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APPROXIMATIONS TO CAPUTO FRACTIONAL DERIVATIVE WITH ARBITRARY KERNELS AND UNIFORM MESHES

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Abstract. The main objective of this paper is to find numerical approximations of the Caputo fractional derivative for $\alpha > 0$ with arbitrary kernels and uniform meshes. These numerical approximations are based on polynomial interpolation. Firstly, we derive three numerical formulas: the fractional rectangular formula (FRF), fractional trapezoidal formula (FTF) and fractional Simpson's formula (FSF). In addition, error estimations for all these rules are analyzed. A test example from the literature is considered to validate the effectiveness of the presented formulas. It is observed that FRF, FTF and FSF yield convergence orders of approximately $O(h)$, $O(h^2)$ and $O(h^3)$, respectively.

Keywords: ψ-Caputo fractional derivative, Approximation, Rectangle formula, Trapezoidal formula, Simpson's formula, Error estimate.

AMS Subject Classification: 26A33, 34A08, 65L05.

1. Introduction

In recent years, increasing attention has been devoted to the subject of fractional calculus (FC). For fundamental concepts in fractional calculus, we refer the reader to the books [15, 21, 26, 32]. The origins of fractional calculus date back to the beginnings of classical differential calculus, as it deals with the generalization of integrals and derivatives to non-integer or even complex orders [21, 25, 28]. For more on its applications across various branches of applied science and engineering, see [10, 13, 20, 22, 27, 30, 31].
Several definitions of fractional calculus have emerged, including those Riemann-Liouville, Caputo, Hilfer, Riesz, Erdelyi-Kober and Hadamard, among others. In particular, R. Almeida [15] proposed generalizations of fractional operators involving arbitrary kernels and a weight function ψ, thus building upon the work of other researchers [14, 15, 19]. Additional properties of this generalized fractional derivative can be found in [1, 2, 3, 4, 5, 6, 7, 8].
From the definition of the Caputo derivative, the αth-order (where $m-1 < \alpha < m$) Caputo derivative of a function $f(t)$ can be viewed as the $(m-\alpha)$th-order fractional integral

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of the function $f^{(m)}(t)$. Numerical integration plays a fundamental role in computing approximate values of definite integrals, particularly when analytical evaluation is difficult. Similarly, numerical integrations for fractional integrals is crucial for developing algorithms to solve applied problems formulated using fractional derivatives and integrals.

Recently, numerical of the fractional integrals and the fractional derivatives have attracted many researchers. Therefore, we can use the numerical methods developed by these researchers, see the following references [9, 16, 17, 18, 23, 24], to simulate the numerical solutions of the Caputo derivative with arbitrary kernels by relying on uniform meshes. For instance, Li and Zeng (2015) presented detailed numerical methods for fractional integrals and derivatives in Chapter 2 of their book [18]. They introduced the rectangle, trapezoidal and Simpson's methods based on polynomial interpolation, and discussed their application to approximation the classical Caputo derivative. Moreover, they analyzed the convergence orders for the case when $\alpha > 0$, offering valuable tools for future reseach.

In 2021, Green and Yan, carried out a detailed error analysis for a fractional Adams method on Caputo–Hadamard fractional differential equations, presented an analysis based on an approximation of the integral. They applied the product trapezoidal quadrature rule using a uniform mesh.

In 2022, Songsanga and Ngiamsunthorn, in their article [29], single-step and multi-step methods for Caputo fractional-order differential equations with arbitrary kernels, they developed four numerical schemes to solve fractional differential equations involving the Caputo fractional derivative with arbitrary (generalized) kernels, where these schemes were expressed in terms of the function ψ. The four schemes are: the explicit product integration rectangular rule (forward Euler method), the implicit product integration rectangular rule (backward Euler method), the implicit product integration trapezoidal rule, and the Adams-type predictor–corrector method.

In 2023, Fan et al. [11] proposed several representative numerical discretization formulas for the Caputo-type fractional derivative with an exponential kernel. These formulas were constructed for orders $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$, including the L1, L1-2, and L2-1 schemes for $\alpha \in (0, 1)$, and the H2N2 and L2 schemes for $\alpha \in (1, 2)$, respectively.

It is worth mentioning some limitations of the present work. First, the proposed numerical methods are constructed under the assumption of uniform meshes and smooth functions, which may restrict their applicability to problems with singularities or non-uniform behavior. Second, although arbitrary kernels are considered, the study is limited to the Caputo fractional derivative, and further generalization to other types of fractional derivatives remains an open question. Moreover, the numerical tests are conducted on benchmark examples with known exact solutions, and future work will aim at extending the applicability to more realistic problems arising in science and engineering.

In this paper, we investigate the numerical formulas (FRF, FTF and FSF) for the ψ-Caputo fractional derivative with uniform meshes. In particular, the efficiency of numerical formulas, error is considered.

The outline of this paper is organized as follows. In Sect. 2, several preliminary knowledge of fractional derivatives and integrals are presented. Sect. 3, the detailed construction of the approximations to ψ-Caputo Derivatives and improved algorithm for this three formulas is presented. These numerical approximations based on polynomial interpolation with uniform meshes and $\alpha > 0$. In Sect. 4, the truncation error analysis of the proposed formulas is derived through a series of Theorems. In Sect. 5, an example is presented to illustrate the performance of our numerical rules. Finally, some conclusions are given at the end of this paper. In this section, we will examine basic definitions and theorems, which will be used to declare and verify our essential results. Let ψ be a continuously differentiable function on $[a,T]$ such that $\psi'(t)>0$, for all $t\in[a,T]$.

2. Preliminaries

In this section, we have the following definitions

Definition 2.1 ([3, 15])**.** *For $\alpha>0$, the left-sided ψ–Riemann–Liouville fractional integral of order α for an integrable function $x:[a,T]\longrightarrow\mathbb{R}$ with respect to another function $\psi:[a,T]\longrightarrow\mathbb{R}$ that is an increasing differentiable function such that $\psi'(t)\neq 0$, for all $t\in[a,T]$ is defined as follows*

$$I_{a^+}^{\alpha,\psi}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}x(s)ds,$$

where Γ is the gamma function.

Definition 2.2 ([3])**.** *Let $n\in\mathbb{N}$ and let $\psi,x\in C^n([a,T],\mathbb{R})$ be two functions such that y is increasing and $\psi'(t)\neq 0$, for all $t\in[a,b]$. The left-sided ψ-Riemann–Liouville fractional derivative of a function x of order α is defined by*

$$\begin{aligned}D_{a^+}^{\alpha,\psi}x(t)&=\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^nI_{a^+}^{n-\alpha,\psi}x(t)\\&=\frac{1}{\Gamma(n-\alpha)}\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^n\int_a^t\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}x(s)ds,\end{aligned}$$

where $n=[\alpha]+1$.

Definition 2.3 ([3])**.** *Let $n\in\mathbb{N}$ and let $\psi,x\in C^n([a,T],\mathbb{R})$ be two functions such that y is increasing and $\psi'(t)\neq 0$, for all $t\in[a,T]$. The left-sided ψ-Caputo fractional derivative of x of order α is defined by*

$${}^cD_{a^+}^{\alpha,\psi}x(t)=I_{a^+}^{n-\alpha,\psi}\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^nx(t),$$

where $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$, $n=\alpha$ for $\alpha\in\mathbb{N}$.
To simplify notations, we will use the abbreviated symbol

$$x_\psi^{[n]}(t)=\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^nx(t).$$

From the definition 3, it is clear that

$${}^cD_{a^+}^{\alpha,\psi}x(t)=\begin{cases}\displaystyle\int_a^t\frac{\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}x_\psi^{[n]}(s)ds, & \text{if } \alpha\notin\mathbb{N},\\[2ex] x_\psi^{[n]}, & \text{if } \alpha\in\mathbb{N}.\end{cases}$$

Also, the ψ-Caputo fractional derivative of order α of x is defined as

$${}^cD_{a^+}^{\alpha,\psi}x(t)={}^cD_{a^+}^{\alpha,\psi}\left[x(t)-\sum_{k=0}^{n-1}\frac{x_\psi^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k\right].$$

For more details see ([3], Theorem 3).

Remark 2.1 ([3])**.** *In particular, the ψ-Caputo fractional derivative is a generalization of the fractional derivatives. That is, the classical Caputo fractional derivative for $\psi(t)=t$, the Caputo-Hadamard fractional derivative for $\psi(t)=\ln(t)$ and the Caputo-Erdélyi-Kober fractional derivative for $\psi(t)=t^\sigma$.*

3. Approximations to ψ-Caputo Derivatives

In this section, we will present similar numerical approaches used in the section previous to approach fractional derivatives of order $\alpha > 0$ in the sense of ψ-Caputo. Indeed, for $f \in C([a,T])$, let N be a positive integer and let $a = t_0 < t_1 < t_2 < \ldots < t_N = T$ be a partition on $[a,T]$. We define the following uniform mesh on $[\psi(a), \psi(T)]$ by

$$\psi(a) = \psi(t_0) < \psi(t_1) < \psi(t_2) < \ldots < \psi(t_N) = \psi(T),$$

such that

$$\frac{\psi(t_j) - \psi(a)}{\psi(t_N) - \psi(a)} = \frac{j}{N},$$

which implies that

$$\psi(t_j) = \psi(a) + (\psi(t_N) - \psi(a))(\frac{j}{N}),$$

when $j = N$ and $j = 0$ we have $\psi(t_N) = \psi(T)$ and $\psi(a) = \psi(t_0)$. Further, we have

$$\begin{aligned}\psi(t_j) &= \psi(t_0) + (\psi(T) - \psi(t_0))(\frac{j}{N}) \\ &= \psi(t_0) + hj \text{ with } h = \frac{\psi(T) - \psi(t_0)}{N}.\end{aligned} \tag{1}$$

