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GENERALIZED FRACTIONAL HEAT EQUATION IN EXTENDED COLOMBEAU ALGEBRAS

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Abstract. In this paper, we use the Colombeau generalized algebra to prove the existence and uniqueness of the solution of fractional heat equation with singular potentials (i.e., singular distributions). The concept of a generalized fractional semigroup was used to prove the result.

Keywords: Heat equation, Caputo derivative, Generalised solution, Semigroup, Singular potential.

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1. Introduction

Classical distribution theory, while powerful for linear analysis, fails to provide a consistent framework for multiplying distributions, particularly in the presence of singularities such as the Dirac delta function or its derivatives. This limitation becomes critical when studying nonlinear partial differential equations (PDEs) with singular initial data or coefficients-common in physical models involving point sources, discontinuous media, or sharp interfaces. To overcome this obstacle, we utilize the framework of Colombeau generalized algebras, which extend distribution theory by embedding distributions into a differential algebra where multiplication is well-defined and consistent with smooth functions. This approach not only preserves the operational structure of classical analysis but also allows for rigorous treatment of nonlinear operations involving singularities. In particular, it enables us to define and analyze generalized solutions of fractional heat equations with singular potentials-problems that are otherwise inaccessible using standard methods.

The heat equation is examined in this work with the temporal derivative of the first order replaced with a caputo derivative. The study on Convolution-type derivatives has become a focus area of research because some of those dynamical models could be more

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precisely explained with fractional derivatives compared to those that have integer-order derivatives [26, 27, 28].

In recent years many researchers have centered on the study of phenomena whose modeling is given by nonlinear differential equations with a singularity, to address this, it is necessary to define the multiplication of two distributions in a manner that is consistent with the standard multiplication, the thing that led us automatically to do the study in Colombeau's algebra. This algebra is commutative, associative, and differential, and allows embedding of the space of distributions so that the product of the infinitely differentiable functions and the regular derivative are satisfied [18, 1, 10, 6].

In extended Colombeau algebra, the current research investigates the solution of heat equation with Caputo fractional derivative. The introduction of Caputo derivative into algebra of generalized functions was motivated by the opportunity of solving nonlinear issues with singularities and derivatives of any real order. In order to offer a sense of our situation, we use a special space of Colombeau algebra type which is a commutative, associative differential algebra where we are able to inject D' (space of distributions) so that the smooth function product as well as the normal distribution derivative have been preserved [20, 17, 9].

The following is how the paper is structured, after this introduction, we will discuss various notions related to Colombeau's algebra. In section 3, we will give and demonstrate the existence of Caputo derivative in Colombeau algebra. Section 4, introduce the concept of generalized fractional semigroup. The existence and uniqueness of the solution are discussed in section 5.

2. Preliminaries

In the section that follows, we will mention the Colombeau generalized function (see also [14, 16]).

Definition 1. *$\mathcal{A}_0(\mathbb{R}^n)$ is a set of functions ϕ in $C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\phi(t)dt = 1$. For $q \in \mathbb{N}, \mathcal{A}_q(\mathbb{R}^n) = \left\{\phi \in \mathcal{A}_0 : \int_{\mathbb{R}^n} t^i \phi(t)dt = 0, 0 < |i| \leq q\right\}$, where $t^i = t_1^{i_1}\cdots t_n^{i_n}$.*

In [16] sets

$$\overline{\mathcal{A}}_q(\mathbb{R}^n) = \{\Phi(x_1,..,x_n) = \Phi(x_1)..\Phi(x_n) : \phi(x_i) \in \mathcal{A}_q(\mathbb{R})\},$$

are used because of applications to initial value problems. We shall follow the Colombeau original definition.

Obviously, if $\phi \in \mathcal{A}_q, q \in \mathbb{N}_0$, then for every $\varepsilon > 0, \phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q$. If $\phi \in \mathcal{A}_0$, then its support number $d(\phi)$ is defined by

$$d(\phi) = \sup\{|x| : \phi(x) \neq 0\}.$$

$\mathcal{E}(\Omega)$ represents the set of

$$R : \mathcal{A}_0 \times \Omega \to \mathbb{C}, (\Phi, x) \mapsto R(\Phi, x),$$

which are in $C^\infty(\Omega)$ for every fixed ϕ. In the other words elements of $\mathcal{E}$ are functions $R : \mathcal{A}_0 \to C^\infty$. Note that for any $f \in C^\infty$, the mapping

$$(\phi, x) \mapsto f(x), (\phi, x) \in \mathcal{A}_0 \times \Omega,$$

defines an element in $\mathcal{E}(\Omega)$ which does not depend on ϕ. Conversely, if an element F in $\mathcal{E}(\Omega)$ does not depend on $\Phi \in \mathcal{A}_0$, we have:

$$F(\Phi, x) = F(\Psi, x), \quad x \in \Omega, \text{ for every } \Phi, \Psi \in \mathcal{A}_0,$$

then it defines a function $f \in C^{\infty}(\Omega)$,

$$f(x) = F(\Phi, x), x \in \Omega, \text{ for every } \phi \in \mathcal{A}_0.$$

In this sense, we identify $C^{\infty}(\Omega)$ with the corresponding subspace of $\mathcal{E}(\Omega)$.

Definition 2. *A component $R \in \mathcal{E}(\Omega)$ is moderate if $\forall L \subset\subset \Omega$, $\alpha \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\|\partial^{\alpha} R\left(\Phi_{\epsilon}, x\right)\| \leq C\epsilon^{-N} \qquad x \in L, 0 < \epsilon < \eta.$$

The ensemble of all mild components is expressed as $\mathcal{E}_M(\Omega)$.

Definition 3. *An element $R \in \mathcal{E}_0(\mathbb{C})$ is moderate if $\exists N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$, $\exists \eta > 0$, $C > 0$ such that:*

$$\|R\left(\phi_{\varepsilon}\right)\| < C\varepsilon^{-N}, 0 < \varepsilon < \eta.$$

The ensemble of mild components is expressed by $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$).

Definition 4. *A component $R \in \mathcal{E}_M(\Omega)$ is named null if for every $L \subset\subset \Omega$ and every $\alpha \in \mathbb{N}_0^n$, $\exists N \in \mathbb{N}_0$ and $\{a_q\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\|\partial^{\alpha} R\left(\phi_{\varepsilon}, x\right)\| \leq C\varepsilon^{a_q - N} \qquad x \in L, \qquad 0 < \varepsilon < \eta.$$

The ensemble of null components is expressed by $\mathcal{N}(\Omega)$.

