_2))\end{aligned} \tag{9}$$

Thus we have $y_1 * y_2 \in N$.
Now if $y \in N$ then for $x \in G$

$$\begin{aligned}\mathcal{M}(x*y^{-1}*x^{-1}) &= \mathcal{M}((x*y^{-1}*x^{-1})^{-1})\\&= \mathcal{M}(x*y*x^{-1}) = \mathcal{M}(y) = \mathcal{M}(y^{-1})\end{aligned} \tag{10}$$

Hence, $y^{-1} \in N$.
Finally, let $x, z \in G$ and $y \in N$ then

$$\begin{aligned}\mathcal{M}(z*(x*y*x^{-1})*z^{-1}) &= \mathcal{M}((z*x*)y*(z*x*)^{-1})\\&= \mathcal{M}(y) = \mathcal{M}(x*y*x^{-1})\end{aligned} \tag{11}$$

Thus $x*y*x^{-1} \in N$. Hence, N is a normal subgroup.

□

3.4. **Multidimensional Fuzzy Homomorphism and Isomorphism.** In this section, we define homomorphic and isomorphic multidimensional fuzzy structures and study their relations with ordinary morphisms.

Theorem 3.21. *Let $(G, *)$ and $(H, *^{'})$ be two groups and $\mathcal{N}$ be an MDFG on H with respect to a multidimensional t-norm $\mathbb{I}$ and $\phi : G \to H$ be a homomorphism. Then the MDFS $\mathcal{M}$ defined on G by $\mathcal{M} = \mathcal{N} \circ \phi$ is an MDFG on G with respect to $\mathbb{I}$.*

Proof. Let $x, y \in G$ then

$$\begin{aligned}\mathbb{I}(\mathcal{M}(x), \mathcal{M}(y)) &= \mathbb{I}(\mathcal{N}(\phi(x)), \mathcal{N}(\phi(y)))\\&\leq_\infty \mathcal{N}((\phi(x)) *^{'} (\phi(y))) = \mathcal{N} \circ \phi(x*y) = \mathcal{M}(x*y)\end{aligned} \tag{12}$$

Now, if $|\mathcal{M}(x)| = |\mathcal{M}(e)|$ then $|\mathcal{N}(\phi(x))| = |\mathcal{N}(\phi(e))| = |\mathcal{N}(e^{'})|$ where $e^{'}$ is the identity in H. Hence

$$\mathcal{M}(x) = \mathcal{N}(\phi(x)) \leq_\infty \mathcal{N}(e^{'}) = \mathcal{N}(\phi(e)) = \mathcal{M}(e) \tag{13}$$

Now, for $x \in G$ we have

$$\mathcal{M}(x^{-1}) = \mathcal{N}(\phi(x^{-1})) = \mathcal{N}((\phi(x))^{-1}) = \mathcal{N}(\phi(x)) = \mathcal{M}(x) \tag{14}$$

□

Corollary 3.2. *Let $(G, *)$ and $(H, *^{'})$ be two groups, and $\mathcal{N}$ and $\mathcal{M}$ are defined as above. Then if $\mathcal{N}$ is an MDFGP(MDFM) then so is $\mathcal{M}$*

Theorem 3.22. *Let $(G, *)$ and $(H, *^{'})$ be two groups and $\mathcal{N}$ be an SMDFG(SMDFGP/SMDFM) on H with respect to a multidimensional t-conorm $\mathbb{U}$ and $\phi : G \to H$ be a homomorphism. Then the MDFS $\mathcal{M}$ defined on G by $\mathcal{M} = \mathcal{N} \circ \phi$ is an SMDFG(SMDFGP/SMDFM) on G with respect to $\mathbb{U}$.*

Definition 3.9. *Let $(G, *)$ and $(H, *^{'})$ be two groups. Let $\mathcal{M}$ be an MDFG on G and $\mathcal{N}$ be an MDFG on H respectively with respect to the same multidimensional t-norm. Then $\mathcal{M}$ said to have a multidimensional fuzzy homomorphism with $\mathcal{N}$ in a subgroup of H, if there is a homomorphism ϕ from $G \to H$ such that $\mathcal{M} = \mathcal{N} \circ \phi$. If there is an isomorphism ϕ from $G \to H$ such that $\mathcal{M} = \mathcal{N} \circ \phi$ then we say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