Next, we study numerically the values of the following integral with $m = \lceil \alpha \rceil + 1$

$${}^C D_{t_0}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{m-\alpha-1} f_\psi^{[m]}(s) ds$$

We put $t = t_n$, so

$$[{}^C D_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} = \frac{1}{\Gamma(m-\alpha)} \int_0^{t_n} \psi'(s)(\psi(t_n) - \psi(s))^{m-\alpha-1} f_\psi^{[m]}(s) ds$$

and

$$[{}^C D_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} = \frac{1}{\Gamma(m-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n) - \psi(s))^{m-\alpha-1} f_\psi^{[m]}(s) ds. \tag{2}$$

3.1. **Fractional Rectangular Formula (FRF).** For each subinterval $[t_k, t_{k+1}]$, $k = 0, 1, \ldots, n-1$, the function $f_\psi^{[m]}(t)$ is approximated by a constant, i.e.,

$$f_\psi^{[m]}(t)|_{[t_k,t_{k+1}]} \approx f_\psi^{[m]}(t_k) = \Big(\frac{1}{\psi'(t)} \frac{d}{dt}\Big)^m f(t_k), \; t \in [t_k, t_{k+1}]. \tag{3}$$

In this case Equation (2) can be written as
$[{}^{C}D_{t_0}^{\alpha,\psi}f(t)]_{t=t_n}$

$$\begin{aligned}
&\approx \frac{1}{\Gamma(m-\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}f_{\psi}^{[m]}(t_k)ds\\
&=\frac{1}{\Gamma(m-\alpha)}\sum_{k=0}^{n-1}f_{\psi}^{[m]}(t_k)\int_{t_k}^{t_{k+1}}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}ds\\
&=\frac{1}{\Gamma(m-\alpha)}\sum_{k=0}^{n-1}f_{\psi}^{[m]}(t_k)\left[\frac{(\psi(t_n)-\psi(s))^{m-\alpha}}{m-\alpha}\right]_{t_k}^{t_{k+1}}\\
&=\frac{1}{\Gamma(m-\alpha+1)}\sum_{k=0}^{n-1}f_{\psi}^{[m]}(t_k)\left[(\psi(t_n)-\psi(t_k))^{m-\alpha}-(\psi(t_n)-\psi(t_{k+1}))^{m-\alpha}\right]\\
&=\frac{1}{\Gamma(m-\alpha+1)}\sum_{k=0}^{n-1}f_{\psi}^{[m]}(t_k)\left[(nh-kh)^{m-\alpha}-(nh-(k+1)h)^{m-\alpha}\right]\\
&=\frac{h^{m-\alpha}}{\Gamma(m-\alpha+1)}\sum_{k=0}^{n-1}f_{\psi}^{[m]}(t_k)\left[(n-k)^{m-\alpha}-(n-k-1)^{m-\alpha}\right]\\
&=\sum_{k=0}^{n-1}v_{n-k-1}f_{\psi}^{[m]}(t_k)=[{}^{C}\mathscr{D}_{t_0}^{\alpha,\psi}f(t)]_{t=t_n}, \qquad (4)
\end{aligned}$$

where the coefficients are given by

$$v_k=\frac{h^{m-\alpha}}{\Gamma(m-\alpha+1)}\big((k+1)^{m-\alpha}-k^{m-\alpha}\big).$$

So, (4) is called the **the left fractional rectangular formula (LFRF)**.
Similarly, when $f(x)$ is approximated by the following piecewise constant function

$$f_{\psi}^{[m]}(t)|_{[t_k,t_{k+1}]}\approx f_{\psi}^{[m]}(t_{k+1})=\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^m f(t_{k+1}),\ t\in[t_k,t_{k+1}], \qquad (5)$$

then **the right fractional rectangular formula (RFRF)** in given by

$$[{}^{C}D_{t_0}^{\alpha,\psi}f(t)]_{t=t_n}=\sum_{k=0}^{n-1}V_{n-k-1}f_{\psi}^{[m]}(t_{k+1})=[{}^{C}\mathscr{D}_{t_0}^{\alpha,\psi}f(t)]_{t=t_n}. \qquad (6)$$

3.2. **Fractional Trapezoidal Formula (FTF).** For each subinterval $[t_k,t_{k+1}]$, $f_{\psi}^{[m]}(t)$ is approximated by the following piecewise polynomial with degree of order one

$$\begin{aligned}
f_{\psi}^{[m]}(t)|_{[t_k,t_{k+1}]}&\approx\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^m f_{\psi}(t)|_{[t_k,t_{k+1}]}\\
&=\frac{\psi(t_{k+1})-\psi(t)}{\psi(t_{k+1})-\psi(t_k)}f_{\psi}^{[m]}(t_k)+\frac{\psi(t)-\psi(t_k)}{\psi(t_{k+1})-\psi(t_k)}f_{\psi}^{[m]}(t_{k+1}),. \qquad (7)
\end{aligned}$$

Replacing (7) in (2), we have
$[^C D^{\alpha,\psi}_{t_0} f(t)]_{t=t_n}$

$$
\begin{aligned}
&\approx \frac{1}{\Gamma(m-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \Big[\frac{\psi(t_{k+1})-\psi(s)}{\psi(t_{k+1})-\psi(t_k)} f^{[m]}_{\psi}(t_k) \\
&+ \frac{\psi(s)-\psi(t_k)}{\psi(t_{k+1})-\psi(t_k)} f^{[m]}_{\psi}(t_{k+1}) \Big] ds \\
&= \frac{1}{\Gamma(m-\alpha)} \Bigg[\int_{t_0}^{t_1} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \frac{\psi(t_1)-\psi(s)}{\psi(t_1)-\psi(t_0)} f^{[m]}_{\psi}(t_0) ds \\
&+ \int_{t_0}^{t_1} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \frac{\psi(s)-\psi(t_0)}{\psi(t_1)-\psi(t_0)} f^{[m]}_{\psi}(t_1) ds \\
&+ \int_{t_1}^{t_2} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \frac{\psi(t_2)-\psi(s)}{\psi(t_2)-\psi(t_1)} f^{[m]}_{\psi}(t_1) ds \\
&+ \int_{t_1}^{t_2} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \frac{\psi(s)-\psi(t_1)}{\psi(t_2)-\psi(t_1)} f^{[m]}_{\psi}(t_2) ds \\
&+ \\
&\vdots \\
&+ \int_{t_{n-1}}^{t_n} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \frac{\psi(t_n)-\psi(s)}{\psi(t_n)-\psi(t_{n-1})} f^{[m]}_{\psi}(t_{n-1}) ds \\
&+ \int_{t_{n-1}}^{t_n} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \frac{\psi(s)-\psi(t_{n-1})}{\psi(t_n)-\psi(t_{n-1})} f^{[m]}_{\psi}(t_n) ds \Bigg] \\
&= \sum_{k=0}^{n} W_{k,n} f^{[m]}_{\psi}(t_k) = [^C \mathscr{D}^{\alpha,\psi}_{t_0} f(t)]_{t=t_n},
\end{aligned} \tag{8}
$$

where
$W_{k,n}$

$$
= \frac{1}{h\Gamma(m-\alpha)} \begin{cases} \int_{t_0}^{t_1} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(t_1)-\psi(s)) ds, & k=0, \\ \int_{t_{k-1}}^{t_k} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{k-1})) ds \\ + \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(t_{k+1})-\psi(s)) ds, & 1 \le k \le n-1, \\ \int_{t_{n-1}}^{t_n} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{n-1})) ds, & k=n. \end{cases}
$$

For $k=0$, we have

$$
W_{0,n} = \frac{1}{h\Gamma(m-\alpha)} \int_{t_0}^{t_1} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(t_1)-\psi(s)) ds, \tag{9}
$$

by virtue of integration by parts, we put

$$
u = \psi(t_1)-\psi(s) \implies u' = -\psi'(s)
$$

$$
v' = \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \implies v = \frac{-1}{m-\alpha}(\psi(t_n)-\psi(s))^{m-\alpha}
$$

one obtains

$$
\begin{aligned}
W_{0,n} &= \frac{1}{h\Gamma(m-\alpha)}\Bigg\{\Big[\frac{-1}{m-\alpha}(\psi(t_1)-\psi(s))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_0}^{t_1} \\
&- \frac{1}{m-\alpha}\int_{t_0}^{t_1}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha}ds\Bigg\} \\
&= \frac{1}{h\Gamma(m-\alpha+1)}\Bigg\{n^{m-\alpha}h^{m-\alpha+1} + \Big[\frac{1}{m-\alpha+1}(\psi(t_n)-\psi(s))^{m-\alpha+1}\Big]_{t_0}^{t_1}\Bigg\} \\
&= \frac{h^{m-\alpha}}{\Gamma(m-\alpha+2)}\Bigg[(n-1)^{m-\alpha+1} - n^{m-\alpha}(n-m+\alpha-1).\Bigg] \qquad (10)
\end{aligned}
$$

For $1 \leq k \leq n-1$, we also have

$$
\begin{aligned}
W_{k,n} &= \frac{1}{\Gamma(m-\alpha)}\Bigg\{\int_{t_{k-1}}^{t_k}\frac{\psi(s)-\psi(t_{k-1})}{\psi(t_k)-\psi(t_{k-1})}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}ds \\
&+ \int_{t_k}^{t_{k+1}}\frac{\psi(t_{k+1})-\psi(s)}{\psi(t_{k+1})-\psi(t_k)}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}ds\Bigg\} \\
&= \frac{1}{h\Gamma(m-\alpha)}\Bigg\{\underbrace{\int_{t_{k-1}}^{t_k}\psi'(s)(\psi(s)-\psi(t_{k-1}))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds}_{I_1} \\
&+ \underbrace{\int_{t_k}^{t_{k+1}}\psi'(s)(\psi(t_{k+1})-\psi(s))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds}_{I_2}\Bigg\}. \qquad (11)
\end{aligned}
$$