Definition 5. *The spaces of generalized functions $\mathcal{G}(\Omega)$ expressed by*

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

The following description describes what the term "association" means in colombeau algebra.

Definition 6. [14] *Let $f, g \in \mathcal{G}(\mathbb{R})$.*
We said that f,g are associated if $\forall$ $h(\varphi_{\epsilon}, x)$ and $m(\varphi_{\epsilon}, x)$ and arbitrary $\xi \in \mathcal{D}(\mathbb{R})$ there is a $n \in \mathbb{N}$ such that $\forall \varphi(x) \in \mathcal{A}_n(\mathbb{R})$, we have:

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \|h(\varphi_{\epsilon}, x) - m(\varphi_{\epsilon}, x)\| \xi(x) dx = 0,$$

and we denoted by $f \approx g$.

3. Caputo derivative in Colombeau algebras

In this section we will introduce the various definitions and features that we will need in the following.

A fractional integral is defined by: [16]

$$I^{\alpha} f(r) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} f(s) ds \quad \alpha \in \mathbb{R}^+.$$

In the Caputo meaning, the fractional derivative of order $\alpha > 0$ is defined as: [16]

$$D^{\alpha} f(r) = \frac{1}{\Gamma(m-\alpha)} \int_0^r \frac{f^{(m)}(s) ds}{(r-s)^{\alpha+1-m}} \quad , \quad m - 1 < \alpha < m.$$

Let (f_{ε}) be a representative of f in $\mathcal{G}([0, +\infty[)$, so:

$$\begin{aligned} D^{\alpha} f_{\epsilon}(r) &= \frac{1}{\Gamma(1-\alpha)} \int_0^r \frac{f'(s)}{(r-s)^{\alpha}} ds \quad 0<\alpha<1 \\ \sup_{t\in[0,T]} \|D^{\alpha} f_{\varepsilon}(r)\| &\leq \frac{1}{\Gamma(1-\alpha)} \sup_{r\in[0,T]} \| \int_0^r \frac{f'(s)ds}{(r-s)^{\alpha}} \| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left\|f'\right\|_{L^{\infty}([0,T])} \sup_{t\in[0,T]} \int_0^r \frac{ds}{(r-s)^{\alpha}} ds \\ &\leq \frac{1}{\Gamma(1-\alpha)} \epsilon^{-N} \frac{T^{1-\alpha}}{1-\alpha} \leq C_{\alpha,T} \epsilon^{-N}. \end{aligned}$$

Generally [16], for $\alpha \in (m-1, m)$

$$\begin{aligned} \sup_{r\in[0,T]} \|D^{\alpha} f_{\epsilon}(r)\| &\leq \frac{1}{\Gamma(m-\alpha)} \sup_{r\in[0,T]} \int_0^r \frac{\|f^{(m)}(s)\|}{(r-s)^{\alpha+1-m}} ds \\ &\leq \frac{1}{\Gamma(m-\alpha)} \left\|f^{(m)}\right\|_{L^{\infty}([0,T])} \sup_{r\in[0,T]} \int_0^r \frac{1}{(r-s)^{\alpha+1-m}} ds \\ &\leq \frac{1}{\Gamma(m-\alpha)} \epsilon^{-N} \frac{T^{m-\alpha}}{m-\alpha} \leq C_{\alpha,T} \epsilon^{-N}. \end{aligned}$$

The constant $C_{\alpha,T}$ depends on two factors α and T.

Proposition 1. [16] *Let $(\omega_{\epsilon}(t))_{\epsilon}$ be a representative of $\omega(t) \in \mathcal{G}([0,+\infty))$. The regularized Caputo αth fractional derivative of $(\omega_{\epsilon}(t))_{\epsilon}$, $\alpha > 0$, is defined by*

$$i_{frac} : \begin{cases} \mathcal{G}\left([0,+\infty)\right) \to \mathcal{G}\left([0,+\infty)\right) \\ \omega \to \left[\left(\widetilde{D^{\alpha}} \omega_{\epsilon}\right)_{\epsilon>0} \right] = \left[\left(D^{\alpha} \omega * \varphi_{\epsilon}\right)_{\epsilon>0} \right]. \end{cases}$$

Proposition 2.

$$\left(\left(\widetilde{D^{\alpha}} \omega_{\epsilon}\right)_{\epsilon>0} \right) \approx \left(\left(D^{\alpha} \omega_{\epsilon}\right)_{\epsilon>0} \right).$$

Proof. Let: $u_{\varepsilon} \in G\left([0,+\infty)\right)$.

We have,

$$\begin{aligned} \|\tilde{D}^{\alpha} u_{\varepsilon}(t)\| &= \|D^{\alpha} u_{\varepsilon} * \varphi_{\varepsilon}(t)\| \\ &= \|\frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u_{\epsilon}^{(2)}(s)ds}{(t-s)^{\alpha-1}} * \varphi_{\epsilon}(t)\| \\ &\leqslant \|\frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u_{\epsilon}^{(2)}(s)}{(t-s)^{\alpha}-1} ds\| \times \|\varphi_{\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)} \\ &\leqslant \|D^{\alpha} U_{\varepsilon}(t)\| \times \|\varphi_{\varepsilon}\| L^{\infty}\left(\mathbb{R}^n\right). \end{aligned}$$

Then:

$$\begin{aligned} \|\widetilde{D^{\alpha}} u_{\varepsilon}(t) - D^{\alpha} u_{\varepsilon}(t)\| &\leqslant \|D^{\alpha} u_{\varepsilon}(t)\| \left(\|\varphi_{\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)} - 1\right) \\ &\leqslant \frac{1}{\Gamma(2-\alpha)} \sup_{t\in[0,T]} \| \int_0^t \frac{u_{\varepsilon}^{(2)}(\tau)}{(t-\tau)^{\alpha-1}} d\tau \| \times \left(\|\varphi_{\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)} - 1\right) \end{aligned}$$

$\leqslant \frac{1}{\Gamma(2-\alpha)} \sup_{t\in[0,T]} \|u_\varepsilon^{(2)}(t)\| \times \frac{T^{2-\alpha}}{2-\alpha} \times \left(\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} - 1\right)$
$\leqslant C_{T,\alpha}\varepsilon^{2-\alpha} \stackrel{\varepsilon\longrightarrow 0}{\longrightarrow} 0.$

We utilize the regularization for $\alpha \in (0,1)$

$$\tilde{\mathcal{D}}^\alpha u_\epsilon(y) = \mathcal{D}^\alpha u_\epsilon * \phi_\epsilon(y).$$

The form of convolution is provided by :

$$\tilde{\mathcal{D}}^\alpha u_\epsilon(y) = \int_{\mathbb{R}} \mathcal{D}^\alpha u_\epsilon(s)\phi_\epsilon(y-\tau)d\tau.$$

We state that $\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon(y) \mid\approx 0$.

