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MOMENTS OF ORDER STATISTICS AND K-RECORD VALUES ARISING FROM THE BETA-LOMAX DISTRIBUTION WITH APPLICATION

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Abstract. The concept of k-record values plays a significant role in extreme value theory, which is essential for modeling rare events. In this paper, we focus on k-record values from the Beta-Lomax distribution(BLD), a flexible probability distribution widely used to model extreme events in various fields. We begin by introducing the BLD, highlighting its key properties and characteristics. We then define the notion of k-record values, which represent the maximum of k consecutive observations from a given dataset. The k-record values provide valuable insights into the extreme behavior of the data and are particularly useful for estimating tail probabilities and quantiles. Next, we discuss the statistical properties of k-record values from the BLD, including their moments, distributional properties, and asymptotic behavior. We explore various methodologies for estimating the parameters of the BLD using maximum likelihood estimation (MLE) and discuss strategies for validating the goodness-of-fit of the resulting models. Furthermore, we present applications of k-record values from the BLD in real-world scenarios. These include assessing the risk associated with extreme events, such as natural disasters or financial market crashes, and making informed decisions regarding prevention, mitigation, or insurance coverage. Finally, we conclude by summarizing the key findings and contributions of this study. The analysis of k-record values from the BLD provides valuable insights into extreme event modeling and enhances our understanding of rare occurrences. The results presented in this paper can help practitioners in diverse fields accurately assess and manage risks associated with extreme events, leading to more robust decision-making processes.

Keywords: Beta-Lomax distribution, k-record values, order statistics, MLE, extreme value theory.

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1. Introduction

The BLD, additionally referred to as the Lomax distribution or the Pareto kind II distribution, is a heavy-tailed distribution usually utilized in reliability and survival analysis. It has applications in numerous domain names, particularly when modeling the conduct of strain and power systems. The BLD is bendy in modeling distinct varieties of statistics. It is able to manage a diffusion of shapes, which includes symmetrical and skewed distributions. Since the BLD has heavy tails, it is good for modeling severe values or outliers. In lots of real-international scenarios, mainly in engineering and materials technology, extreme activities (like sudden stress disasters) are huge. The BLD debts for those instances effectively, offering a more sensible illustration of strain and strength traits.
A k-record value represents the maximum value among k consecutive observations from a dataset. It provides insights into the occurrence and magnitude of extreme events within a given sequence. By considering different values of k, one can investigate the behavior of extreme events at varying scales and time intervals. Estimating the parameters of the BLD is essential for accurate modeling. In particular, the k-record values, defined as the smallest k order statistics from a random sample of a given distribution, have received considerable attention in recent years due to their usefulness in extreme value analysis, survival analysis, and reliability theory. In this paper, we focus on the k-record values from the BLD and derive their exact expressions for the probability density function, cumulative distribution function, and moments. We also investigate some statistical properties of k-record values from the BLD, such as their asymptotic behavior, estimation, and inference. Our results provide valuable insights into the behavior of k-record values from the BLD, which can be useful for researchers and practitioners working in various fields. MLE is a commonly used technique to estimate the α and λ parameters based on observed data. Efficient estimation ensures that the distribution adequately captures the extreme behavior of the dataset.
Suppose the random variables $X_1, \ldots, X_n$ are a sample extracted from a statistical population with the density f and the CDF F. If we arrange them in order of size, then we reach to the corresponding order statistics (OS) of the sample denoted by $X_{1:n} \leq \cdots \leq X_{n:n}$. The density of the ith OS, $X_{i:n}$, is [14]

$$f_{i:n}(x) = i\binom{n}{i}\big[F(x)\big]^{i-1} f(x)\big[1 - F(x)\big]^{n-i}. \tag{1}$$

The $OS's$ arises in many practical and theoretical issues such as the parameter estimation methods, e.g. *L-moments*, best linear unbiased estimation for the location-scale family of distributions [10],goodness-of-fit tests, characterizations of probability distributions, analyze of censored samples and reliability theory. There are also some statistics which are constructed by employing the $OS's$. Record values and their generalization, k-record values, belong to this class. To have a background for the former topic, let $\{X_i,\ i \geq 1\}$ be a sequence of random variables coming from a population with density f and CDF, F. An observation X_j is called an upper record value if it exceeds all previous observations, i.e., X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition deals with the lower record statistics: X_j is a lower record if $X_j < X_i$ for every $i < j$. For convenience, assume that the first upper U_1 and lower L_1 records be taken as $U_1 = L_1 \equiv X_1$, and the nth ($n \geq 1$) upper and lower records as U_n and L_n, respectively. The density of U_n is of the form