Now, we calculate the above integrals separately, we use integration by parts again.

$$
I_1 = \int_{t_{k-1}}^{t_k}\psi'(s)(\psi(s)-\psi(t_{k-1}))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds.
$$

We put $u = \psi(s) - \psi(t_{k-1})$, so $u' = \psi'(s)$ and $v' = \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}$ implies $v = \frac{-1}{m-\alpha}(\psi(t_n)-\psi(s))^{m-\alpha}$. We get

$$
\begin{aligned}
I_1 &= \Big[\frac{-1}{m-\alpha}(\psi(s)-\psi(t_{k-1})))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_{k-1}}^{t_k} \\
&+ \frac{1}{m-\alpha}\int_{t_{k-1}}^{t_k}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha}ds \\
&= \frac{1}{m-\alpha)}\Bigg\{-(n-k)^{m-\alpha}h^{m-\alpha+1} - \Big[\frac{1}{m-\alpha+1}(\psi(t_n)-\psi(s))^{m-\alpha+1}\Big]_{t_{k-1}}^{t_k}\Bigg\} \\
&= \frac{h^{m-\alpha+1}}{(m-\alpha)(m-\alpha+1)}\Bigg[-(m-\alpha+1)(n-k)^{m-\alpha} - (n-k)^{m-\alpha+1} \\
&+ (n-k+1)^{m-\alpha+1}\Bigg]. \qquad (12)
\end{aligned}
$$

In a similar manner, we calculate

$$I_2 = \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_{k+1}) - \psi(s))(\psi(t_n) - \psi(s))^{m-\alpha-1} ds.$$

Putting $u = \psi(t_{k+1}) - \psi(s)$, so $u' = -\psi'(s)$ and $v' = \psi'(s)(\psi(t_n) - \psi(s))^{m-\alpha-1}$ implies $v = \frac{-1}{m-\alpha}(\psi(t_n) - \psi(s))^{m-\alpha}$. We get

$$\begin{aligned}
I_2 &= \Big[\frac{-1}{m-\alpha}(\psi(t_{k+1} - \psi(s)))(\psi(t_n) - \psi(s))^{m-\alpha}\Big]_{t_k}^{t_{k+1}} \\
&- \frac{1}{m-\alpha}\int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n) - \psi(s))^{m-\alpha} ds \\
&= \frac{1}{m-\alpha)}\left\{(n-k)^{m-\alpha}h^{m-\alpha+1} + \Big[\frac{1}{m-\alpha+1}(\psi(t_n) - \psi(s))^{m-\alpha+1}\Big]_{t_k}^{t_{k+1}}\right\} \\
&= \frac{h^{m-\alpha+1}}{(m-\alpha)(m-\alpha+1)}\left[(m-\alpha+1)(n-k)^{m-\alpha} + (n-k-1)^{m-\alpha+1},\right]. \qquad (13)
\end{aligned}$$

Combining Eqs. (12), (13), we find that

$$\begin{aligned}
W_{k,n} &= \frac{1}{h(m-\alpha)}[I_1 + I_2] \\
&= \frac{h^{m-\alpha}}{(m-\alpha+2)}\left[(n-k+1)^{m-\alpha+1} + (n-k-1)^{m-\alpha+1} - 2(n-k)^{m-\alpha+1}\right]. \qquad (14)
\end{aligned}$$

For $k = n$, we have

$$W_{n,n} = \frac{1}{h\Gamma(m-\alpha)}\int_{t_{n-1}}^{t_n} \psi'(s)(\psi(t_n) - \psi(s))^{m-\alpha-1}(\psi(s) - \psi(t_{n-1}))ds,$$

using integration by parts, let us put $u = \psi(s) - \psi(t_{n-1})$, so $u' = \psi'(s)$ and $v' = \psi'(s)(\psi(t_n) - \psi(s))^{m-\alpha-1}$ implies $v = \frac{-1}{m-\alpha}(\psi(t_n) - \psi(s))^{m-\alpha}$. Thus, we have

$$\begin{aligned}
W_{n,n} &= \frac{1}{h\Gamma(m-\alpha)}\left\{\Big[\frac{-1}{m-\alpha}(\psi(s) - \psi(t_{n-1}))(\psi(t_n) - \psi(s))^{m-\alpha}\Big]_{t_{n-1}}^{t_n}\right. \\
&\left. + \frac{1}{m-\alpha}\int_{t_{n-1}}^{t_n} \psi'(s)(\psi(t_n) - \psi(s))^{m-\alpha} ds\right\} \\
&= \frac{1}{h\Gamma(m-\alpha+1)}\left[\frac{-1}{(m-\alpha+1)}(\psi(t_n) - \psi(s))^{m-\alpha+1}\right]_{t_{n-1}}^{t_n} \\
&= \frac{h^{m-\alpha}}{\Gamma(m-\alpha+2)}. \qquad (15)
\end{aligned}$$

Hence
$W_{k,n}$

$$= \frac{h^{m-\alpha}}{\Gamma(m-\alpha+2)}\begin{cases} (n-1)^{m-\alpha+1} - n^{m-\alpha}(n-m+\alpha-1), & k = 0, \\ (n-k+1)^{m-\alpha+1}(n-k-1)^{m-\alpha+1} - 2(n-k)^{m-\alpha+1}, & 1 \le k \le n-1, \\ 1, & k = n. \end{cases}$$

3.3. **Fractional Simpson's Formula (FSF).** In this method, we will approximate $f_{\psi}^{[m]}(t)$ by a piecewise quadratic polynomial on each sub interval $[t_k, t_{k+1}]$, denoted by $t_{k+\frac{1}{2}} = \frac{t_k+t_{k+1}}{2}$.

$$\begin{aligned} f_{\psi}^{[m]}(t)|_{[t_k,t_{k+1}]} &\approx \Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^m f_{\psi}(t)|_{[t_k,t_{k+1}]} \\ &= \frac{(\psi(s)-\psi(t_{k+\frac{1}{2}}))(\psi(s)-\psi(t_{k+1}))}{(\psi(t_k)-\psi(t_{k+\frac{1}{2}}))(\psi(t_k)-\psi(t_{k+1}))} f_{\psi}^{[m]}(t_k) \\ &+ \frac{(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+1}))}{(\psi(t_{k+\frac{1}{2}})-\psi(t_k))(\psi(t_{k+\frac{1}{2}})-\psi(t_{k+1}))} f_{\psi}^{[m]}(t_{k+\frac{1}{2}}) \\ &+ \frac{(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+\frac{1}{2}}))}{(\psi(t_{k+1})-\psi(t_k))(\psi(t_{k+1})-\psi(t_{k+\frac{1}{2}}))} f_{\psi}^{[m]}(t_{k+1}). \end{aligned} \tag{16}$$

Replacing (16) in (2), we obtain
$[{}^C D_{t_0}^{\alpha,\psi} f(t)]_{t=t_n}$

$$\begin{aligned} &\approx \frac{1}{\Gamma(m-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \\ &\times \Big[\frac{(\psi(s)-\psi(t_{k+\frac{1}{2}}))(\psi(s)-\psi(t_{k+1}))}{(\psi(t_k)-\psi(t_{k+\frac{1}{2}}))(\psi(t_k)-\psi(t_{k+1}))} f_{\psi}^{[m]}(t_k) \\ &+ \frac{(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+1}))}{(\psi(t_{k+\frac{1}{2}})-\psi(t_k))(\psi(t_{k+\frac{1}{2}})-\psi(t_{k+1}))} f_{\psi}^{[m]}(t_{k+\frac{1}{2}}) \\ &+ \frac{(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+\frac{1}{2}}))}{(\psi(t_{k+1})-\psi(t_k))(\psi(t_{k+1})-\psi(t_{k+\frac{1}{2}}))} f_{\psi}^{[m]}(t_{k+1})\Big] ds \\ &= \frac{2}{h^2\Gamma(m-\alpha)} \Bigg\{ \int_{t_0}^{t_1} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \Big[(\psi(s)-\psi(t_{\frac{1}{2}}))(\psi(s)-\psi(t_1)) f_{\psi}^{[m]}(t_0) \\ &- 2(\psi(s)-\psi(t_0))(\psi(s)-\psi(t_1)) f_{\psi}^{[m]}(t_{\frac{1}{2}}) + (\psi(s)-\psi(t_0))(\psi(s)-\psi(t_{\frac{1}{2}})) f_{\psi}^{[m]}(t_1)\Big] ds \\ &+ \int_{t_1}^{t_2} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \Big[(\psi(s)-\psi(t_{\frac{3}{2}}))(\psi(s)-\psi(t_2)) f_{\psi}^{[m]}(t_1) \\ &- 2(\psi(s)-\psi(t_1))(\psi(s)-\psi(t_2)) f_{\psi}^{[m]}(t_{\frac{3}{2}}) + (\psi(s)-\psi(t_1))(\psi(s)-\psi(t_{\frac{3}{2}})) f_{\psi}^{[m]}(t_2)\Big] ds \\ &+ \\ &\vdots \\ &+ \int_{t_{n-1}}^{t_n} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1} \Big[(\psi(s)-\psi(t_{n-\frac{1}{2}}))(\psi(s)-\psi(t_n)) f_{\psi}^{[m]}(t_{n-1}) \\ &- 2(\psi(s)-\psi(t_{n-1}))(\psi(s)-\psi(t_n)) f_{\psi}^{[m]}(t_{n-\frac{1}{2}}) \\ &+ (\psi(s)-\psi(t_{n-\frac{1}{2}}))(\psi(s)-\psi(t_{n-1})) f_{\psi}^{[m]}(t_n)\Big] ds \Bigg\} \end{aligned}$$