Proof.

$$\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon(y) \mid=\mid \mathcal{D}^\alpha u_\varepsilon * \phi_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon(y) \mid$$
$$\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon(y) \mid=\mid \mathcal{D}^\alpha u_\varepsilon * \phi_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon * \delta(y) \mid$$
$$\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon(y) \mid=\mid \mathcal{D}^\alpha u_\varepsilon * (\phi_\varepsilon(y) - \delta(y)) \mid$$
$$\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon(y) \mid=\mid \int_{\mathbb{R}} \mathcal{D}^\alpha u_\epsilon(y-\tau)\left(\phi_\epsilon(\tau) - \delta(s)\right) d\tau \mid$$
$$\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) - \mathcal{D}^\alpha u_\varepsilon(y) \mid= \int_{\mathbb{R}} \mid \mathcal{D}^\alpha u_\varepsilon(y-\tau) \mid\mid \phi_\varepsilon(\tau) - \delta(\tau) \mid d\tau \longrightarrow 0.$$

Because of $\lim_{\epsilon\longrightarrow 0} \mid \phi_\epsilon(\tau) - \delta(\tau) \mid= 0$, consequently

$$\tilde{\mathcal{D}}^\alpha u_\varepsilon(y) \approx \mathcal{D}^\alpha u_\varepsilon(y).$$

□

By using assumption that $\phi_\epsilon(y)$ has compact support on K_0, the following computations can be made utilizing Holder inequalities:

$$\tilde{\mathcal{D}}^\alpha u_\varepsilon(y) = \mathcal{D}^\alpha u_\epsilon * \phi_\epsilon(y) = \int_{\mathbb{R}} \mathcal{D}^\alpha u_\varepsilon(y-\tau)\phi_\epsilon(\tau)d\tau$$
$$\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) \mid=\mid \int_{\mathbb{R}} \mathcal{D}^\alpha u_\varepsilon(y-\tau)\phi_\epsilon(\tau)d\tau \mid=\mid \int_{K_0} \mathcal{D}^\alpha u_\varepsilon(y-\tau)\phi_\epsilon(\tau)ds \mid$$
$$\mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) \mid= \int_{K_0} \mid \mathcal{D}^\alpha u_\varepsilon(y-\tau) \mid\mid \phi_\varepsilon(\tau) \mid d\tau$$
$$\sup_{y\in K} \mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) \mid= \sup_{y\in K} \left\{ \int_{K_0} \mid \mathcal{D}^\alpha u_\epsilon(y-\tau) \mid\mid \phi_\epsilon(\tau) \mid d\tau \right\}$$

so,

$$\sup_{y\in K_0} \mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) \mid\leq \sup_{y\in K_0} \mid \mathcal{D}^\alpha u_\varepsilon(y) \mid \int_{K_0} \mid \phi_\varepsilon(\tau) \mid d\tau$$
$$\sup_{y\in K} \mid \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) \mid\leq C_1\varepsilon^p.$$

And

$$\frac{d}{dy}\left(\tilde{\mathcal{D}}^\alpha u_\varepsilon(y)\right) = \frac{d}{dy}\left(\mathcal{D}^\alpha u_\varepsilon\right) * \phi_\varepsilon(y) = \mathcal{D}^\alpha u_\varepsilon * \frac{d}{dy}\left(\phi_\varepsilon(y)\right).$$

Then,

$$\sup_{y\in K} \mid \frac{d}{dy}\left(\tilde{\mathcal{D}}^\alpha u_\varepsilon(y)\right) \mid\leq \sup_{y\in K_0} \mid \mathcal{D}^\alpha u_\varepsilon(y) \mid \int_{K_0} \mid \frac{d}{dy}\left(\phi_\varepsilon(\tau)\right) \mid d\tau \leq C_2\varepsilon^p.$$

A similar approach is used to demonstrate moderateness for higher derivatives.

$$\sup_{y\in K} \mid \partial^n \tilde{\mathcal{D}}^\alpha u_\varepsilon(y) \mid \leq C_\epsilon \varepsilon^p.$$

□

By the same principle we define "Generalized caputo semigroup".

4. Generalized Caputo semigroup

Let $(X, \|.\|)$ be a Banach space and $\mathcal{L}(X)$ be the space of all linear continuous mappings . Firstly, we want to see whether we can create a map $A : \mathcal{G} \longrightarrow \mathcal{G}$ using a provided family $(A_\epsilon)_{\epsilon\in(0,1)}$ of $A_\epsilon X \longrightarrow X$, where $A_\epsilon \in \mathcal{L}(X)$. The following are the general requirements:

Lemma 1. *Let $(A_\epsilon)_{\epsilon\in]0,1[}$ be a provided family of maps $A_\epsilon : X \longrightarrow X$. In each case of $(e_\epsilon)_\epsilon \in \mathcal{E}_M(X)$ and $(f_\epsilon)_\epsilon \in \mathcal{N}(X)$, suppose that*
(1) $(A_\epsilon e_\epsilon)_\epsilon \in \mathcal{E}_M(X)$,
(2) $(A_\epsilon (e_\epsilon + f_\epsilon))_\epsilon - (A_\epsilon e_\epsilon)_\epsilon \in \mathcal{N}(X)$.
Then

$$A : \begin{cases} \mathcal{G} \longrightarrow \mathcal{G} \\ e = [e_\epsilon] \mapsto Ae = [A_\epsilon e_\epsilon] \end{cases}$$

is well defined.