$$f_{U_n}(u) = \frac{\big[-\ln \bar{F}(u)\big]^n}{(n-1)!} f(u). \tag{2}$$

Record statistics are of interest and importance in several situations such as industrial stress testing, meteorological data analysis, sporting, and athletic events, oil and mining surveys. The k-record values from a probability distribution are important statistical quantities that provide useful information about extreme events and outliers. The formal examine of document value concept probably started with the pioneering paper by way of [13]. In [6] the varieties of file values, a few distributional properties, and statistical inferences of report values and their packages are reviewed. [19] acquired the ML and empirical Bayes estimate for the parameter of the exponential model based totally on record records. [9] proposed families of premiere self belief regions for the location and scale parameters of the two-parameter exponential distribution based totally on higher facts. [3] considered several distributional properties of the upper facts from the exponential distribution and provided some characterizations of the exponential distribution. [4] mentioned some distributional residences of the report values of non-identically distributed random variables having geometric distributions. One can refer to the useful books [8], [14] and also to the papers [2] and 24, for more details on the OS and the k-records. In [22] some recurrence relations satisfied by single and product moments of k-th upper record values from the exponential-Weibull lifetime distribution are discussed. In [23] discussed the $OS's$ and k-record from the CB distribution. In [25] discusses the properties and applications of k-record values derived from the generalized exponential distribution. In [18] the k-record values of the Weibull distribution discussed. In [11] some properties have been studied about the order statistics and k-record values. In [5] information measurements and simultaneous k-value records are based on bivariate distributions of the Sarmanov family. In [1] the residual extropy of records from any continuous distribution is calculated in terms of the residual extropy of records from the uniform distribution. In [21] the major developments of the past decade in the study of record values, record times, inter-record times, and some related statistics from a series of observations are reviewed. The k-record concept can be seen as a generalization to the ordinary record value. In fact, the upper k-record values are defined in terms of the kth largest data yet seen. More specifically, let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with the CDF F and the density f. Then, $Y_1^{(k)} = X_{1:k}$ is taken as the starting point of the k-record process. For $n \geq 2$, let the record times be given by

$$T_{n+1(k)} = \min\left\{j > T_{n(k)}, X_{j:j+k-1} > X_{T_{n(k)}:T_{n(k)}+k-1}\right\},$$

where $T_{1(k)} = k$. Then $Y_n^{(k)} := X_{T_{n(k)}}$, $n \geq 1$, is the nth upper k-record statistic arising from the X-sequence. The density of $Y_n^{(k)}$ and the joint density of $Y_m^{(k)}$ and $Y_n^{(k)}$ are respectively as follows [15]

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{\Gamma(n)}\left[-\ln \bar{F}(x)\right]^{(n-1)}\left[\bar{F}(x)\right]^{(k-1)} f(x), \qquad n \geq 1,\ k \geq 1, \tag{3}$$

and

$$\begin{aligned} f_{Y_m^{(k)},Y_n^{(k)}}(x,y) &= \frac{k^n}{\Gamma(m)\,\Gamma(n-m)}\left[-\ln \bar{F}(x)\right]^{m-1}\frac{f(x)}{\bar{F}(x)} \\ &\quad \times \left[\ln \bar{F}(x) - \ln \bar{F}(y)\right]^{n-m-1}\left[\bar{F}(y)\right]^{k-1} f(y), \end{aligned} \tag{4}$$

where $x < y,\ 1 \leq m < n,\ k \geq 1, n > 1$, and $\bar{F}(x) = 1 - F(x)$. Comparing densities (2) and (3) confirms that the k-records reduce to the ordinary records for $k = 1$. In particular, the smallest k $OS's$, known as k-record values, have attracted considerable attention in recent years due to their applications in various fields such as reliability theory, survival analysis, and extreme value analysise. The BLD is a flexible probability distribution that has been widely used in modeling data with different skewness and kurtosis levels, heavy tails or outliers. However, there is limited research on the k-record values from the BLD, which hinders its applications in practical problems. Therefore, there is a need for investigating the k-record values from the BLD and deriving their exact expressions for the PDF, CDF, and moments. Moreover, it is also essential to investigate some of the statistical properties of k-record values from the BLD, such as their asymptotic behavior, estimation, and inference. Addressing these research gaps will provide valuable insights into the behavior of k-record values from the BLD and enhance our understanding of this important probability distribution.

2. A Four Parameter Beta Lomax Distribution

Let $G(X)$ denote the CDF of a random variable X. The cumulative distribution function for a generalized class of distribution for the random variable X, as defined by [16], is generated by applying the inverse CDF to the beta distributed random variable to obtain

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1} dt \qquad a > 0,\ b > 0 \tag{5}$$

The corresponding PDF for $F(x)$ is given by

$$f(x) = \frac{1}{B(a,b)} [G(X)]^{a-1} [1-G(X)]^{b-1} \acute{G}(X) \tag{6}$$

In the present study, we let $G(x)$ be the CDF of the Lomax random variable with parameters (λ, α) and density function $g(x) = \frac{\alpha}{\lambda}[1+\frac{x}{\lambda}]^{-(\alpha+1)}$ and CDF $G(x) = 1 - [1+\frac{x}{\lambda}]^{-\alpha}$ for $x \geq 0$. From Equations (5) and (6), the PDF and CDF of the BL random variable is given by respectively,

$$f(x) = \frac{\alpha}{\lambda B(a,b)} \left[1 - \left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{a-1} \left(1+\frac{x}{\lambda}\right)^{-(\alpha b+1)}, \tag{7}$$

and

$$F(x) = \frac{\alpha}{B(a,b)} \sum_{i=0}^{a-1} \binom{a-1}{i} (-1)^i \frac{1}{\alpha(i+b)} \left[1 - \left(1+\frac{x}{\lambda}\right)^{-\alpha(i+b)}\right]. \tag{8}$$

The BLD is a probability distribution that combines characteristics of the beta and Lomax distributions. It is commonly used for modeling extreme events due to its flexibility in capturing heavy tails and skewness. The distribution is defined by two parameters, α (shape) and λ (scale), which control its shape and location.