$$= \sum_{k=0}^{n} e_{k,n} f_{\psi}^{[m]}(t_k) + \sum_{k=0}^{n-1} \hat{e}_{k,n} f_{\psi}^{[m]}(t_{k+\frac{1}{2}}) = [{}^{C}\mathscr{D}_{t_0}^{\alpha,\psi} f(t)]_{t=t_n}, \tag{17}$$

where
$e_{k,n}$

$$= \frac{2}{h^2\Gamma(m-\alpha)} \begin{cases} \int_{t_0}^{t_1} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{\frac{1}{2}}))(\psi(s)-\psi(t_1))ds, \\ k=0, \\ \\ \int_{t_{k-1}}^{t_k} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{k-1}))(\psi(s)-\psi(t_{k-\frac{1}{2}}))ds \\ +\int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{k+\frac{1}{2}}))(\psi(s)-\psi(t_{k+1}))ds, \\ 1 \le k \le n-1, \\ \\ \int_{t_{n-1}}^{t_n} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{n-1}))(\psi(s)-\psi(t_{n-\frac{1}{2}}))ds, \\ k=n, \end{cases}$$

and

$$\hat{e}_{k,n} = \frac{-4}{h^2\Gamma(m-\alpha)} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+1}))ds,$$
$$0 \le k \le n-1.$$

For $k=0$, we have

$$e_{0,n} = \frac{2}{h^2\Gamma(m-\alpha)} \int_{t_0}^{t_1} \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{\frac{1}{2}}))(\psi(s)-\psi(t_1))ds.$$

Using integration by parts, let we put $u=(\psi(s)-\psi(t_{\frac{1}{2}}))(\psi(s)-\psi(t_1))$, so $u' = 2\psi'(s)\psi(s) - \psi(t_{\frac{1}{2}})\psi'(s) - \psi(t_1)\psi'(s)$ and $v' = \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}$ implies $v = \frac{-1}{m-\alpha}(\psi(t_n) - \psi(s))^{m-\alpha}$. We get

$$\begin{aligned} e_{0,n} &= \frac{2}{h^2\Gamma(m-\alpha+1)} \Bigg\{ \Big[-(\psi(s)-\psi(t_{\frac{1}{2}}))(\psi(s)-\psi(t_1))(\psi(t_n)-\psi(s))^{m-\alpha} \Big]_{t_0}^{t_1} \\ &+ \int_{t_0}^{t_1} 2\psi'(s)(\psi(s)-\psi(t_{\frac{1}{2}})-\psi(t_1))(\psi(t_n)-\psi(s))^{m-\alpha} ds \Bigg\} \\ &= \frac{n^{m-\alpha}h^{m-\alpha}}{\Gamma(m-\alpha+1)} - \frac{2}{h^2\Gamma(m-\alpha+2)} \Bigg\{ \Big[(2\psi(s)-\psi(t_{\frac{1}{2}})-\psi(t_1)) \\ &\times (\psi(t_n)-\psi(s))^{m-\alpha+1} \Big]_{t_0}^{t_1} + 2\int_{t_0}^{t_1} (\psi(t_n)-\psi(s))^{m-\alpha+1} ds \Bigg\} \\ &= \frac{h^{m-\alpha}}{\Gamma(m-\alpha+3)} \Bigg\{ 4[n^{m-\alpha+2}-(n-1)^{m-\alpha+2}] - (m-\alpha+2)[3n^{m-\alpha+1} \\ &+ (n-1)^{m-\alpha+1}] + (m-\alpha+1)(m-\alpha+2)n^{m-\alpha} \Bigg\}. \end{aligned} \tag{18}$$

For $1 \leq k \leq n-1$, we see that

$$e_{k,n} = \frac{2}{h^2\Gamma(m-\alpha)}\Bigg\{ \underbrace{\int_{t_{k-1}}^{t_k} \psi'(s)(\psi(s)-\psi(t_{k-1}))(\psi(s)-\psi(t_{k-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds}_{I_3}$$
$$+ \underbrace{\int_{t_k}^{t_{k+1}} \psi'(s)(\psi(s)-\psi(t_{k+\frac{1}{2}}))(\psi(s)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds}_{I_4}\Bigg\}.$$

We calculate the above integrals separately, we use integration by parts again

$$I_3 = \int_{t_{k-1}}^{t_k} \psi'(s)(\psi(s)-\psi(t_{k-1}))(\psi(s)-\psi(t_{k-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds,$$

we put $u = (\psi(s)-\psi(t_{k-1}))(\psi(s)-\psi(t_{k-\frac{1}{2}}))$, so $u' = 2\psi'(s)\psi(s) - \psi(t_{k-1})\psi'(s) - \psi(t_{k-\frac{1}{2}})\psi'(s)$ and $v' = \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}$ implies $v = \frac{-1}{m-\alpha}(\psi(t_n)-\psi(s))^{m-\alpha}$. We observe that

$$\begin{aligned}
I_3 &= \frac{1}{m-\alpha}\Bigg\{\Big[-(\psi(s)-\psi(t_{k-1}))(\psi(s)-\psi(t_{k-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_{k-1}}^{t_k} \\
&\quad + \int_{t_{k-1}}^{t_k} \psi'(s)(2\psi(s)-\psi(t_{k-1})-\psi(t_{k-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha}ds\Bigg\} \\
&= \frac{1}{(m-\alpha)(m-\alpha+1)}\Bigg\{\Big[(2\psi(s)-\psi(t_{k-1})-\psi(t_{k-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha+1}\Big]_{t_{k-1}}^{t_k} \\
&\quad + 2\int_{t_{k-1}}^{t_k} (\psi(t_n)-\psi(s))^{m-\alpha}ds\Bigg\} - \frac{h^{m-\alpha+2}(n-k)^{m-\alpha}}{2(m-\alpha)} \\
&= \frac{h^{m-\alpha+2}}{2(m-\alpha)(m-\alpha+1)(\alpha+2)}\Big[-(m-\alpha+1)(m-\alpha+2)(n-k)^{m-\alpha} \\
&\quad - 3(\alpha+2)(n-k)^{m-\alpha+1} - (m-\alpha+2)(n-k+1)^{m-\alpha+1} - 4(n-k)^{m-\alpha+2} \\
&\quad + 4(n-k+1)^{m-\alpha+2}\Big]. \qquad (19)
\end{aligned}$$

Similarly, we calculate

$$I_4 = \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(s)-\psi(t_{k+\frac{1}{2}}))(\psi(s)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds,$$

we put $u = (\psi(s)-\psi(t_{k+\frac{1}{2}}))(\psi(s)-\psi(t_{k+1}))$, so $u' = 2\psi'(s)\psi(s) - \psi(t_{k+\frac{1}{2}})\psi'(s) - \psi(t_{k+1})\psi'(s)$ and $v' = \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}$ implies $v = \frac{-1}{m-\alpha}(\psi(t_n)-\psi(s))^{m-\alpha}$.

We get

$$
\begin{aligned}
I_4 &= \frac{2}{h^2\Gamma(m-\alpha+1)}\Bigg\{\Big[-(\psi(s)-\psi(t_{k+\frac{1}{2}}))(\psi(s)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_k}^{t_{k+1}}\\
&+\int_{t_k}^{t_{k+1}}\psi'(s)(2\psi(s)-\psi(t_{k+\frac{1}{2}})-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha}ds\Bigg\}\\
&=-\frac{1}{(m-\alpha)(m-\alpha+1)}\Bigg\{\Big[(2\psi(s)-\psi(t_{k+\frac{1}{2}})-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_k}^{t_{k+1}}\\
&+2\int_{t_k}^{t_{k+1}}(\psi(t_n)-\psi(s))^{m-\alpha+1}ds\Bigg\}+\frac{h^{m-\alpha+2}(n-k)^{m-\alpha}}{2(m-\alpha)}\\
&=\frac{h^{m-\alpha+2}}{2(m-\alpha)(m-\alpha+1)(m-\alpha+2)}\Big[(m-\alpha+1)(m-\alpha+2)(n-k)^{m-\alpha}\\
&+4(n-k)^{m-\alpha+2}-(m-\alpha+2)(n-k-1)^{m-\alpha+1}-4(n-k-1)^{m-\alpha+2}\\
&-3(m-\alpha+2)(n-k)^{m-\alpha+1}\Big].
\end{aligned} \tag{20}
$$

Combining Eqs. (19), (20), we find that

$$
\begin{aligned}
e_{k,n} &= I_3+I_4\\
&=\frac{h^{m-\alpha}}{\Gamma(m-\alpha+3)}\Bigg\{-(m-\alpha+2)\Big[(n-k+1)^{m-\alpha+1}+(n-k-1)^{m-\alpha+1}\\
&+6(n-k)^{m-\alpha+1}\Big]+4\Big[(n-k+1)^{m-\alpha+2}-(n-k-1)^{m-\alpha+2}\Big]\Bigg\}.
\end{aligned}
$$