Proof. We can see from the first attribute that the class $[(A_\epsilon e_\epsilon)_\epsilon] \in \mathcal{G}$.
Let $e_\epsilon + f_\epsilon$ be an additional member of $e = [e_\epsilon]$. We have from the second property:

$$(A_\epsilon (e_\epsilon + f_\epsilon))_\epsilon - (A_\epsilon e_\epsilon)_\epsilon \in \mathcal{N}(X),$$

and $[(A_\epsilon (e_\epsilon + f_\epsilon))_\epsilon] = [(A_\epsilon e_\epsilon))_\epsilon]$ in $\mathcal{G}$.
Thus A is properly defined. □

Definition 7. *We define*
$\mathcal{E}_M^S([0, +\infty[, \mathcal{L}(X)) = \{S_\epsilon : [0, +\infty[\longrightarrow \mathcal{L}(X)), \quad \epsilon \in]0, 1[,$ *such that* $\forall T > 0, \quad \exists a \in \mathbb{R},$ *we have*

$$\sup_{r\in[0,T]} \left\| S_\epsilon \left(r^{\frac{1}{\alpha}} \right) \right\| = O_{\epsilon\to 0} (\epsilon^a)\},$$

and
$\mathcal{N}^S([0, +\infty[, \mathcal{L}(X)) = \{N_\epsilon : [0, +\infty[\longrightarrow \mathcal{L}(X)), \quad \epsilon \in]0, 1[,$ *such that* $\forall T > 0, \quad \forall b \in \mathbb{R},$ *we have*

$$\sup_{r\in[0,T]} \left\| N_\epsilon \left(r^{\frac{1}{\alpha}} \right) \right\| = O_{\epsilon\to 0} \left(\epsilon^b \right)\}.$$

With the following characteristics:
1) $\exists s > 0$ and $\exists a \in \mathbb{R}$ such that

$$\sup_{t<s} \left\| \frac{N_\epsilon \left(r^{\frac{1}{\alpha}} \right)}{r} \right\| = O_{\epsilon\to 0} (\epsilon^a),$$

2) $\exists (H_\epsilon)_\epsilon$ in $\mathcal{L}(X)$ and $\epsilon \in]0, 1[$ such that

$$\lim_{s\to 0} \frac{N_\epsilon \left(s^{\frac{1}{\alpha}} \right)}{s} e = H_\epsilon e, \quad e \in X,$$

For every $b > 0$,

$$\|H_\epsilon\| = O_{\epsilon\to 0} \left(\epsilon^b \right),$$

Proposition 1. *$\mathcal{E}_M^S([0,+\infty[,\mathcal{L}(X))$ is algebra in terms of composition and $\mathcal{N}^S([0,+\infty[,\mathcal{L}(X))$ is an ideal of $\mathcal{E}_M^S([0,+\infty[,\mathcal{L}(X))$.*

Proof. Let $(S_\epsilon)_\epsilon \in \mathcal{E}_M^S([0,+\infty[,\mathcal{L}(X))$ and $(N_\epsilon)_\epsilon \in \mathcal{N}^S([0,+\infty[,\mathcal{L}(X))$.
We shall simply establish the second statement, specifically,

$$\left(S_\epsilon\left(r^{\frac{1}{\alpha}}\right)N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right)_\epsilon, \left(N_\epsilon\left(r^{\frac{1}{\alpha}}\right)S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right)_\epsilon \in \mathcal{N}^S([0,+\infty[,\mathcal{L}(X)).$$

Where $S_\epsilon\left(r^{\frac{1}{\alpha}}\right)N_\epsilon\left(r^{\frac{1}{\alpha}}\right)$ represents the composition.
By (1) and the definition of $\mathcal{N}^S$ from the previous definition, we have:

$$\left\|S_\epsilon\left(r^{\frac{1}{\alpha}}\right)N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\| \leq \left\|S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\| = O_{\epsilon\to 0}\left(\epsilon^{a+b}\right),$$

The same is also true for $\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right)S_\epsilon\left(r\frac{1}{\alpha}\right)\right\|$.
Furthermore, (1) and (2) provide

$$\begin{aligned}\sup_{t<s}\left\|\frac{S_\epsilon\left(r^{\frac{1}{\alpha}}\right)N_\epsilon\left(r^{\frac{1}{\alpha}}\right)}{r}\right\| &\leq \sup_{r<s}\left\|S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\|\sup_{r<s}\left\|N_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\| \\ &= O_{\epsilon\to 0}\left(\epsilon^{a}\right).\end{aligned}$$

In some situations $s>0$. We have,

$$\sup_{r>s}\left\|\frac{N_\epsilon\left(r^{\frac{1}{\alpha}}\right)S_\epsilon\left(r^{\frac{1}{\alpha}}\right)}{r}\right\| = O_{\epsilon\to 0}\left(\epsilon^{a}\right),$$

For some $s>0$ and $a\in\mathbb{R}$. Let now $\epsilon\in]0,1[$ be fixed. We have

$$\begin{aligned}&\left\|\frac{S_\epsilon(r^{\frac{1}{\alpha}})N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - S_\epsilon(0)H_\epsilon x\right\| \\ &= \left\|S_\epsilon(r^{\frac{1}{\alpha}})\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - S_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x + S_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - S_\epsilon(0)H_\epsilon x\right\| \\ &\leq \left\|S_\epsilon(r^{\frac{1}{\alpha}})\right\|\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon x\right\| + \left\|S_\epsilon(r^{\frac{1}{\alpha}})H_\epsilon x - S_\epsilon(0)H_\epsilon x\right\|.\end{aligned}$$

According to (1) and (2), in addition to the continuity of $t\mapsto S_\epsilon(r^{\frac{1}{\alpha}})(H_\epsilon x)$ at 0, the final expression becomes zero as $r\to 0$. Likewise, we have:

$$\begin{aligned}\left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})S_\epsilon(r^{\frac{1}{\alpha}})}{r}x - H_\epsilon S_\epsilon(0)x\right\| &= \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}S_\epsilon(r^{\frac{1}{\alpha}})x - \frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}S_\epsilon(0)x\right. \\ &\quad \left.+\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}S_\epsilon(0)x - H_\epsilon S_\epsilon(0)x\right\| \\ &\leq \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\right\|\left\|S_\epsilon(r^{\frac{1}{\alpha}})x - H_\epsilon(t)S_\epsilon(0)x\right\| \\ &\quad + \left\|\frac{N_\epsilon(r^{\frac{1}{\alpha}})}{r}\left(S_\epsilon(0)x\right) - H_\epsilon\left(S_\epsilon(0)x\right)\right\|.\end{aligned}$$

Assertions (1) and (2) require that the final expression goes to zero since $r \mapsto 0$. As a result, the proposition is proven in both circumstances. □

The factor algebra is now defined as Colombeau type algebra by:

$$\mathcal{G}^S([0,+\infty[,\mathcal{L}(X)) = \mathcal{E}_M^S([0,+\infty[,\mathcal{L}(X))/\mathcal{N}^S([0,+\infty[,\mathcal{L}(X)).$$

Components of $\mathcal{G}^S([0,+\infty[,\mathcal{L}(X))$ will be represented by $S=[S_\epsilon]$, where $(S_\epsilon)_e$ is a member of the preceding class.