The plot of the PDF of four parameter Beta Lomax Distribution for difrent parameters are given in Figure (1)-(3). Moreover, it exhibits that as the value of α increases the cumulative probability of failure increases sharply which is a similar characteristic to Pareto distribution. The expression for the reliability Function associated with four parameter BLD is given by

$$R(x) = 1 - \frac{\alpha}{B(a,b)} \sum_{i=0}^{a-1} \binom{a-1}{i} \frac{1}{\alpha(b+i)} \left[1 - \left(1+\frac{x}{\lambda}\right)^{-\alpha(i+b)}\right]. \tag{9}$$

There is an inverse relationship between the shape parameter α and the reliability function. The hazard function of the five parameter BLD can be obtained by the relation $h(x) = \frac{f(x)}{R(x)}$

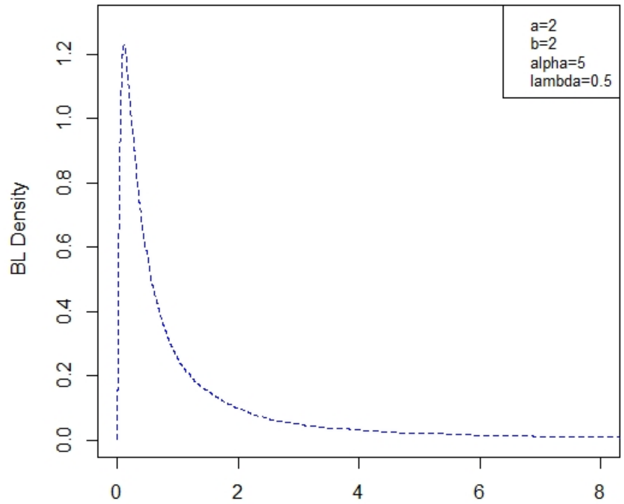

FIGURE 1. Plot of the PDF for Beta-Lomax based on (7).

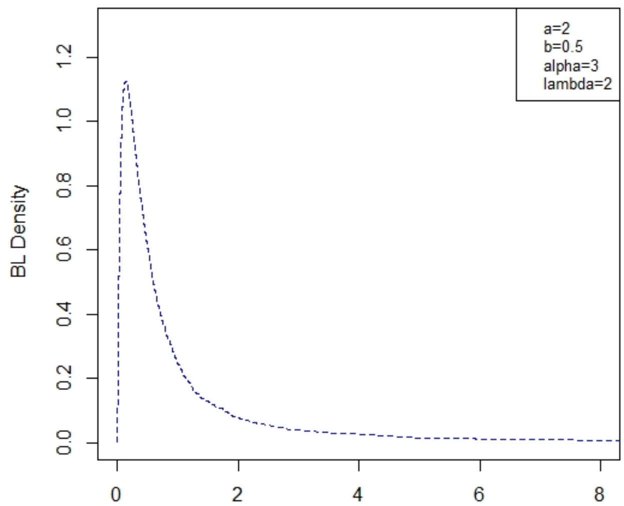

FIGURE 2. Plot of the PDF for Beta-Lomax based on (7).

and is given by

$$h(x) = \frac{\frac{\alpha}{\lambda B(a,b)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{a-1}\left(1+\frac{x}{\lambda}\right)^{-(\alpha b+1)}}{1-\frac{\alpha}{B(a,b)}\sum_{i=0}^{a-1}\binom{a-1}{i}\frac{1}{\alpha(b+i)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha(i+b)}\right]}. \tag{10}$$

Therefore, it can be said that the four parameter BLD can be used for modeling such failure times in which initial failure probability is higher and it decreases during the aging process. The expression for the cumulative hazard function corresponding to the proposed density function of four parameter BLD is given by

$$H(x) = -\ln\left[1-\frac{\alpha}{B(a,b)}\sum_{i=0}^{a-1}\binom{a-1}{i}\frac{1}{\alpha(b+i)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha(i+b)}\right]\right]. \tag{11}$$

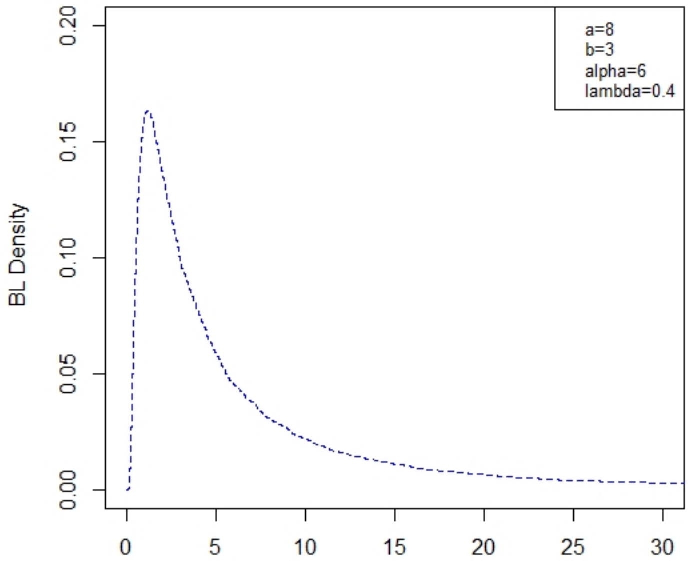

FIGURE 3. Plot of the PDF for Beta-Lomax based on (7).