For $k=n$

$$
e_{n,n}=\frac{2}{h^{m-\alpha}\Gamma(m-\alpha)}\int_{t_{n-1}}^{t_n}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}(\psi(s)-\psi(t_{n-1}))(\psi(s)-\psi(t_{n-\frac{1}{2}}))ds,
$$

using integration by parts, let us put $u=(\psi(s)-\psi(t_{n-1}))(\psi(s)-\psi(t_{n-\frac{1}{2}}))$, so $u'=2\psi'(s)\psi(s)-\psi(t_{n-1})\psi'(s)-\psi(t_{n-\frac{1}{2}})\psi'(s)$ and $v'=\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}$ implies $v=\frac{-1}{m-\alpha}(\psi(t_n)-\psi(s))^{m-\alpha}$, we have

$$
\begin{aligned}
e_{n,n} &= \frac{2}{h^{m-\alpha}\Gamma(m-\alpha+1)}\Big[-(\psi(s)-\psi(t_{n-1}))(\psi(s)-\psi(t_{n-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_{n-1}}^{t_n}\\
&+\frac{2}{h^{m-\alpha}\Gamma(m-\alpha+1)}\int_{t_{n-1}}^{t_n}\psi'(s)(2\psi(s)-\psi(t_{n-1})-\psi(t_{n-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha}ds\\
&=\frac{2}{h^{m-\alpha}\Gamma(m-\alpha+2)}\Big[-(2\psi(s)-\psi(t_{n-1})-\psi(t_{n-\frac{1}{2}}))(\psi(t_n)-\psi(s))^{m-\alpha+1}\Big]_{t_{n-1}}^{t_n}\\
&+\frac{4}{h^{m-\alpha}\Gamma(m-\alpha+2)}\int_{t_{n-1}}^{t_n}\psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha+1}ds\\
&=\frac{2}{h^{m-\alpha}\Gamma(m-\alpha+2)}\Bigg\{\frac{-h^{m-\alpha+2}}{2}-\Big[\frac{2}{m-\alpha+2}(\psi(t_n)-\psi(s))^{m-\alpha+2}\Big]_{t_{n-1}}^{t_n}\Bigg\}\\
&=\frac{h^{m-\alpha}}{\Gamma(m-\alpha+3)}(2-m+\alpha).
\end{aligned} \tag{21}
$$

For $0 \leq k \leq n-1$

$$\hat{e}_{k,n} = \frac{-4}{h^2\Gamma(m-\alpha)} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha-1}ds,$$

using integration by parts twice, let us put $u = (\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+1}))$, so $u' = 2\psi'(s)\psi(s)-\psi(t_k)\psi'(s)-\psi(t_{k+1})\psi'(s)$ and $v' = \psi'(s)(\psi(t_n)-\psi(s))^{m-\alpha-1}$ implies $v = \frac{-1}{m-\alpha}(\psi(t_n)-\psi(s))^{m-\alpha}$, we get

$$\begin{aligned}
\hat{e}_{k,n} &= \frac{4}{h^2\Gamma(m-\alpha+1)}\Big[(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_k}^{t_{k+1}} \\
&- \frac{4}{h^2\Gamma(m-\alpha+1)} \int_{t_k}^{t_{k+1}} \psi'(s)(2\psi(s)-\psi(t_k)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha}ds \\
&= \frac{4}{h^2\Gamma(m-\alpha+1)}\Bigg\{\Big[(\psi(s)-\psi(t_k))(\psi(s)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha}\Big]_{t_k}^{t_{k+1}} \\
&- \int_{t_k}^{t_{k+1}} \psi'(s)(2\psi(s)-\psi(t_k)-\psi(t_{k+1}))(\psi(t_n)-\psi(s))^{m-\alpha}ds\Bigg\} \\
&= \frac{4h^{m-\alpha}}{\Gamma(m-\alpha+3)}\Bigg\{(m-\alpha+2)\Big[(n-k)^{m-\alpha+1}+(n-k-1)^{m-\alpha+1}\Big] \\
&- 2\Big[(n-k)^{m-\alpha+2}-(n-k-1)^{m-\alpha+2}\Big]\Bigg\}. \qquad (22)
\end{aligned}$$

Therefore,

$e_{k,n}$

$$= \frac{h^{m-\alpha}}{\Gamma(m-\alpha+3)} \begin{cases} 4\Big[n^{m-\alpha+2}-(n-1)^{m-\alpha+2}\Big]-(m-\alpha+2)\Big[3n^{m-\alpha+1} \\ +(n-1)^{m-\alpha+1}\Big]+(m-\alpha+1)(m-\alpha+2)n^{m-\alpha}, & k=0, \\ \\ -(m-\alpha+2)\Big[(n-k+1)^{m-\alpha+1}+(n-k-1)^{m-\alpha+1} \\ +6(n-k)^{m-\alpha+1}\Big]+4\Big[(n-k+1)^{m-\alpha+2} \\ -(n-k-1)^{m-\alpha+2}\Big], & 1 \leq k \leq n-1, \\ \\ 2-m+\alpha, & k=n. \end{cases}$$

For $0 \leq k \leq n-1$, we have

$$\hat{e}_{k,n} = \frac{4h^{m-\alpha}}{\Gamma(m-\alpha+3)}\Bigg\{(m-\alpha+2)\Big[(n-k)^{m-\alpha+1}+(n-k-1)^{m-\alpha+1}\Big]-2\Big[(n-k)^{m-\alpha+2} - (n-k-1)^{m-\alpha+2}\Big]\Bigg\}. \qquad (23)$$

4. Error analysis

In this section, we present the error analysis for our numerical algorithms. In order to fulfill error analysis, some Theorems are mentioned below.

4.1. **Truncation error analysis for the main algorithm.** We first discuss the errors of rectangle, trapezoidal and Simpson's formula.
Given $f \in C^{r+1}[a,T]$, $r \in \mathbb{N}$. Let $P_r \in \mathbb{P}_r$ (a polynomial of degree r) interpolate the function f in $r+1$ distinct nodes $t_j \in [a,T]$ with $a = t_0$ and $T = t_N$. The function $f(t)$ at $t = t_0$ can be expanded in the following form,

$$\begin{aligned} f(t) &= f_\psi(t_0) + f_\psi^{[1]}(t_0)(\psi(t)-\psi(t_0)) + \frac{1}{2!} f_\psi^{[2]}(t_0)(\psi(t)-\psi(t_0))^2 + \ldots + \\ &+ \frac{1}{r!} f_\psi^{[r]}(t_0)(\psi(t)-\psi(t_0))^r + \frac{1}{(r+1)!} f_\psi^{[r+1]}(\xi)(\psi(t)-\psi(t_0))^{r+1}, \end{aligned} \tag{24}$$

which is just the Taylor expansion in the span $\{1, \psi(t), \psi^2(t), \ldots\}$ with $f_\psi^{[r]}(s) = \Big(\frac{1}{\psi'(s)}\frac{d}{ds}\Big)^r f(s)$, the truncation error in the following form

$$f(t) - P_r(t) = \frac{f_\psi^{[r+1]}(\xi_r)}{(r+1)!} \prod_{j=0}^{r} (\psi(t) - \psi(t_j)). \tag{25}$$

where ξ_r is between t and t_0, see [11].

4.2. **Rectangular Formula.** In this subsection, we prove the following error estimate for the rectangular formula over the given uniform mesh.

Theorem 4.1. *For $0 < \alpha < 1$ and $f_\psi^{[1]} \in C[t_0, T]$, then the following truncation error R_α^{Re}, where $h = \psi(t_{k+1}) - \psi(t_k) = \dfrac{\psi(T) - \psi(t_0)}{N}$ has the estimate below*

$$|R_\alpha^{Re}| \le C_\alpha^{Re} (\psi(T) - \psi(t_0))^\alpha h, \tag{26}$$

where $C_\alpha^{Re} = \frac{M_1}{\Gamma(1-\alpha)}$, $M_1 = \max_{\xi_0 \in [t_0, t_n]} |f_\psi^{[1]}(\xi_0)|$ and $\xi_0 \in [t_0, t_n]$.

Proof. The truncation error given

$$\begin{aligned} R_\alpha^{Re} &= [{}^C D_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} - [{}^C \mathscr{D}_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} \\ &\le \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n) - \psi(s))^{-\alpha} [f(s) - P_0(s)]_\psi^{[1]}(s) ds. \end{aligned} \tag{27}$$

Using integration by parts, for $u = (\psi(t_n) - \psi(s))^{-\alpha}$, so $u' = \alpha\psi'(s)(\psi(t_n) - \psi(s))^{-\alpha-1}$ and $v' = \psi'(s)[f(s) - P_0(s)]_\psi^{[1]}$ implies $v = [f(s) - P_0(s)]_\psi$. We get

$$\begin{aligned}
R_\alpha^{Re} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \Bigg\{ \Big[[f(s) - P_0(s)]_\psi (\psi(t_n) - \psi(s))^{-\alpha} \Big]_{t_k}^{t_{k+1}} \\
&\quad - \alpha \int_{t_k}^{t_{k+1}} \psi'(s)[f(s) - P_0(s)]_\psi (\psi(t_n) - \psi(s))^{-\alpha-1} ds \Bigg\} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \Bigg\{ \underbrace{\frac{f_\psi^{[1]}(\xi_0)}{1!} (\psi(s) - \psi(t_0))(\psi(t_n) - \psi(s))^{-\alpha} \Big|_{t_k}^{t_{k+1}}}_{=0} \\
&\quad - \alpha \int_{t_k}^{t_{k+1}} \frac{f_\psi^{[1]}(\xi_0)}{1!} \psi'(s)(\psi(s) - \psi(t_0))(\psi(t_n) - \psi(s))^{-\alpha-1} ds \Bigg\} \\
&= \frac{-\alpha}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_\psi^{[1]}(\xi_0)\psi'(s)(\psi(s) - \psi(t_0))(\psi(t_n) - \psi(s))^{-\alpha-1} ds, \qquad (28)
\end{aligned}$$