Definition 8. *$S \in \mathcal{G}([0,+\infty[,\mathcal{L}(X))$ is referred to as a Colombeau C_0Semigroup if it has a member $(S_\epsilon)_\epsilon$ such that, for $\epsilon > 0, S_\epsilon$ is a C_0Semigroup.*

When ϵ low sufficient, we will only utilize members $(S_\epsilon)_\epsilon$ of a Colombeau C_0semigroup.

Proposition 2. *Let $\left(\tilde{S}_\epsilon\right)_\epsilon$ and $(S_\epsilon)_\epsilon$ be members of a Colombeau C_0semigroup $(S_\epsilon)_\epsilon$, with the infinitesimal generators $\tilde{A}_\epsilon, \quad \epsilon < \tilde{\epsilon}_0$, and $A_\epsilon, \quad \epsilon < \epsilon_0$, respectively, where $\tilde{\epsilon}_0$ and ϵ_0 correspond (in the sense of definition 5.6) to $(S_\epsilon)_\epsilon$ and $(S_\epsilon)_\epsilon$, respectively.*

Then, $D\left(D\left(\tilde{A}_\epsilon\right) = A_\epsilon\right)$, $\forall \tilde{\epsilon} = \min\{\epsilon_0, \tilde{\epsilon}_0\} > \epsilon$ and $\tilde{A}_\epsilon - A_\epsilon$ can be prolonged to a component of$\mathcal{L}(X)$. Moreover, for every $a \in]-\infty,+\infty[$,

$$\left\|\tilde{A}_\epsilon - A_\epsilon\right\| = O_{\epsilon\to 0}(\epsilon^a),$$

Proof. Indicate $N_\epsilon = \left(S_\epsilon - \tilde{S}_\epsilon\right)_\epsilon \in \mathcal{N}^S([0,+\infty[,\mathcal{L}(X))$.
Let $\epsilon < \tilde{\epsilon}_0$ be fixed and $y \in X$. we have

$$\frac{S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t} - \frac{\tilde{S}_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t} = \frac{N_\epsilon\left(t^{\frac{1}{\alpha}}\right)}{t}y.$$

This indicates that by allowing $ttomapto0: D(A_\epsilon) = D(\bar{A}_\epsilon)$.
After that, we have

$$\begin{aligned}\left(A_\epsilon - (\tilde{A})_\epsilon\right)y &= \lim_{t\to 0}\frac{S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t} - \lim_{t\to 0}\frac{S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - y}{t}\\ &= \lim_{t\to 0}\frac{N_\epsilon\left(t^{\frac{1}{\alpha}}\right)}{t}y = H_\epsilon y, \quad y \in D(A_\epsilon),\end{aligned}$$

since $D(\bar{A}_e) = X$ and characteristics (1), (2) and definition(7) imply that $\forall a \in \mathbb{R}$,

$$\left\|A_\epsilon - \tilde{A}_\epsilon\right\| = O_{\epsilon\to 0}(\epsilon^a).$$

□

The following definition makes sense because of Proposition 2.

Definition 9. *If there exists a representative $(A_\epsilon)_\epsilon$ of A such that A_ϵ is the infinitesimal generator of S_ϵ for ϵ small sufficiently, then A is the infinitesimal generator of a Colombeau C_0semigroup S.*

By Pazy we presente the following suggestion:

Proposition 3. *Assume S is a Colombeau C_0semigroup with an infinitesimal generator A.*
Then $\exists\epsilon_0 \in]0,1[$ such that:
1) Mapping $r \longmapsto S_\epsilon\left(r^{\frac{1}{\alpha}}\right)y : [0,+\infty[\longrightarrow X$ is continuous $\forall y \in X$ and $\epsilon < \epsilon_0$.
2)

$$\lim_{h\to 0}\int_r^{r+h} S_\epsilon\left(s^{\frac{1}{\alpha}}\right) y ds_\alpha = S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y, \quad \epsilon<\epsilon_0, \quad y\in X.$$

3)

$$\int_0^r S_\epsilon\left(s^{\frac{1}{\alpha}}\right) y ds_\alpha \in D\left(A_\epsilon\right), \quad \epsilon<\epsilon_0, \quad y\in X.$$

4) $\forall y \in D\left(A_\epsilon\right)$ and $r\geq 0$ $\quad S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y \in D\left(A_\epsilon\right)$ and

$$\frac{d^\alpha}{dt^\alpha}S_\epsilon\left(r^{\frac{1}{\alpha}}\right)y = A_\epsilon S_\epsilon\left(r^{\frac{1}{\alpha}}\right)y = S_\epsilon\left(r^{\frac{1}{\alpha}}\right)A_\epsilon y, \quad \epsilon<\epsilon_0.$$

5) Take $\left(S_\epsilon\right)_\epsilon$ and $\left(\tilde{S}_\epsilon\right)_\epsilon$ be representatives of Colombeau C_0semigroup S, with infinitesimal generators A_ϵ and $\tilde{A}_\epsilon$, $\epsilon<\epsilon_0$, accordingly. So, $\forall a\in\mathbb{R}$, $r\geq 0$, we have:

$$\left\|\frac{d^\alpha}{dt^\alpha}S_\epsilon\left(r^{\frac{1}{\alpha}}\right) - A_\epsilon S_\epsilon\left(r^{\frac{1}{\alpha}}\right)\right\| = O\left(\epsilon^a\right).$$

6) $\forall y\in D\left(A_\epsilon\right)$ and $t,r\geq 0$,

$$S_\epsilon\left(t^{\frac{1}{\alpha}}\right)y - S_\epsilon\left(s^{\frac{1}{\alpha}}\right)y = \int_r^t S_\epsilon\left(e^{\frac{1}{\alpha}}\right)A_\alpha y de_\alpha = \int_s^t A_\epsilon S_\epsilon\left(e^{\frac{1}{\alpha}}\right) y de_\alpha.$$

Theorem 1. *Let S and $\tilde{S}$ be Colombeau C_0semigroup with infinitesimal generators A and $\bar{A}$, respectively. If $A=\tilde{A}$ then $S=\tilde{S}$.*

Proof. Applying the previous properties will be easy to proof the theorem. □

5. Main result

This section describes the use of Colombeau Caputo C_0-semigroup in the solution of a family of heat equations with singular data and potentials. We considering the next issue,
Before we explore the subject, we will create some working areas.
We put $\|\cdot\|_{L^2(\mathbb{R})} = \|\cdot\|_2$.