3. Expression for the rth Moment of the Four Parameter Beta Lomax distribution

The probability density function of the four parameter BLD can be written as

$$f(x) = \frac{\alpha}{\lambda B(a,b)} \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]^{a-1} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha b+1)}. \tag{12}$$

Then we have,

$$E\left(1 + \frac{X}{\lambda}\right) = \int_0^\infty \left(1 + \frac{x}{\lambda}\right) \frac{\alpha}{\lambda B(a,b)} \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]^{a-1} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha b+1)} dx.$$

By setting $y = (1 + \frac{x}{\lambda})^{-\alpha}$, the above integration becomes

$$E\left(1 + \frac{X}{\lambda}\right) = \int_0^1 \frac{1}{B(a,b)} (1-y)^{a-1} y^{b-\frac{1}{\alpha}-1} dy = \frac{B\left(b - \frac{1}{\alpha}, a\right)}{B(a,b)}.$$

Now the $E(X)$ about origin is defined as

$$E(X) = \left[\frac{B\left(b - \frac{1}{\alpha}, a\right)}{B(a,b)} - 1\right] \lambda. \tag{13}$$

Table (1) presents numerical values of $E(X)$ for BLD given by (13) for some selected values of a, b and $\alpha = 2.5, \lambda = 1$, up to tow decimals. Table (2) presents numerical values of $E(X)$ for BLD given by (13) for some selected values of a, b and $\alpha = 3, \lambda = 1.5$, up to tow decimals. Table (3) presents numerical values of $E(X)$ for BLD given by (13) for some selected values of a, b and $\alpha = 5, \lambda = 0.5$, up to tow decimals. In the following, We use the method of changing the distribution function to obtain higher Moments.

TABLE 1. The numerical values of $E(X)$ for the BLD for some choices of a, b and $\alpha = 2.5, \lambda = 1$.

	a				
b	0.2	0.5	1	2	5
0.5	1.32	2.60	4.00	5.81	9.06
1	0.17	0.39	0.66	1.08	1.91
2	0.05	0.14	0.25	0.44	0.86
5	0.02	0.05	0.09	0.16	0.36
10	0.00	0.02	0.04	0.08	0.19

TABLE 2. The numerical values of $E(X)$ for the BLD for some choices of a, b and $\alpha = 3, \lambda = 1.5$.

	a				
b	0.2	0.5	1	2	5
0.5	1.02	1.98	3.00	4.28	6.47
1	0.20	0.44	0.75	1.20	2.05
2	0.07	0.16	0.30	0.52	1.01
5	0.02	0.06	0.11	0.20	0.44
10	0.01	0.03	0.05	0.10	0.23

Theorem 3.1. *Suppose that $X \sim BL(a, b, \alpha, \lambda)$, then we have*

$$E(X^r) = \frac{\lambda^r}{B(a,b)} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{i} B\left(a, b - \frac{j}{\alpha}\right). \tag{14}$$

Proof. According to the main definition of the distribution density function BG we have:

$$E(X^r) = \frac{1}{B(a,b)} \int_0^\infty x^s G(x)^{a-1} [1 - G(x)]^{b-1} g(x) dx, \tag{15}$$

Where $G(x)$ represents the $Lomax(\alpha, \lambda)$ distribution. let $G(x) = u$, then we have $x = \lambda[(1-u)^{\frac{-1}{\alpha}} - 1]$ and we can write,

$$E(X^r) = \frac{1}{B(a,b)} \int_0^\infty [(1-u)^{\frac{-1}{\alpha}} - 1]^r u^{a-1} [1-u]^{b-1} du, \tag{16}$$

Use the binomial formal we have,

$$\begin{aligned} E(X^r) &= \tfrac{\lambda^r}{B(a,b)} \textstyle\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{i} \int_0^\infty u^{a-1} \left[1-u\right]^{b-\frac{j}{\alpha}-1} du \\ &= \tfrac{\lambda^r}{B(a,b)} \textstyle\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{i} B\left(a, b - \frac{j}{\alpha}\right). \end{aligned} \tag{17}$$

□

Remark 3.1. *The expressions for the second four raw moments are given by $r = 2$ in (14)*

$$E(X^2) = \frac{2\lambda}{B(a,b)} \left[B(a,b) - 2B\left(a, b - \frac{1}{\alpha}\right) + B\left(a, b - \frac{2}{\alpha}\right) \right]. \tag{18}$$

Remark 3.2. *By using appropriate moment expressions, the variance* σ_X^2 *as*

$$\begin{aligned}\mathrm{Var}(X) &= \sigma_X^2 \\ &= \frac{2\lambda}{B(a,b)}\left[B(a,b) - 2B\left(a, b-\frac{1}{\alpha}\right) + B\left(a, b-\frac{2}{\alpha}\right)\right] \\ &- \left(\left[\frac{B\left(b-\frac{1}{\alpha}, a\right)}{B(a,b)} - 1\right]\lambda\right)^2.\end{aligned} \tag{19}$$

Table (3) presents numerical values of $E\left(X^2\right)$ for BL given by (18) for some selected values of a, b and $\alpha = 5, \lambda = 0.5$, up to tow decimals.