We estimate R_α^{Re} and obtain

$$\begin{aligned}
|R_\alpha^{Re}| &\leq \frac{\alpha}{\Gamma(1-\alpha)} |f_\psi^{[1]}(\xi_0)||(\psi(\xi_0) - \psi(t_0)))| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n) - \psi(s))^{-\alpha-1} ds \\
&\leq \frac{\alpha}{\Gamma(1-\alpha)} \max_{\xi_0 \in [t_0, t_n]} |f_\psi^{[1]}(\xi_0)| h \sum_{k=0}^{n-1} \left[\frac{(\psi(t_n) - \psi(s))^{-\alpha}}{-\alpha} \Bigg|_{t_k}^{t_{k+1}} \right] \\
&\leq \frac{\alpha}{\Gamma(1-\alpha)} \max_{\xi_0 \in [t_0, t_n]} |f_\psi^{[1]}(\xi_0)| h \sum_{k=0}^{n-1} \left[\frac{(\psi(t_n) - \psi(t_k))^{-\alpha}}{-\alpha} - \frac{(\psi(t_n) - \psi(t_{k+1}))^{-\alpha}}{-\alpha} \right] \\
&\leq \frac{M_1}{\Gamma(1-\alpha)} (\psi(t_n) - \psi(t_0))^{-\alpha} h \\
&\leq C_\alpha^{Re} (\psi(T) - \psi(t_0))^{-\alpha} h, \quad (\text{because}, t_n \leq T \text{ as } t \in [t_0, T]), \qquad (29)
\end{aligned}$$

where $C_\alpha^{Re} = \frac{M_1}{\Gamma(1-\alpha)}$ is a constant that depends only on α, $M_1 = \max_{\xi_0 \in [t_0, t_n]} |f_\psi^{[1]}(\xi_0)|$ and $\xi_0 \in [t_0, t_n]$. □

Remark 4.1. *It is easy also to see that rectangular formula convergent with order $O(h)$, when $1 < \alpha < 2$.*

4.3. **Trapezoidal Formula.** In this subsection, we prove the following error estimate for the trapezoidal formula over the given uniform mesh.

Theorem 4.2. *For $0 < \alpha < 1$ and $f_\psi^{[2]} \in C[t_0, T]$, then the following truncation error R_T^α, where $h = \psi(t_{k+1}) - \psi(t_k) = \dfrac{\psi(T) - \psi(t_0)}{N}$ has the estimate below*

$$|R_\alpha^{Tr}| \leq C_\alpha^{Tr} (\psi(T) - \psi(t_0))^\alpha h^2, \qquad (30)$$

where $C_\alpha^{Tr} = \frac{M_3}{2\Gamma(1-\alpha)}$, $M_2 = \max_{\xi_1 \in [t_0, t_n]} |f_\psi^{[2]}(\xi_1)|$ and $\xi_1 \in [t_0, t_n]$.

Proof. The truncation error given

$$
\begin{aligned}
R_\alpha^{Tr} &= [{}^C D_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} - [{}^C \mathscr{D}_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} \\
&\leq \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{-\alpha}[f(s)-P_1(s)]_\psi^{[1]}(s)ds, \qquad (31)
\end{aligned}
$$

In the same manner, we get

$$
\begin{aligned}
&R_\alpha^{Tr} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \Bigg\{ \Big[[f(s)-P_1(s)]_\psi (\psi(t_n)-\psi(s))^{-\alpha} \Big]_{t_k}^{t_{k+1}} \\
&- \alpha \int_{t_k}^{t_{k+1}} \psi'(s)[f(s)-P_1(s)]_\psi (\psi(t_n)-\psi(s))^{-\alpha-1} ds \Bigg\} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \Bigg\{ \underbrace{\frac{f_\psi^{[2]}(\xi_1)}{2!}(\psi(s)-\psi(t_0))(\psi(s)-\psi(t_1))(\psi(t_n)-\psi(s))^{-\alpha} \Big|_{t_k}^{t_{k+1}}}_{=0} \\
&- \alpha \int_{t_k}^{t_{k+1}} \frac{f_\psi^{[2]}(\xi_1)}{2!} \psi'(s)(\psi(s)-\psi(t_0))(\psi(s)-\psi(t_1))(\psi(t_n)-\psi(s))^{-\alpha-1} ds \Bigg\} \\
&= \frac{-\alpha}{2\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_\psi^{[2]}(\xi_1)\psi'(s)(\psi(s)-\psi(t_0))(\psi(s)-\psi(t_1)) \\
&\times (\psi(t_n)-\psi(s))^{-\alpha-1} ds, \qquad (32)
\end{aligned}
$$

we estimate R_α^{Tr} and obtain

$$
\begin{aligned}
|R_\alpha^{Tr}| &\leq \frac{\alpha}{2\Gamma(1-\alpha)} |f_\psi^{[2]}(\xi_1)| |(\psi(\xi_1)-\psi(t_0))(\psi(\xi_1)-\psi(t_1))| \\
&\times \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{-\alpha-1} ds \\
&\leq \frac{\alpha}{2\Gamma(1-\alpha)} \max_{\xi_1 \in [t_0,t_n]} |f_\psi^{[2]}(\xi_1)| h^2 \sum_{k=0}^{n-1} \left[\frac{(\psi(t_n)-\psi(s))^{-\alpha}}{-\alpha} \Big|_{t_k}^{t_{k+1}} \right] \\
&\leq \frac{\alpha}{2\Gamma(1-\alpha)} \max_{\xi_1 \in [t_0,t_n]} |f_\psi^{[2]}(\xi_1)| h^2 \sum_{k=0}^{n-1} \left[\frac{(\psi(t_n)-\psi(t_k))^{-\alpha}}{-\alpha} - \frac{(\psi(t_n)-\psi(t_{k+1}))^{-\alpha}}{-\alpha} \right] \\
&\leq \frac{M_2}{2\Gamma(1-\alpha)} (\psi(t_n)-\psi(t_0))^{-\alpha} h^2 \\
&\leq C_\alpha^{Tr} (\psi(T)-\psi(t_0))^{-\alpha} h^2, \quad (\text{because}, t_n \leq T \text{ as } t \in [t_0,T]), \qquad (33)
\end{aligned}
$$

where $C_\alpha^{Tr} = \frac{M_2}{2\Gamma(1-\alpha)}$ is a constant that depends only on α, $M_2 = \max_{\xi_1 \in [t_0,t_n]} |f_\psi^{[2]}(\xi_1)|$ and $\xi_1 \in [t_0, t_n]$. □

Remark 4.2. *It is easy also to see that trapezoidal formula convergent with order $O(h^2)$, when $1 < \alpha < 2$.*

4.4. **Simpson's Formula.** In this subsection, we prove the following error estimate for the Simpson's formula over a given uniform mesh.

Theorem 4.3. *For $0 < \alpha < 1$ and $f_{\psi}^{[3]} \in C[t_0, T]$, then the following truncation error R_S^{α}, where $h = \psi(t_{k+1}) - \psi(t_k) = \dfrac{\psi(T) - \psi(t_0)}{N}$ has the estimate below*

$$|R_{\alpha}^{Si}| \leq C_{\alpha}^{Si}(\psi(T) - \psi(t_0))^{\alpha} h^3, \tag{34}$$

where $C_{\alpha}^{Si} = \frac{M_1}{6\Gamma(1-\alpha)}$, $M_3 = \max_{\xi_2 \in [t_0, t_n]} |f_{\psi}^{[3]}(\xi_2)|$ and $\xi_2 \in [t_0, t_n]$.

Proof. The truncation error given

$$\begin{aligned} R_{\alpha}^{Si} &= [{}^{C}D_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} - [{}^{C}\mathscr{D}_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} \\ &\leq \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n) - \psi(s))^{-\alpha} [f(s) - P_2(s)]_{\psi}^{[1]}(s) ds. \end{aligned} \tag{35}$$

In a manner similar
R_{α}^{Si}

$$\begin{aligned} &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \Bigg\{ \Big[[f(s) - P_2(s)]_{\psi} (\psi(t_n) - \psi(s))^{-\alpha} \Big]_{t_k}^{t_{k+1}} \\ &- \alpha \int_{t_k}^{t_{k+1}} \psi'(s) [f(s) - P_2(s)]_{\psi} (\psi(t_n) - \psi(s))^{-\alpha - 1} ds \Bigg\} \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \Bigg\{ \underbrace{\frac{f_{\psi}^{[3]}(\xi_2)}{3!} (\psi(s) - \psi(t_0))(\psi(s) - \psi(t_1))(\psi(s) - \psi(t_2))(\psi(t_n) - \psi(s))^{-\alpha}}_{=0} \Bigg|_{t_k}^{t_{k+1}} \\ &- \alpha \int_{t_k}^{t_{k+1}} \frac{f_{\psi}^{[3]}(\xi_2)}{3!} \psi'(s)(\psi(s) - \psi(t_0))(\psi(s) - \psi(t_1))(\psi(s) - \psi(t_2)) \\ &\times (\psi(t_n) - \psi(s))^{-\alpha - 1} ds \Bigg\} \\ &= \frac{-\alpha}{6\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_{\psi}^{[3]}(\xi_2) \psi'(s)(\psi(s) - \psi(t_0))(\psi(s) - \psi(t_1))(\psi(s) - \psi(t_2)) \\ &\times (\psi(t_n) - \psi(s))^{-\alpha - 1} ds. \end{aligned} \tag{36}$$