Definition 10. *We indicate H_α^2 by the set of a function $u \in L^2(\mathbb{R})$ with, $\widetilde{D}^\alpha u \in L^2(\mathbb{R})$*
In accordance with the norm

$$\|u\|_{H_\alpha^2} = \sqrt{\|u\|_2^2 + \left\|\widetilde{D}^\alpha u\right\|_2^2}.$$

The following is the definition of the Colombeau algebra type:

$$\mathcal{G}_{H_\alpha^2} = \mathcal{E}_M\left(H_\alpha^2\right)/\mathcal{N}\left(H_\alpha^2\right),$$

where $\mathcal{E}_M\left(H_\alpha^2\right) = \{\left(G_\varepsilon\right)_\varepsilon \in H_\alpha^2, \forall T>0 \quad \exists a\in\mathbb{R} : \|G_\varepsilon\|_{H_\alpha^2} = O\left(\varepsilon^a\right)\}$,
And $\mathcal{N}\left(H_\alpha^2\right) = \{\left(G_\varepsilon\right)_\varepsilon \in H_\alpha^2, \forall T>0 \quad \forall b\in\mathbb{R} : \|G_\varepsilon\|_{H_\alpha^2} = O\left(\varepsilon^b\right)\}$.

Definition 11. $\mathcal{E}_{C^1,H_\alpha^2}([0,T],\mathbb{R}) = \{G_\varepsilon \in C\left([0,T],H_\alpha^2\right)\cap C^1\left([0,T],L^2(\mathbb{R})\right), \forall T>0 \quad \exists a\in\mathbb{R}$:
$\max\left\{\sup_{t\in[0,T_1]}\|G_\varepsilon\|_{H_\alpha^2}, \sup_{t\in[0,T]}\left\|\widetilde{D}^\alpha G_\varepsilon\right\|_{L^2(\mathbb{R})}\right\} = O_{\varepsilon\to 0}\left(\varepsilon^a\right)\}$.

And,
$\mathcal{N}_{C^1,H^2_\alpha}([0,T],\mathbb{R})) = \{G_\varepsilon \in C\left([0,T],H^2_\alpha\right) \cap C^1\left([0,T],L^2(\mathbb{R})\right), \forall T>0 \quad \forall b \in \mathbb{R}:$
$\max\left\{\sup_{t\in[0,T_1]} \|G_\varepsilon\|_{H^2_\alpha}, \sup_{t\in[0,T]} \left\|\widetilde{D}^\alpha G_\varepsilon\right\|_{L^2(\mathbb{R})}\right\} = O_{\varepsilon\to 0}\left(\varepsilon^b\right)\}.$

Then the Colombeau type vector space, define by:

$$\mathcal{G}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R}) = \mathcal{E}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R})/\mathcal{N}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R}).$$

Proposition 4. *Let $v \in \mathcal{G}_{H^2_\alpha}$ and $x \in \mathcal{G}_{C^1,H^2_\alpha}(\mathbb{R}^+,\mathbb{R})$ which is proposed as the solution to the following problem:*

$$\begin{cases} \partial_t^\alpha x(t,y) = (\Delta - v(y))x(t,y) & y \in \mathbb{R}, t \in \mathbb{R}^+ \\ x(0,y) = x_0(y) = \delta(y) \\ v(y) = \delta(y). \end{cases} \tag{1}$$

Then the multiplication v(y).x(t,y) makes sense.

Definition 12. *A generalized function $G \in \mathcal{G}_{C^1,H^2_\alpha}$ is considered to be a solution to the equation $\widetilde{D}^\alpha G = AG$. With A is characterized by a nets $(A_\epsilon)\varepsilon$ of linear operators with the consistent framework $H^2_\alpha(\mathbb{R})$ and values in $L^2(\mathbb{R})$, if and only if*

$$\sup_{t\in[0,T]} \left\|\widetilde{D}^\alpha G_\epsilon(t,.) - A_\varepsilon G_\varepsilon(t,.)\right\|_2 = O\left(\epsilon^a\right), \epsilon \to 0 \quad \forall a \in \mathbb{R}.$$

Definition 13. *An component $U \in \mathcal{G}_{H^2_\alpha}$ is logarithmic type if it has an identification $(U_\epsilon)_\epsilon \in \mathcal{E}_{C^1,H^2_\alpha}$ with,*

$$\|U_\varepsilon\|_{H^2_\alpha} = O_{\varepsilon\to 0}\left(\ln \varepsilon^{-1}\right).$$

An componen $U \in \mathcal{G}_{H^2_\alpha}$ is claimed to be log-log type if it has a identification $(U_\epsilon)_\epsilon \in \mathcal{E}_{C^1,H^2_\alpha}$ with,

$$\|U_\epsilon\|_{H^2_\alpha} = O\left(\ln^a \ln \varepsilon^{-1}\right) \quad \epsilon \to 0.$$

We put

$$E(t,y) = \frac{1}{2\sqrt{\pi t}} \exp^{\frac{-|y|^2}{4t}}.$$

5.1. **Existence and uniqueness in the Colombeau algebra.**

Theorem 2. *Consider a function v that belongs to the set $\mathcal{G}H^2_\alpha$ and is logarithmic type. (1) The infinitesimal generators of Caputo semigroups T_ϵ $\forall \epsilon > 0$ is given by $(\Delta - v)u = A_\epsilon u$, with $u \in H^2_\alpha$. The collection of these semigroups $(T_\epsilon)_\epsilon$, is a representative of a Colombeau Caputo C_0-semigroup.*

$$T(t) \in G\mathcal{S}\left([0,+\infty[,\mathcal{L}\left(L^2\right)\right).$$

(2) Consider T_ϵ be as in (1) and let v be a member of the set $G_{H^2_\alpha}$.
Then, $\forall T > 0$, the issue 1 has unique solution in $\mathcal{G}H^2_\alpha$.