TABLE 3. values of $E(X^2)$ for the BLD for some choices of a, b and $\alpha = 5, \lambda = 0.5$.

	a				
b	0.2	0.5	1	2	5
0.5	0.86	1.71	2.66	3.97	6.42
1	0.04	0.09	0.17	0.31	0.64
2	0.00	0.01	0.03	0.06	0.09
5	0.00	0.00	0.00	0.00	0.03
10	0.00	0.00	0.00	0.00	0.01

Table (4) presents numerical values of $E(X^2)$ for BL given by (18) for some selected values of a, b and $\alpha = 3, \lambda = 1.5$, up to tow decimals.

TABLE 4. The numerical values of $E(X^2)$ for the BLD for some choices of a, b and $\alpha = 3, \lambda = 1.5$.

	a				
b	0.2	0.5	1	2	5
0.8	2.55	5.88	10.71	18.96	33.96
1	0.63	1.55	3.00	7.70	12.83
2	0.05	0.13	0.30	0.69	1.98
5	0.01	0.02	0.03	0.19	0.32
10	0.00	0.00	0.01	0.02	0.09

4. Some Extensions and Properties

In this section we present some representations of the CDF of BLD. The mathematical relation given in below will be useful in this, and next, section.
Proposition 1. We can express (5) as a mixture of distribution function of BLD as follows: If $a; b; \lambda; \alpha > 0$ we have,

$$F_X(x) = I_{G(x)}(a,b) = 1 - I_{1-G(x)}(b,a) \tag{20}$$

which $G(x)$ is the Lomax distribution with parameters λ, α, and $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$ denotes the incomplete beta function ratio, and

$$B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1}dw.$$

denotes the incomplete beta function.
Read More, we need to summarize the content of the complementary beta distribition (CBD) article [23] because in the next section we will draw on the results of this article. To calculate the raw moments of the CBD, we need to tackle the following integral

$$\mathbb{M}^{(s)}(a,b,k) = \int_0^1 \big[I_u(a,b)\big]^s u^k \,\mathrm{d}u \qquad k=0,1,\dots,\ s=1,2,\dots, \tag{21}$$

where $I_u(a,b) = F_B(u)$ denote the incomplete beta rate function. More precisely, the following fact can be readily concluded

$$X \sim CB(a,b) \implies E(X^s) = \mathbb{M}^{(s)}(a,b,0).$$

The integral (21) will also be used for computing the moments of $OS's$ and k-records in the next two sections. The following lemma provides a recurrence solution for the above integral.

Lemma 4.1. *Assume that a and b are two positive numbers and k be a non-negative integer value, then the integral (21) is derived from the recurrence form*

$$\mathbb{M}^{(s)}(a,b,k) = \frac{1}{k+1}\left[1 - \frac{s}{B(a,b)}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j \mathbb{M}^{(s-1)}(a,b,a+k+j)\right], \qquad s \geq 1,$$

Obviously, if b is an integer value, then the upper bound of the above summation will stops at $j = b-1$.

Proof.

$$\begin{aligned}\mathbb{M}^{(s)}(a,b,k) &= \int_0^1 \big[I_u(a,b)\big]^s u^k \,\mathrm{d}u \\ &= \frac{u^{k+1}}{k+1}\big[I_u(a,b)\big]^s\Big|_0^1 - \frac{s}{(k+1)B(a,b)}\int_0^1 u^{k+1}\big[I_u(a,b)\big]^{s-1}u^{a-1}(1-u)^{b-1}du,\end{aligned}$$

we consider the series expansion [17, formula 1.110]

$$(1-x)^q = \sum_{j=0}^{\infty}\binom{q}{j}(-1)^j x^j.$$

If q is neither a natural number nor zero, the series converges absolutely for $|x| < 1$ and diverges. If $q = n$ is a natural number, this series is reduced to the finite sum $(1-x)^n = \sum_{j=0}^{n}\binom{n}{j}(-1)^j x^j$. Therefore, knowing that $|u| < 1$, the above series can be used to convert $(1-u)^{b-1}$ for $b > 0$. With the starting point for $s = 1$ we have

$$\begin{aligned}\mathbb{M}^{(1)}(a,b,k) &= \int_0^1 I_u(a,b)u^k \mathrm{d}u = \int_0^1 \frac{u^a(1-u)^b}{aB(a,b)}{}_2F_1(a+b,1;a+1;u)u^k\mathrm{d}u \\ &= \frac{B(a+k+1,b+1)}{aB(a,b)}{}_3F_2\left(\begin{array}{c}a+b,1,a+k+1\\ a+1,a+b+k+2\end{array};1\right).\end{aligned}$$

We used $B_z(a,b) = \frac{z^a(1-z)^b}{a}{}_2F_1(a+b,1;a+1;z)$ from [20] and [17]. The ${}_3F_2$ is a generalized hypergeometric function. [17, formula 7.512.5].