We estimate R_α^{Si} and obtain

$$\begin{aligned}
|R_\alpha^{Si}| &\leq \frac{\alpha}{6\Gamma(1-\alpha)}|f_\psi^{[3]}(\xi_1)||(\psi(\xi_2)-\psi(t_0))(\psi(\xi_2)-\psi(t_1))(\xi_2)-\psi(t_2))| \\
&\times \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \psi'(s)(\psi(t_n)-\psi(s))^{-\alpha-1}ds \\
&\leq \frac{\alpha}{6\Gamma(1-\alpha)} \max_{\xi_2\in[t_0,t_n]} |f_\psi^{[3]}(\xi_2)|h^3 \sum_{k=0}^{n-1}\left[\frac{(\psi(t_n)-\psi(s))^{-\alpha}}{-\alpha}\Bigg|_{t_k}^{t_{k+1}}\right] \\
&\leq \frac{\alpha}{6\Gamma(1-\alpha)} \max_{\xi_2\in[t_0,t_n]} |f_\psi^{[3]}(\xi_1)|h^3 \sum_{k=0}^{n-1}\left[\frac{(\psi(t_n)-\psi(t_k))^{-\alpha}}{-\alpha}\right. \\
&\left. - \frac{(\psi(t_n)-\psi(t_{k+1}))^{-\alpha}}{-\alpha}\right] \\
&\leq \frac{M_3}{6\Gamma(1-\alpha)}(\psi(t_n)-\psi(t_0))^{-\alpha}h^3 \\
&\leq C_\alpha^{Si}(\psi(T)-\psi(t_0))^{-\alpha}h^3, \quad (\text{because}, t_n \leq T \text{ as } t\in[t_0,T]), \qquad (37)
\end{aligned}$$

where $C_\alpha^{Si} = \frac{M_3}{6\Gamma(1-\alpha)}$ is a constant that depends only on α, $M_3 = \max_{\xi_2\in[t_0,t_n]}|f_\psi^{[3]}(\xi_2)|$ and $\xi_2 \in [t_0, t_n]$. □

Remark 4.3. *It is easy also to see that simpson's formula convergent with order $O(h^3)$, when $1 < \alpha < 2$.*

5. Numerical example

In this section, to illustrate the maximum absolute errors and the experimental order of convergence (EOC) of FRF, FTF and FSF proposed above, we present some numerical example.

we assume that $J = [a, T]$ and $\psi \in C(J)$ be a differentiable function such that $\psi'(t) > 0$, $t \in J$, $\alpha > 0$. Moreover, we solve the numerical example by using MATLAB(v2016a) software and investigate different choices of suitable functions ψ in the numerical example as below.

Let N be a positive integer and let $\psi(a) = \psi(t_0) < \psi(t_1) < \ldots < \psi(t_N) = \psi(T)$ be the uniform mesh on the interval $[\psi(a), \psi(T)]$. such that $\psi(t) = \psi(a) + hj$ with $h = \dfrac{\psi(T)-\psi(a)}{N}$, $j = 0, 1, \ldots, N$.

Therefore, the maximum absolute errors

$$\|e_N\|_\infty = \max_{0\leq j\leq N} |[^C D_{t_0}^{\alpha,\psi} f(t)]_{t=t_n} - [^C \mathscr{D}_{t_0}^{\alpha,\psi} f(t)]_{t=t_n}|.$$

For this example the experimental order of convergence or EOC as

$$\log_2\left(\frac{\|e_N\|_\infty}{\|e_{2N}\|_\infty}\right).$$

The main purpose is to check the order of convergence of the numerical method with respect to the fractional order α. For various choices of $\alpha > 0$, we computed the errors.

We can now plot this graph such that $y = \log_2\|e_N\|$ and $x = \log_2(h)$ and $h = 1/(5\times 2^l)$, $l = 1, 2, 3, 4, 5, 6, 7$. Doing this, we get that the gradient of the graph would equal the

EOC. To compare this to the theoretical order of convergence, we have also plotted the straight line $y = x$, $y = 2x$ and $y = 3x$.

Example. Let $f(t) = (\psi(t) - \psi(0))^5$, $[t_0, T] = [0, 1]$. Then one has

$$^{C}D_{t_0}^{\alpha,\psi} f(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)} (\psi(t) - \psi(0))^{5-\alpha}. \tag{38}$$

	α	0.25		0.75		1.25		1.75	
	N	Error	EOC	Error	EOC	Error	EOC	Error	EOC
FRF	10	3.52e-01	-	3.66e-01	-	1.32e-00	-	1.25e-00	-
	20	1.76e-01	1.00	2.06e-01	0.83	6.62e-01	1.00	7.02e-01	0.83
	40	8.86e-02	0.99	1.14e-01	0.85	3.33e-01	0.99	3.85e-01	0.87
	80	4.46e-02	1.00	6.18e-02	0.88	1.67e-01	1.00	2.08e-01	0.89
	160	2.24e-02	0.99	3.31e-02	0.90	8.41e-02	0.99	1.10e-01	0.92
	320	1.12e-02	1.00	1.75e-02	0.92	4.22e-02	0.99	5.80e-02	0.92
	640	5.63e-03	0.99	9.13e-03	0.94	2.11e-02	1.00	3.02e-02	0.94
FTF	10	2.19e-02	-	2.44e-02	-	6.08e-02	-	5.88e-02	-
	20	5.55e-03	1.98	6.71e-03	1.86	1.53e-02	1.99	1.59e-02	1.89
	40	1.40e-03	1.99	1.80e-03	1.90	3.86e-03	1.99	4.23e-03	1.91
	80	3.51e-04	2.00	4.77e-04	1.92	9.67e-04	2.00	1.11e-03	1.93
	160	8.80e-05	2.00	1.25e-04	1.93	2.42e-04	2.00	2.88e-04	1.95
	320	2.20e-05	2.00	3.23e-05	1.95	6.06e-05	2.00	7.42e-05	1.96
	640	5.51e-06	2.00	8.32e-06	1.97	1.52e-05	2.00	1.90e-05	1.97
FSF	10	3.03e-05	-	1.39e-04	-	3.05e-05	-	1.48e-04	-
	20	2.42e-06	3.65	1.52e-05	3.19	2.43e-06	3.65	1.57e-05	3.24
	40	1.90e-07	3.67	1.63e-06	3.22	1.91e-07	3.67	1.67e-06	3.23
	80	1.48e-08	3.68	1.74e-07	3.22	1.48e-08	3.69	1.76e-07	3.25
	160	1.14e-09	3.70	1.84e-08	3.24	1.14e-09	3.70	1.85e-08	3.25
	320	8.72e-11	3.71	1.94e-09	3.25	8.82e-11	3.69	1.96e-09	3.24
	640	6.72e-12	3.70	2.06e-10	3.24	7.02e-12	3.65	2.07e-10	3.24

Table 1: Maximum errors and EOC for (5.1) with $\psi(t) = t$

	α	0.25		0.75		1.25		1.75	
	N	Error	E O C	Error	E O C	Error	E O C	Error	E O C
	10	6.16e-02	-	7.71e-02	-	3.33e-01	-	3.82e-01	-
	20	3.09e-02	1.00	4.34e-02	0.83	1.67e-01	1.00	2.13e-01	0.84
F	40	1.55e-02	1.00	2.40e-02	0.85	8.42e-02	0.99	1.17e-01	0.86
R	80	7.81e-03	0.99	1.30e-02	0.88	4.24e-02	0.99	6.31e-02	0.89
F	160	3.92e-03	0.99	6.96e-03	0.90	2.13e-02	0.99	3.35e-02	0.91
	320	1.97e-03	0.99	3.68e-03	0.92	1.07e-02	0.99	1.76e-02	0.93
	640	9.87e-04	1.00	1.92e-03	0.94	5.34e-03	1.00	9.18e-03	0.94
	10	3.83e-03	-	5.13e-03	-	1.54e-02	-	1.79e-02	-
	20	9.73e-04	1.98	1.41e-03	1.86	3.88e-03	1.99	4.83e-03	1.89
F	40	2.45e-04	1.99	3.80e-04	1.89	9.76e-04	1.99	1.28e-03	1.92
T	80	6.16e-05	1.99	1.01e-04	1.91	2.45e-04	1.99	3.37e-04	1.93
F	160	1.54e-05	2.00	2.63e-05	1.94	6.13e-05	2.00	8.75e-05	1.95
	320	3.86e-06	2.00	6.81e-06	1.95	1.53e-05	2.00	2.26e-05	1.95
	640	9,67e-07	2.00	1.75e-06	1.96	3.84e-06	1.99	5.78e-06	1.97
	10	5.32e-06	-	2.93e-05	-	7.72e-06	-	4.49e-05	-
	20	4.25e-07	3.65	3.20e-06	3.19	6.15e-07	3.64	4.78e-06	3.23
F	40	3.34e-08	3.67	3.44e-07	3.22	4.82e-08	3.67	5.06e-07	3.24
S	80	2.59e-09	3.69	3.66e-08	3.23	3.74e-09	3.69	5.35e-08	3.24
F	160	2.00e-10	3.69	3.88e-09	3.24	2.88e-10	3.70	5.63e-09	3.25
	320	1.53e-11	3.71	4.10e-10	3.24	2.23e-11	3.69	5.95e-10	3.24
	640	1.18e-12	3.70	4.33e-11	3.24	1.78e-12	3.65	6.30e-11	3.24