Proof. (1) Put $\epsilon > 0$ as small as possible. The operator A_ϵ is the infinitesimal generator of the associated semigroup according to the Feynman-Kac formula.

$$T_\epsilon(s)\phi(y) = \int_\Omega \left(\exp^{\left(-\int_0^{s^\alpha} v_\epsilon(\omega(e))de\right)}\right) \phi\left(\omega\left(s^\alpha\right)\right) d\mu_{y(\omega)},$$

for $\phi \in L^2(\mathbb{R})$, $\Omega = \mathbb{R}$ and μ_y is is the Wiener measure centered at $x \in \mathbb{R}$. Since v is of logarithmic type, $\exists C > 0$, such that

$$| T_\epsilon(s)\phi(y) | \leq \exp^{\left(s^\alpha \sup_{e\in\mathbb{R}}|v_\epsilon(e)|\right)} \int_\Omega | \phi(\omega(s)) | d\mu_{y(\omega)}$$

$$\leq \epsilon^{Cs^{\alpha}} \frac{1}{2\sqrt{\pi s^{\alpha}}} \int_{\mathbb{R}} \exp^{-\frac{|y-e|^2}{4s^{\alpha}}} \mid \phi(e) \mid de.$$

Consequently, $\exists C_0 > 0$, such that

$$\sup_{s\in(0,T]} \|T_\epsilon(s)\phi(y)\|_2 \leq C_0 \epsilon^{Cs^{\alpha}} \|\phi\|_2.$$

Then $(T_\epsilon)_\epsilon \in G\mathcal{S}\left([0,+\infty[, \mathcal{L}\left(L^2(\mathbb{R})\right)\right)$.

(2) **Existence**

By the principle of Duhamel , then the solution $x_\epsilon(t,y)$ of issue 1 satisfies:

$$\begin{aligned} x_\epsilon(s,y) = &\int_{\mathbb{R}} E\left(s^{\alpha}, y-e\right) b_\epsilon(e) de \\ &+ \int_0^s \int_{\mathbb{R}} E\left((s-t)^{\alpha}, y-e\right) v_\epsilon(e) x_\epsilon(t,e) de dt. \end{aligned} \quad (2)$$

By Young's inequality, we have:

$$\|x_\epsilon(s,.)\|_2 \leq \|b_\epsilon\|_2 + \int_0^s \|v_\epsilon(.)\|_{L^\infty} \|x_\epsilon(t,.)\|_2 \, dt.$$

By Gronwall's inequality, we have:

$$\|x_\epsilon(s,.)\|_2 \leq \|b_\epsilon\|_2 \exp^{\int_0^s \|v_\epsilon(.)\|_{L^\infty} \|x_\epsilon(t,.)\|_2 dt}, \forall t \in [0,T].$$

Since $v \in GH_\alpha^2$ is logarithmic type and $(x_\epsilon)_\epsilon \in \mathcal{E}H_\alpha^2$, it follows that $\sup_{s\in[0;T]} \|x_\epsilon(s,.)\|_2$ has a moderate bound.

For the first derivative to $y_j, j \in 1,...,n$ we obtain

$$\begin{aligned} \frac{d}{dy} x_\epsilon(s,y) = &\int_{\mathbb{R}} E\left(s^{\alpha}, x\right) \frac{d}{dy} b_{\epsilon(y-x)} dx \\ &+ \int_0^s \int_{\mathbb{R}} E\left((s-t)^{\alpha}, x\right) \frac{d}{dy} v(y-x) x_\epsilon(t, y-x) \\ &+ v(y-x) \frac{d}{dy} x_\epsilon(t, y-x) dx dt. \end{aligned}$$

Then

$$\left\|\frac{d}{dy} x_\epsilon(s,.)\right\|_2 \leq \left\|\frac{d}{dy} b_\epsilon\right\|_2 + \int_0^s \left\|\frac{d}{dy} v_\epsilon(.)\right\|_{L^\infty} \|x_\epsilon(s,.)\|_2 + \|v_\epsilon(\cdot)\|_{L^\infty} \|\frac{d}{dy} x_\epsilon(s,.))\|_2,$$

consequently by Gronwall's inequality implies that $\sup_{s\in[0,T]} \left\|\frac{d}{dy} x_\epsilon(s,.)\right\|_2$ is moderate.

With the same proof, we can prove that all types of derivatives exists in $\mathcal{E}_{C^1, H_\alpha^2}$.

So

$$(x_\epsilon)_\epsilon \in \mathcal{E}_{C^1, H_\alpha^2}.$$

Uniqueness:

Let x_ϵ and z_ϵ two solution of issue (1).

We set $G_\epsilon = x_\epsilon - z_\epsilon$, we get

$$\begin{aligned} G_\epsilon(s,y) = &\int_{\mathbb{R}} E\left(s^\alpha, y-e\right) N_\epsilon(e) de \\ &+ \int_0^s \int_{\mathbb{R}} E\left((s-t)^\alpha, y-e\right) v_\epsilon(e) G_\epsilon(t,e) dedt \\ &+ \int_0^s \int_{\mathbb{R}} E\left((s-t)^\alpha, y-x\right) N_\epsilon(e) dedt, \end{aligned}$$

where $N_\epsilon(y) = G(0,y)$, and $N_\epsilon = \frac{d^\alpha}{ds^\alpha} G_\epsilon - (\Delta - v) G_\epsilon$.
So Young's and Gronwall's inequalities imply that:

$$\|G_\epsilon(s,.)\|_2 \leq \|N_\epsilon\|_2 + \int_0^s \|v_\epsilon(t,.)\|_{L^\infty} \|G_\epsilon(t,.)\|_2 \, dt + \int_0^s \|N_\epsilon(t,.)\|_2 \, dt.$$

Thus

$$G_\epsilon \in \mathcal{N} H^2_\alpha.$$

□

Remark 1 (Application of Theorem 2). *To illustrate the applicability of the main result, consider the following initial value problem:*

$$\begin{cases} \partial_t^\alpha u(t,x) = \left(\Delta - \delta(x)\right) u(t,x), \quad x \in \mathbb{R}, \ t > 0, \\ u(0,x) = \delta(x), \end{cases}$$

where $\delta(x)$ denotes the Dirac delta distribution and ∂_t^α is the Caputo fractional derivative of order $\alpha \in (0,1)$. This equation models anomalous diffusion in media with highly concentrated singular potentials, such as point interactions in quantum mechanics or heat propagation with localized defects.

In classical distribution theory, the product $\delta(x) \cdot u(t,x)$ is not well-defined. However, by embedding the problem in the Colombeau algebra GH^2_α, both the singular initial data and the singular potential become tractable. Theorem 2 guarantees the existence and uniqueness of a generalized solution within this framework, demonstrating the power and necessity of Colombeau algebras for analyzing fractional PDEs with strong singularities.

5.2. **Existence and uniqueness in the extension of Colombeau algebra.** We prove the existence and uniqueness solution for the problem 1, in this case initial data and the equation controlled by the fractional derivative in the framework of the extended algebra type of generalized functions $\mathcal{G}H^2_\alpha$ with $\alpha \in \mathbb{R}$.