Definition 3.10. *An MDFG $\mathcal{N}$ on G is said to commutative if $\mathcal{N}(x * y) = \mathcal{N}(y * x)$ for every $x, y \in G$.*

Theorem 3.23. *Let $\mathcal{N}$ be commutative MDFG on H and $\mathcal{N}$ is an MDFG on G such that $\mathcal{M} = \mathcal{N} \circ \phi$ then $\mathcal{M}$ is also commutative.*

Proof. Let $x, y \in G$. Then $\mathcal{M}(x*y) = \mathcal{N} \circ \phi(x*y) = \mathcal{N}((\phi(x)) * (\phi(y))) = \mathcal{N}((\phi(y)) * (\phi(x))) = \mathcal{N} \circ \phi(y * x) = \mathcal{M}(y * x)$. Thus $\mathcal{N}$ is commutative. □

Definition 3.11. *Let $f : \mathcal{W} \to \mathcal{Y}$ be a function and $\mathcal{N}$ be an MDFS on $\mathcal{Y}$. Then the inverse image of $\mathcal{N}$ under f is again an MDFS and is defined by $f^{-1}(\mathcal{N})(x) = \mathcal{N}(f(x))$*

Theorem 3.24. *Let $(G, *)$ and $(H, *^{'})$ be two groups and $\phi : G \to H$ be a homomorphism. Let $\mathcal{N}$ be an MDFG on H with respect to a multidimensional t-norm $\mathbb{I}$. Then $\phi^{-1}(\mathcal{N})$ is an MDFG on G with respect $\mathbb{I}$.*

Proof. Let $x, y \in G$ then

$$\begin{aligned}\mathbb{I}(\phi^{-1}(\mathcal{N}(x)), \phi^{-1}(\mathcal{N}(y)) &= \mathbb{I}(\mathcal{N}(\phi(x)), \mathcal{N}(\phi(y))) \\ &\leq_\infty \mathcal{N}(\phi(x) *^{'} \phi(y)) \\ &= \mathcal{N}(\phi(x * y)) = \phi^{-1}(\mathcal{N}(x * y)) \end{aligned} \tag{15}$$

Now, $|\phi^{-1}(\mathcal{N}(x)| = |\phi^{-1}(\mathcal{N}(e)| \Rightarrow |\mathcal{N}(\phi(x))| = |\mathcal{N}(\phi(e))| = |\mathcal{N}(e^{'})|$. Thus

$$\mathcal{N}(\phi(x)) \leq_\infty \mathcal{N}(e^{'}) = \mathcal{N}(\phi(e)) \Rightarrow \phi^{-1}(\mathcal{N}(x)) \leq_\infty \phi^{-1}(\mathcal{N}(e)) \tag{16}$$

Finally, consider

$$\begin{aligned}\phi^{-1}(\mathcal{N}(x^{-1})) &= \mathcal{N}(\phi(x^{-1})) = \mathcal{N}((\phi(x))^{-1}) \\ &= \mathcal{N}(\phi(x)) = \phi^{-1}(\mathcal{N}(x))\end{aligned}$$

Thus $\phi^{-1}(\mathcal{N})$ is an MDFG on G with respect $\mathbb{I}$. □

Theorem 3.25. *Let $(G, *)$ and $(H, *^{'})$ be two groups and $\phi : G \to H$ be a homomorphism. Let $\mathcal{N}$ be an SMDFG on H with respect to a multidimensional t-conorm $\mathbb{U}$. Then $\phi^{-1}(\mathcal{N})$ is an SMDFG on G with respect to $\mathbb{U}$.*

4. Some Equivalence Relations on the Collection of All MDFS and Examples of Some Monoids and Groups in it

Next, we define some equivalence relations on MDFS, and using these relations, we will construct some monoids and groups later.