Table 2: Maximum errors and EOC for (5.1) with $\psi(t) = \log(t + 1)$

	α	0.25		0.75		1.25		1.75	
	N	Error	EOC	Error	EOC	Error	EOC	Error	EOC
	10	1.55e-01	-	1.76e-01	-	6.90e-01	-	7.18e-01	-
	20	7.76e-02	1.00	9.90e-02	0.83	3.47e-01	0.99	4.01e-01	0.84
F	40	3.90e-02	0.99	5.47e-02	0.86	1.74e-01	1.00	2.20e-01	0.87
R	80	1.96e-02	0.99	2.97e-02	0.88	8.76e-02	0.99	1.18e-01	0.99
F	160	9.86e-03	0.99	1.59e-02	0.90	4.40e-02	0.99	6.30e-02	0.91
	320	4.95e-03	0.99	8.39e-03	0.92	2.21e-02	0.99	3.31e-02	0.93
	640	2.48e-03	1.00	4.39e-03	0.93	1.11e-02	0.99	1.72e-02	0.94
	10	9.63e-03	-	1.17e-02	-	3.18e-02	-	3.36e-02	-
	20	2.44e-03	1.98	3.22e-03	1.86	8.03e-03	1.99	9.08e-03	1.89
F	40	6.16e-04	1.99	8.66e-04	1.89	2.02e-03	1.99	2.41e-03	1.91
T	80	1.55e-04	1.99	2.29e-04	1.92	5.06e-04	2.00	6.32e-04	1.93
F	160	3.88e-05	2.00	5.99e-05	1.93	1.27e-04	1.99	1.64e-04	1.95
	320	9.70e-06	2.00	1.55e-05	1.95	3.17e-05	2.00	4.24e-05	1.95
	640	2.43e-06	2.00	3.99e-06	1.96	7.94e-06	2.00	1.09e-05	1.96
	10	1.34e-05	-	6.69e-05	-	1.60e-05	-	8.44e-05	-
	20	1.07e-06	3.65	7.30e-06	3.20	1.27e-06	3.66	8.98e-06	3.23
F	40	8.39e-08	3.67	7.84e-07	3.22	9.98e-08	3.67	9.51e-07	3.24
S	80	6.51e-09	3.69	8.34e-08	3.23	7.75e-09	3.69	1.00e-07	3.25
F	160	5.02e-10	3.70	8.84e-09	3.24	5.97e-10	3.70	1.06e-08	3.24
	320	3.84e-11	3.71	9.34e-10	3.24	4.62e-11	3.69	1.12e-09	3.24
	640	2.96e-12	3.70	9.88e-11	3.24	3.67e-12	3.65	1.18e-10	3.25

Table 3: Maximum errors and EOC for (5.1) with $\psi(t) = \sin(t)$

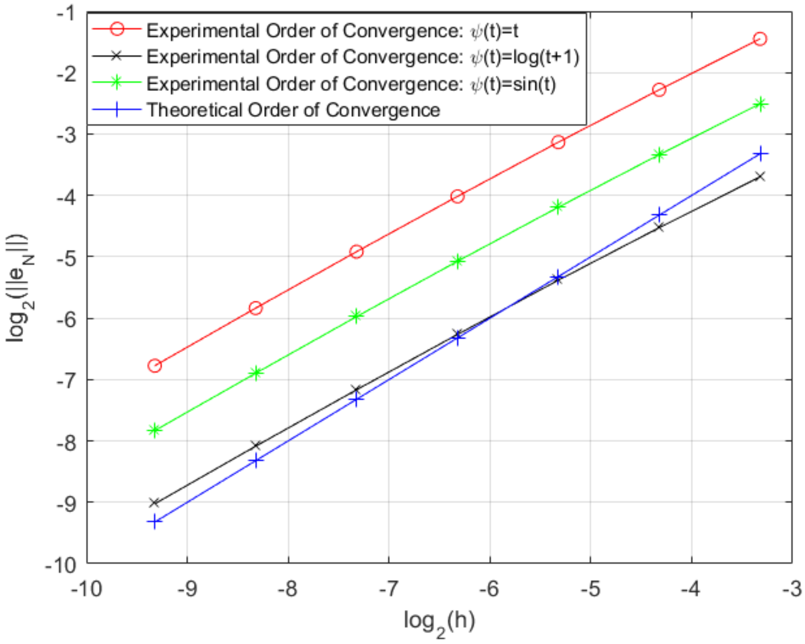

FIGURE 1. The experimentally determined orders of convergence (EOC) at $T = 1$ using FRF with arbitrary kernels $\psi(t)$ and $\alpha = 0.75$

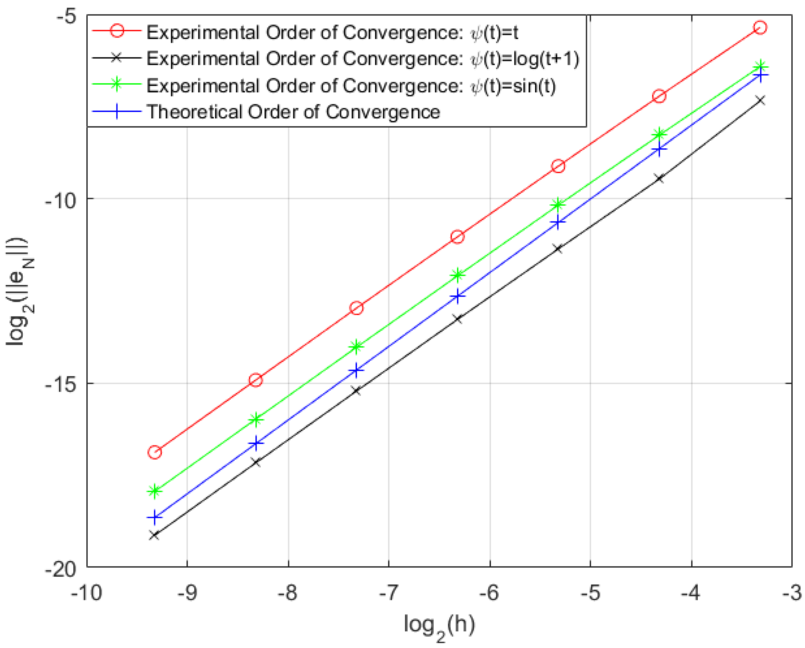

FIGURE 2. The experimentally determined orders of convergence (EOC) at $T = 1$ using FTF with arbitrary kernels $\psi(t)$ and $\alpha = 0.75$

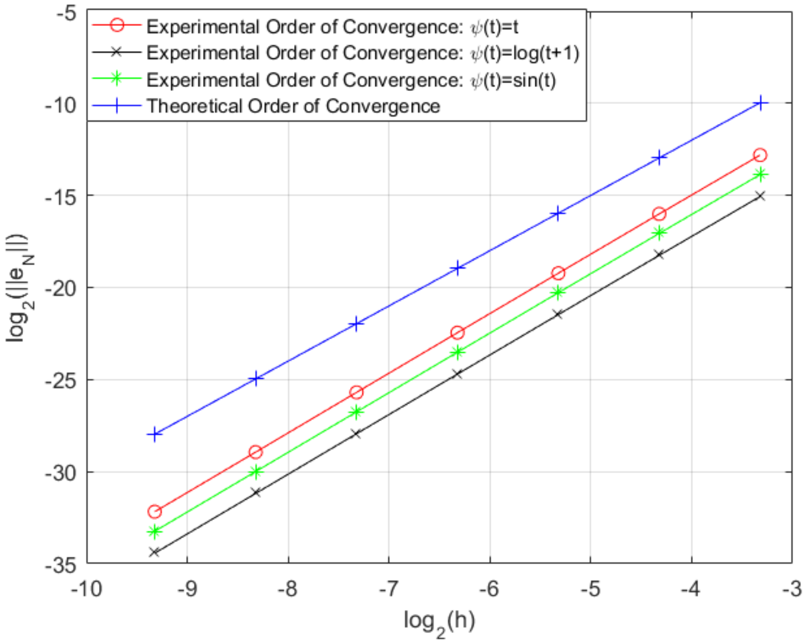

FIGURE 3. The experimentally determined orders of convergence (EOC) at $T = 1$ using FSF with arbitrary kernels $\psi(t)$ and $\alpha = 0.75$

In Tables 1, 2 and 3 we notice that if we change the value of α from 0.25 to 1.75, the maximum error is not affected almost. Also, whenever we change the function kernels $\psi(t)$ from (t, $\log(t+1)$, $\sin(t)$), the maximum error does not change. We can see that the experimental results for the order of convergence (EOC) are consistent with the theoretical results (Theorem 4.1, Theorem 4.2, Theorem 4.3).
In Figures 1, 2, and 3, for $\alpha = 0.75$ we can see that the three lines for each the function kernels $\psi(t)$ are almost parallel to the line $y = x$ for FRR, parallel to the line $y = 2x$ for FTR, and parallel to the line $y = 3x$ for FSR. Hence, we can conclude that the order of convergence (EOC) of this FRR, FTR and FSR is almost $O(h)$, $O(h^2)$ and $O(h^3)$ respectively, such that $y = \log_2 \|e_N\|$ and $x = \log_2(h)$, where $h = 1/(5 \times 2^l)$, $l = 1, 2, \ldots, 7$.

6. Conclusions

In this work, three numerical methods have been proposed: the fractional rectangular formula (FRF), the fractional trapezoidal formula (FTF), and the fractional Simpson's formula (FSF). These methods are general and efficient methods for approximating the generalized Caputo fractional derivative with random kernels, using uniform grids. Numerical experiments reveals that changing the kernel function $\psi(t)$ has a negligible effect on the accuracy of the results whether in terms of the maximum error or the experimentally observed orders of convergence (EOC). The same remains true for variations in the fractional order $\alpha > 0$, as no significant impact on the accuracy was observed. Moreover, the experimentally observed orders of convergence are in complete agreement with the theoretical predictions outlined in Theorems 4.1, 4.2, and 4.3. Specifically, the observed convergence rates for the FRF, FTF, and FSF formulas are $O(h)$, $O(h^2)$ and $O(h^3)$, respectively. These are the same convergence rates previously obtained for approximating the classical Caputo derivative, as reported in reference [18]. In conclusion, the obtained results confirm the consistency, reliability, and effectiveness of the proposed numerical methods in approximating fractional derivatives with arbitrary kernels.

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