□

Theorem 4.1. *Suppose that $X_1, \ldots, X_n$ are iid random variables from $CB(a,b)$ distribution. Then the density of the ith $OS's$, say $X_{i:n}$, from this sample will be simplified as*

$$f_{i:n}(x) = \frac{B(a,b)}{B(i,n-i+1)} F_C(x)^{i-a}\big[1 - F_C(x)\big]^{n-i-b+1}, \qquad 0 < x < 1,\ i = 1, \ldots, n. \quad (22)$$

Let $M^{(s)}_{i:n}$ $(1 \le i \le n)$ denotes the sth moment of the $X_{i:n}$ arising the CB distribution. We can then compute it with the use of (22) as the following recurrence relation

$$M^{(s)}_{(i:n)} = \frac{1}{B(i,n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \mathbb{M}^{(s)}(a, b, i+j-1), \quad (23)$$

where $\mathbb{M}^{(s)}(a, b, i+j-1)$ complies with Lemma (1).

5. Moments of BLD $OS's$ based the CB distribution

Moments of order statistics play an important role in evaluating control quality and relability. For example, if a product has high relability, the time will be expensive in time and money for all products to fail the life test. Therefore, practitioners must predict the failure of their plans based on the failure of some early failures. These estimates are mostly based on time analysis.
Let $X_1, X_2, ..., X_n$ be a random sample of size n from $BL(a, b, \lambda, \alpha)$. Then the pdf and cdf of the ith order statistic, say $X_{(i:n)}$, are given by (1), where $F(x)$ and $f(x)$ denote the (20) and (6) respectively.

Theorem 5.1. *Suppose that $X_1, X_2, ..., X_n$ are iid random variables from $BL(a, b, \lambda, \alpha)$ with distribution (7). Then the $\mu^{(s)}_{i:n}$ denote the sth moment of the ith $OS's$, from this sample will be simplified as*

$$\begin{aligned} \mu^{(s)}_{i:n} &= E(X^s_{i:n}) \\ &= i\lambda^s \binom{n}{i} \sum_{j=0}^{s} \sum_{p=0}^{i-1} \binom{s}{j} \binom{i-1}{p} \frac{(-1)^{p+1} B\left(b - \frac{1}{\alpha}(2+j), a\right)}{B(a,b)} \\ &\quad \times M^{(n-i+p)}_{\left(b-\frac{1}{\alpha}(2+j):a+b-\frac{1}{\alpha}(2+j)-1\right)}(U). \end{aligned} \quad (24)$$

Where $U \sim CB(b,a)$ and $\mathbb{M}^{(s)}(a, b, i+j-1)$ complies with Lemma (1) and $M^{(j)}_{(i:n)}(u)$ denot the sth moment of the ith $OS's$ from the CB distribution (23) which is fully and accurately stated and proven in [23].

Proof.

$$\begin{aligned} \mu^{(s)}_{i:n} &= i\tbinom{n}{i} \int_0^\infty x^s f_{i:n}(x)\mathrm{dx} = i\tbinom{n}{i} \int_0^\infty x^s [F(x)]^{i-1}[1 - F(x)]^{n-i} f(x)\mathrm{dx} \\ &= i\tbinom{n}{i} \int_0^\infty x^s [I_w(b,a)]^{n-i}[1 - I_w(b,a)]^{i-1} \tfrac{\alpha}{\lambda B(a,b)} [1-w]^{a-1} w^{(b-\frac{1}{\alpha})} dx \end{aligned}$$

Where $w = 1 - G(x) = (1 + \frac{x}{\lambda})^{-\alpha}$ and $x = \lambda(w^{\frac{-1}{\alpha}} - 1)$. In the continuation of the proof, we write x as w and we have $\mathrm{dx} = \frac{-\lambda}{\alpha} w^{-(1+\frac{1}{\alpha})}\mathrm{dw}$, then continue the proof.

□

Remark 5.1. *Placing $s = 1$ in (24) specifies the expectation of $X_{i:n}$ as*

$$\mu_{i:n} = i\lambda\binom{n}{i}\sum_{p=0}^{i-1}\sum_{j=0}^{1}(-1)^{j+p}\binom{i-1}{p}\binom{1}{j}\frac{B\left(b-\frac{1}{\alpha}(2+j),a\right)}{B(a,b)} \times M^{(n-i+p)}_{\left(b-\frac{1}{\alpha}(2+j):a+b-\frac{1}{\alpha}(2+j)-1\right)}(U). \quad (25)$$

And we can writh

$$\mu_{i:n} = \frac{i\lambda}{B(a,b)}\binom{n}{i}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p} \times \left[M^{(n-i+p)}_{(b-\frac{2}{\alpha}:a+b-\frac{2}{\alpha}-1)}(U)B\left(b-\frac{2}{\alpha},a\right) - M^{(n-i+p)}_{(b-\frac{3}{\alpha}:a+b-\frac{3}{\alpha}-1)}(U)B\left(b-\frac{3}{\alpha},a\right)\right]. \quad (26)$$

Where $U \sim CB(b,a)$.