Definition 4.1. $\mathbb{M}$ *denotes the collection of all MDFS on $\mathcal{X}$. For $\mathcal{M}, \mathcal{N} \in \mathbb{M}$ define a relation $\sim$ by $\mathcal{M} \sim \mathcal{N}$ if and only if elements in $\mathcal{M}(x)$ and $\mathcal{N}(x)$ are the same for every $x \in \mathcal{X}$. Then the relation defined above is an equivalence relation. For $\mathcal{N} \in \mathbb{M}$ let $[\mathcal{N}]$ denote the equivalence class containing $\mathcal{N}$.*

Definition 4.2. *For $\mathcal{M}, \mathcal{N} \in \mathbb{M}$ define a relation $\sim^{'}$ by $\mathcal{M} \sim^{'} \mathcal{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are equicardinal. Then, clearly, this relation is an equivalence relation. For $\mathcal{M} \in \mathbb{M}$ let $[\mathcal{M}]^{'}$ denote the equivalence class containing $\mathcal{M}$.*

Theorem 4.1. *Let* $\mathcal{A}, \mathcal{B} \in [\mathcal{N}] \cap [\mathcal{N}]^{'}$ *then* $\mathcal{A} \cap \mathcal{B}, \mathcal{A} \cup \mathcal{B} \in [\mathcal{N}] \cap [\mathcal{N}]^{'}$ *where* $\mathcal{A} \cap \mathcal{B}$ *and* $\mathcal{A} \cup \mathcal{B}$ *are with respect to standard min and max operators.*

Proof. Let $\mathcal{A}, \mathcal{B} \in [\mathcal{N}] \cap [\mathcal{N}]^{'}$ then clearly $\mathcal{A} \cap \mathcal{B} \in [\mathcal{N}]^{'}$. So it is enough to show that $\mathcal{A}, \mathcal{B} \in [\mathcal{N}]$. Let $x \in \mathcal{X}$ and let $\mathcal{A}(x) = (a_1, a_2, ...a_n)$, $\mathcal{B}(x) = (b_1, b_2, ...b_n)$ and $\mathcal{A}(x) \cap \mathcal{B}(x) = (c_1, c_2, ...c_n)$. By the definition of c_i we have $\{c_1, c_2, ...c_n\} \subseteq \{a_1, a_2, ...a_n\}$. Now, $\mathcal{A}, \mathcal{B} \in [\mathcal{N}] \Rightarrow a_1 = b_1$ and $a_n = b_n$. Hence, $c_1 = a_1, c_n = a_n \in \{a_1, a_2, ...a_n\} = \{b_1, b_2, ...b_n\}$. Now we show that $a_2 \in \{c_1, c_2, ...c_n\}$. If $c_2 \neq a_2$ then $c_2 = \min\{a_2, b_2\} = b_2$. So $b_2 < a_2$. Since $\{a_1, a_2, ...a_n\} = \{b_1, b_2, ...b_n\}$ we have $a_2 = b_k$ for some $k > 2$. Now we have $a_2 \leq a_k$ and $c_k = \min\{a_k, b_k\} = \min\{a_k, a_2\} = a_2$. Thus $a_2 \in \{c_1, c_2, ...c_n\}$. Continuing like this we can show that $\{a_1, a_2, ...a_n\} \subseteq \{c_1, c_2, ...c_n\}$ and thus $\{b_1, b_2, ...b_n\} = \{a_1, a_2, ...a_n\} = \{c_1, c_2, ...c_n\}$. Similarly, we can show that $\mathcal{A}(x)$ and $\mathcal{B}(x)$ have the same elements for every $x \in \mathcal{X}$. So $\mathcal{A} \cap \mathcal{B} \in [\mathcal{N}] \cap [\mathcal{N}]^{'}$.
Similarly, we can show that $\mathcal{A} \cup \mathcal{B} \in [\mathcal{N}] \cap [\mathcal{N}]^{'}$. □

Following are some examples of monoids in the collection of all equivalence classes defined above:

Example 1

Let $\mathbb{E}$ denote the collection of all equivalence classes over the relation $\sim$. Let $[\mathcal{A}], [\mathcal{B}] \in \mathbb{E}$ define the binary operation $\oplus$ on $\mathbb{E}$ by $[\mathcal{A}] \oplus [\mathcal{B}] = [\mathcal{C}]$ where $\mathcal{C}(x)$ is defined as follows:

Let $x \in \mathcal{X}$ and $\mathcal{A}(x) = (A_1(x), A_2(x), A_3(x)...A_n(x))$,
$\mathcal{B}(x) = (B_1(x), B_3(x), B_3(x)...B_m(x))$ then we define
$\mathcal{C}(x) = (C_1(x), C_3(x), C_3(x)...C_{mn}(x))$ where

$$C_k(x) = A_i(x) + B_j(x) - (A_i(x) + B_j(x)), k = ij = 1, 2, 3...mn$$

(where the monotonicity is assumed and it does not affect $[\mathcal{C}]$)

If $\mathcal{A}^{'} \in [\mathcal{A}]$ and $\mathcal{B}^{'} \in [\mathcal{B}]$ then for each $x \in \mathcal{X}$ we have $\mathcal{A}(x)$ and $\mathbf{A}^{'}(\mathbf{x})$ has same elements. Similarly for B and $B^{'}$. Thus $[\mathcal{C}]$ is well defined.
Now it can be easily verified that $(\mathbb{E}, \oplus)$ is a commutative monoid with identity $[e]$ where $e(x) = 0$ for every x.

Note: A similar monoid can be generated in the collection of all equivalence classes generated by $\sim^{'}$ using the same binary operation used above.

Example 2

Let $[\mathcal{A}], [\mathcal{B}] \in \mathbb{E}$ define the binary operation $\odot$ on $\mathbb{E}$ by $[\mathcal{A}] \odot [\mathcal{B}] = [\mathcal{C}]$ where $\mathcal{C}(x)$ is defined as follows:
Let $x \in \mathcal{X}$ and $\mathcal{A}(x) = (A_1(x), A_2(x), A_3(x)...A_n(x))$,
$\mathcal{B}(x) = (B_1(x), B_3(x), B_3(x)...B_m(x))$ then we define
$\mathcal{C}(x) = (C_1(x), C_3(x), C_3(x)...C_{mn}(x))$ where $C_k(x) = A_i(x).B_j(x)), k = ij = 1, 2, 3...mn$. Then $(\mathbb{E}, \odot)$ is a commutative monoid.

Example 3

The following is an example of a group in a sub-collection of orderless multidimensional fuzzy sets.

Let $\mathbb{E}^{'}$ denote the collection of all orderless multidimensional fuzzy sets on a set $\mathcal{X}$ whose membership values do not contain "1." Or, in another sense, we represent the crisp values "0"

and "1" by 0. Choose an equivalence class $[\mathcal{N}]^{'}$ after defining $\sim^{'}$ on $\mathbb{E}^{'}$. Let $\mathcal{A},\mathcal{B} \in [\mathcal{N}]^{'}$. We define a binary operation $\circledast$ on $[\mathcal{N}]^{'}$ by $\mathcal{A} \circledast \mathcal{B} = \mathcal{C}$ where $\mathcal{C}(x) = (C_1(x), C_2(x), ...C_n(x)$ and $C_i(x) = A_i(x)+B_i(x)$ if $A_i(x)+B_i(x) < 1$ and $C_i(x) = 1-(A_i(x)+B_i(x))$ if $1 \leq A_i(x)+B_i(x)$. Clearly, $\circledast$ is a commutative binary operation.
Now, if $(\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C} = \mathcal{D}$ then $\mathcal{D}(x)$ is the decimal part of $\mathcal{A}(x) + \mathcal{B}(x) + \mathcal{C}(x)$. Hence, $(\mathcal{A} \circledast \mathcal{B}) \circledast \mathcal{C} = \mathcal{A} \circledast (\mathcal{B} \circledast \mathcal{C})$
Consider $\mathcal{E} \in [\mathcal{N}]^{'}$ having $\mathcal{E}(x) \in \overline{0}$ for every x. Then $\mathcal{A} \circledast \mathcal{E} = \mathcal{A}$ for every $\mathcal{A} \in [\mathcal{N}]^{'}$.
Now, for $\mathcal{A} \in [\mathcal{N}]^{'}$ define