Theorem 3. *We consider T_ϵ be as in Theorem 2 and let v be a member of the set $G_{H^2_\alpha}$. Then, the problem 1 has unique solution in $\mathcal{G}H^2_\alpha$.*

Proof. We shall prove only the fractional part since the entire part is already proved in the theorem (1). Consider the fractional derivative D^β with $0 < \beta < 1$.
Without loss of generality, the same holds for $m-1 < \beta < m$, $m \in \mathbb{N}$.
Existence

By the principle of Duhamel , the solution $x_\epsilon(t,y)$ of issue (1) satisfies:

$$\begin{aligned} x_\epsilon(s,y) = &\int_{\mathbb{R}} E\left(s^\alpha, y-e\right) b_\epsilon(e) de \\ &+ \int_0^s \int_{\mathbb{R}} E\left((s-t)^\alpha, y-e\right) v_\epsilon(e) x_\epsilon(t,e) dedt. \end{aligned} \tag{3}$$

Take the fractional derivative to the spatial variable, we have

$$\begin{aligned} D^{\beta}x_{\epsilon}(s,y) &= \int_{\mathbb{R}} E\left(s^{\alpha},x\right)D^{\beta}b_{\epsilon(y-x)}dx \\ &+ \int_0^s\int_{\mathbb{R}} E\left((s-t)^{\alpha},x\right)D^{\beta}v(y-x)x_{\epsilon}(t,y-x) \\ &+ v(y-x)D^{\beta}x_{\epsilon}(t,y-x)dxdt. \end{aligned}$$

Then Young's inequality implies that

$$\left\|D^{\beta}x_{\epsilon}(s,.)\right\|_2 \leq \left\|D^{\beta}b_{\epsilon}\right\|_2 + \int_0^s \left\|D^{\beta}v_{\epsilon}(.)\right\|_{L^{\infty}} \|x_{\epsilon}(s,.)\|_2 + \|v_{\epsilon}(\cdot)\|_{L^{\infty}} \|D^{\beta}x_{\epsilon}(s,.))\|_2.$$

Consequently by Gronwall's inequality, we have $\sup_{s\in[0,T]}\left\|D^{\beta}x_{\epsilon}(s,.)\right\|_2$ is moderate.
Uniqueness:

Let $u_{1\epsilon}$ and $u_{2\epsilon}$ two solution of issue (1).

We set $G_{\epsilon}=u_{1\epsilon}-u_{2\epsilon}$, we get

$$\begin{aligned} G_{\epsilon}(s,y) &= \int_{\mathbb{R}} E\left(s^{\alpha},y-e\right)N_{\epsilon}(e)de \\ &+ \int_0^s\int_{\mathbb{R}} E\left((s-t)^{\alpha},y-e\right)v_{\epsilon}(e)G_{\epsilon}(t,e)dedt \\ &+ \int_0^s\int_{\mathbb{R}} E\left((s-t)^{\alpha},y-x\right)N_{\epsilon}(e)dedt, \end{aligned}$$

where $N_{\epsilon}(y)=G(0,y)$, and $N_{\epsilon}=\frac{d^{\alpha}}{ds^{\alpha}}G_{\epsilon}-(\Delta-v)G_{\epsilon}$.
Take the fractional derivative to the spatial variable, we have

$$\begin{aligned} D^{\beta}\left(x_{1\varepsilon}(s,y)-x_{2\varepsilon}(s,y)\right) &= \int_{\mathbb{R}^n} E(s^{\alpha},y-e)D^{\beta}N_{0,\varepsilon}(e)de \\ &+ \int_0^t\int_{\mathbb{R}^n} E((s-\tau)^{\alpha},y-e)D^{\beta}v_{\varepsilon}(e)\left(x_{1\varepsilon}(\tau,e)-x_{2\varepsilon}(\tau,e)\right)ded\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} E((s-\tau)^{\alpha},y-e)v_{\varepsilon}(e)D^{\beta}\left(x_{1\varepsilon}(\tau,e)-x_{2\varepsilon}(\tau,e)\right)ded\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} E((s-\tau)^{\alpha},y-e)D^{\beta}N_{\varepsilon}(\tau,e)ded\tau, \end{aligned}$$

we get

$$\begin{aligned} \left\|D^{\beta}\left(x_{1\varepsilon}(s,y)-x_{2\varepsilon}(s,y)\right)\right\|_{L^{\infty}(\mathbb{R}^n)} &\leq \|E(s^{\alpha},y-.)\|_{L^1}\left\|D^{\beta}N_{0,\varepsilon}(.)\right\|_{L^{\infty}(\mathbb{R}^n)} \\ &+ \|E((s-\tau)^{\alpha},y-.)\|_{L^1} \\ &\times \int_0^t \|v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\left\|D^{\beta}\left(x_{1\varepsilon}(s,y)-x_{2\varepsilon}(s,y)\right)\right\|_{L^{\infty}(\mathbb{R}^n)}d\tau \\ &+ \|E((s-\tau)^{\alpha},y-e)\|_{L^1} \\ &\times \int_0^t \|v_{\varepsilon}(.)\|\, L^{\infty}\left(\mathbb{R}^n\right)\|D^{\beta}\left(x_{1\varepsilon}(s,y)-x_{2\varepsilon}(s,y)\right\|_{L^{\infty}(\mathbb{R}^n)}d\tau \\ &+ \|E((s-\tau)^{\alpha},y-e)\|_{L^1}\int_0^t\left\|D^{\beta}N_{\varepsilon}(\tau,.)\right\|_{L^{\infty}(\mathbb{R}^n)}d\tau. \end{aligned}$$

Then Gronwall's inequality imply that

$$\begin{aligned}\left\|D^{\beta}\left(u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\right)\right\|_{L^{\infty}(\mathbb{R}^n)} \leq C\Big(&\|N_{0,\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)} \\ &+c_1\left\|D^{\beta}v_{\varepsilon}(.)\right\|_{L^{\infty}(\mathbb{R}^n)}\|x_{1\varepsilon}(s,y)-x_{2\varepsilon}(s,y)\|_{L^{\infty}}\,d\tau \\ &+\left\|D^{\beta}N_{\varepsilon}\right\|_{L^{\infty})}\exp\Big(CT\|v_{\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)}\Big).\end{aligned}$$

Finally,

$$\left\|D^{\beta}\left(u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\right)\right\|_{L^{\infty}(\mathbb{R}^n)} \leq C\varepsilon^{q}, \forall q\in\mathbb{N}.$$

Thus

$$G_{\epsilon}\in\mathcal{N}H^2_{\alpha}.$$

□

Declaration of Competing Interests

The author declares no conflict of interest.

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