The (27) with $i = 1$ reduces to

$$\mu_{1:n} = \frac{n\lambda}{B(a,b)}\left[B(b-\frac{2}{\alpha},a)M^{(n-1)}_{(b-\frac{2}{\alpha}:a+b-\frac{2}{\alpha}-1)}(U) - B(b-\frac{3}{\alpha},a)M^{(n-1)}_{(b-\frac{3}{\alpha}:a+b-\frac{3}{\alpha}-1)}(U)\right].$$

Where $U \sim CB(b,a)$, And with $i = n$ reduces to

$$\mu_{n:n} = \frac{n\lambda}{B(a,b)}\left[B(a,b-\frac{3}{\alpha})M^{(n-1)}_{(a:a+b-\frac{3}{\alpha}-1)}(U) - B(a,b-\frac{2}{\alpha})M^{(n-1)}_{(a:a+b-\frac{2}{\alpha}-1)}(U)\right].$$

Where $U \sim CB(a,b)$.

The values of $\mu^{(1)}_{(i:8)}$ and $\mu^{(2)}_{(i:8)}$ $(1 \le i \le 8)$ calculated by (24) are shown in Tables (5) and (6) for some selected values of parameters up to three decimals.

TABLE 5. The values of $\mu^{(1)}_{(i:8)}$ given by (24) for some selected values of parameters and i.

$b=7, a=2, \alpha=0.5$	i							
λ	1	2	3	4	5	6	7	8
0.5	0.045	0.072	0.096	0.121	0.139	0.162	0.188	0.221
1	0.087	0.145	0.193	0.239	0.284	0.332	0.379	0.430
2	0.178	0.287	0.383	0.274	0.567	0.660	0.759	0.860
$b=7, a=4, \alpha=0.5$	i							
λ	1	2	3	4	5	6	7	8
0.5	0.048	0.062	0.072	0.079	0.085	0.090	0.095	0.099
1	0.096	0.125	0.144	0.158	0.169	0.179	0.189	0.197
2	0.196	0.251	0.289	0.315	0.339	0.359	0.378	0.396

Table 6. The values of $\mu_{(i:8)}^{(2)}$ given by (24) for some selected values of parameters and i

$b=5, a=3, \alpha=1$	i							
λ	1	2	3	4	5	6	7	8
0.5	0.0004	0.0006	0.0009	0.0011	0.0013	0.0016	0.0018	0.0020
1	0.0016	0.0027	0.0037	0.0046	0.0055	0.0068	0.0074	0.0082
2	0.0064	0.0112	0.0154	0.0193	0.0226	0.0262	0.0298	0.0331
$b=5, a=3, \alpha=0.3$	i							
λ	1	2	3	4	5	6	7	8
0.5	0.0051	0.0087	0.0123	0.0155	0.0189	0.0223	0.0255	0.0287
1	0.0202	0.0355	0.0498	0.0635	0.0767	0.0897	0.1029	0.1163
2	0.0810	0.1435	0.1990	0.2508	0.3039	0.3561	0.4100	0.4628

6. Moments of k-record statistics based the CB distribution

As already mentioned in the introduction section, the k-record concept can be seen as a generalization to the ordinary record value. In fact, the upper k-record values are defined in terms of the kth largest data yet seen. More specifically, let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with the cdf F and the density f. Then, $Y_1^{(k)} := X_{1:k}$ is taken as the starting point of the k-record process. For $n \geq 2$, let the record times be given by

$$T_{n+1(k)} = \min\big\{j > T_{n(k)}, X_{j:j+k-1} > X_{T_{n(k)}:T_{n(k)}+k-1}\big\},$$

where $T_{1(k)} = k$. Then $Y_n^{(k)} := X_{T_{n(k)}}$, $n \geq 1$, is the nth upper k-record statistic arising from the X-sequence. The density of $Y_n^{(k)}$ and the joint density of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (3) and (4) respectively by [15], where $\bar{F}(x) = 1 - F(x)$. Comparing densities (2) and (3) confirms that the k-records reduce to the ordinary records for $k = 1$. In the next theorem we obtain a recurrence relation for the single moments of the k-records arising from the BLD.

Theorem 6.1. *If $Y_n^{(k)}$ is the nth upper k-record statistic extracted from a random sequence following the $BLD(a, b, \lambda, \alpha)$ with the density (7), then, its moments will be obtained as*

$$\begin{aligned} E\big[Y_n^{(k)}\big] &= \frac{\lambda k^n}{B(a,b)\Gamma(n)} \sum_{s=0}^{\infty}\sum_{j=0}^{k-1} \binom{k-1}{j} a_{s(n-1)}(-1)^j \\ &\times \Big[B(a, b-\frac{2}{\alpha})M_{(a:b+a-\frac{2}{\alpha}-1)}^{(n+s+j-1)}(U) - B(a, b-\frac{3}{\alpha})M_{(a:b+a-\frac{3}{\alpha}-1)}^{(n+s+j-1)}(U)\Big], \end{aligned} \tag{27}$$

where $U \sim CB(a,b)$, $M_{(i:n)}^{(j)}(u)$ and $a_{s(n-1)}$ are given by (23) and (28), respectively.