$\mathcal{A}^{-1}$ by $\mathcal{A}^{-1}(x) = ((1-\mathcal{A}_1^{-1}(x), (1-\mathcal{A}_2^{-1}(x), ...(1-\mathcal{A}_n^{-1}(x)))$. Then clearly, $\mathcal{A}^{-1} \in [\mathcal{N}]^{'}$ and $\mathcal{A} \circledast \mathcal{A}^{-1} = \mathcal{E}$
Hence, $([\mathcal{N}]^{'}, \circledast)$ forms a group.

5. Comparison of multidimensional fuzzy sets with some other fuzzy models

Although Zadeh fuzzy sets were introduced to address data with uncertainty, this structure is inadequate for managing additional uncertainty issues that have arisen throughout time. For instance, it did not adequately represent the resistance of an object in precise set notation (non-membership). In numerous instances, the non-membership value of an object is as significant as its membership value. Fuzzy extensions, including intuitionistic fuzzy sets, Pythagorean fuzzy sets, and spherical fuzzy sets, were developed to address scenarios where non-membership and neutral values must be considered[2, 27]. However, fuzzy extensions like interval-valued fuzzy sets and hesitant fuzzy sets complicate operations and various measurements due to their tedious structure and lack of compactness in presentation whereas n-dimensional and m-polar fuzzy sets do not provide personal attention to each object by applying a uniform dimension across all studied objects while assigning membership grades [4]. Consequently, it is essential to create a fuzzy framework capable of addressing all these issues simultaneously, leading to a more refined and simplified variant of fuzzy sets, referred to in literature as multidimensional fuzzy sets. The importance of MDFS is evident in scenarios where each element may possess a varying number of qualities, necessitating the separate consideration of each property. Examine a more complex scenario in which neurotransmitters convey differing amounts of impulses to neurons. To quantify the bond strength of each chemical they release, we may use the representations $(0.50, 0.55, 0.60)/(0.49, 0.53, 0.65, 0.69)/(0.40, 0.45)$ or $(0.56_a, 0.41_b, 0.54_c)$ (unordered), resulting in a MDFS. Nonetheless, in an m-polar fuzzy collection, the representation must be confined to a single dimension m. Utilizing a 3-polar fuzzy set or a 3-dimensional fuzzy set to characterize the aforementioned information necessitates that each data member possesses a membership value of $(0.50, 0.52, 0.53)/(0.55, 0.56, 0.58)$, derived by disregarding certain chemicals emitted by neurons and assigning a constant dimension to each element. A comparable scenario occurs when modeling an AI-based image processing system using fuzzy variations like $m-$polar or $m-$dimensional fuzzy sets, since the fixed dimension influences the unrestricted image processing by leveraging all of its attributive properties. This adversely affects the authentic representation of the facts, when viewed by the researcher conducting the study. Thus, the data representation using MDFS has various advantages over the existing fuzzy models. Hence, the study of algebraic structures in an MDFS environment has an important role in completing the literature of algebra.

6. Conclusion

This paper extended the concepts of fuzzy algebra into multidimensional fuzzy sets in the usual sense and in a strong sense using multidimensional t-norms and multidimensional t-conorms. This work explored multidimensional fuzzy algebra, focusing on the algebraic structures of groupoids, monoids, and groups. We introduced multidimensional characteristic functions to provide a comprehensive characterization of these structures. The generality of our results stemmed from the foundational use of multidimensional t-norms and t-conorms. We delved into the relationship between multidimensional fuzzy groups and normal sets, as well as the concept of homomorphisms. Additionally, we introduced equivalence relations on the collection of all multidimensional fuzzy sets and constructed examples of monoids. Finally, we demonstrated how a group structure can be imposed on a subcollection of orderless multidimensional fuzzy sets. Future research will explore the potential of multidimensional fuzzy structures, such as rings and fields.

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