Proof. In view of [10], note that

$$\big[-\ln(1-t)\big]^j = \Big(\sum_{p=1}^{\infty}\frac{t^p}{p}\Big)^j = \sum_{p=0}^{\infty} a_{p(j)} t^{j+p}, \qquad |t| < 1, \tag{28}$$

$a_{p(j)}$ is the coefficient of t^{j+p} in the expansion of $(\sum_{p=1}^{\infty}\frac{t^p}{p})^j$ [17, p.17. formula AD-6361]. □

Remark 6.1. *Setting $k = 1$ in (27) then we have the nth upper record, the sth moments of nth upper record will be obtained as*

$$E[U_n]^s = \frac{\lambda^s}{B(a,b)\Gamma(n)} \sum_{j=0}^{s} \sum_{p=0}^{\infty} a_{p(n-1)} \binom{s}{j} (-1)^{j+s} B(a, b - \frac{2+j}{\alpha}) M^{(n+p-1)}_{(a:b+a-\frac{2+j}{\alpha}-1)}(U). \quad (29)$$

Where $U \sim CB(b,a)$, $M^{(j)}_{(i:n)}(u)$ and $a_{s(n-1)}$ are given by (23) and (28), respectively.

In Table (7), we provide $E[U_n]$ for $n = 1, \cdots, 5$ and some selected values of a and b up to three decimals.

TABLE 7. The values of $E[U_n]$ in (29) for some selected values of parameters.

$a=2, b=3, \alpha=0.5$	n				
λ	1	2	3	4	5
0.5	0.137	0.174	0.189	0.195	0.198
2	0.249	0.304	0.325	0.334	0.338
5	0.338	0.402	0.425	0.432	0.435
$a=2, b=3, \alpha=1$	n				
λ	1	2	3	4	5
0.5	0.198	0.251	0.273	0.285	0.289
2	0.335	0.435	0.463	0.476	0.481
5	0.478	0.564	0.591	0.601	0.606
$a=2, b=3, \alpha=2$	n				
λ	1	2	3	4	5
0.5	0.286	0.362	0.392	0.406	0.412
2	0.499	0.604	0.639	0.652	0.661
5	0.647	0.748	0.778	0.789	0.794

A recurrence relation for the mean of the product of nth and mth upper k-record values coming from the BLD is derived in the sequel.

Theorem 6.2. *If $Y_m^{(k)}$ and $Y_n^{(k)}$ are two upper k-records extracted from a random sequence of $CB(a,b)$ distribution, then for $1 \le m < n,\ k \ge 1$, we have,*

$$E\big[(Y_m^{(k)})(Y_n^{(k)})\big] =$$

$$\frac{Q_{k,m,n}}{aB(a,b)} \sum_{t=0}^{p} \sum_{l=0}^{\infty} \binom{p}{t} (-1)^p d_{l(n-t-2)} \Bigg[\Gamma(t+k) E(U_{t+k}) B(n+a-t+l-1, b)$$

$$\times\, {}_3F_2 \left(\begin{matrix} a+b, 1, n+a-t+l-1 \\ a+1, n+a-t+l+b-1 \end{matrix} ; 1 \right)$$

$$- \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{B(r-b+1, a+b) B(s+k+2a+n+l+r-2, b) (a)_r a_{s(t+k-1)} (-1)^{(t+k-1)}}{\Gamma(r+1)\Gamma(b)\Gamma(1-b)(a+s+t+k)}$$

$$\times\, {}_4F_3 \left(\begin{matrix} a, 1-b, a+s+t+k, s+k+2a+n+l+r-2 \\ a, 1-b, a+s+t+k+1, s+k+2a+n+l+r+b-2 \end{matrix} ; 1 \right) \Bigg],$$

where $Q_{k,m,n} = \frac{k^n}{\Gamma(m)\Gamma(n-m)}$, $p = n-m-1$, $a_{p(j)}$ and $d_{p(j)}$ are introduced in (28), $\mu_{i:n}$ and $E[U_n]$ are computed according to (25) and (29), respectively.

The recent expectation can be shortened for two consecutive upper k-records.

Remark 6.2. *Using Theorem 6.2, the product moments of* $Y_m^{(k)}$ *and* $Y_{m+1}^{(k)}$ *for* $k, m \geq 1$ *is obtained as follows*

$$E\big[(Y_m^{(k)})(Y_{m+1}^{(k)})\big] =$$

$$\frac{k^{m+1}}{a\Gamma(m)B(a,b)}\sum_{l=0}^{\infty} d_{l(m-t-1)}\Bigg[\Gamma(k)E(U_k)B(m+a+l,b)\,{}_3F_2\left(\begin{array}{c} a+b,1,m+a+l \\ a+1,m+a+l+b\end{array};1\right)$$

$$-\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{B(r-b+1,a+b)B(s+k+2a+m+l+r-1,b)(a)_r a_{s(k-1)}(-1)^{(k-1)}}{\Gamma(r+1)\Gamma(b)\Gamma(1-b)(a+s+k)}$$

$$\times\,{}_4F_3\left(\begin{array}{c} a,1-b,a+s+k,s+k+2a+m+l+r-1 \\ a,1-b,a+s+k+1,s+k+2a+m+l+r+b-1\end{array};1\right)\Bigg].$$

In the rest of the section, the lower record values of the BLD are considered.

To this end, let $X_i, i \geq 1$ be a sequence of iid random variables with the F and the density f, and L_m be the associated mth lower record statistic. Then the density of L_m is [7],

$$f_{L_m}(x) = \frac{\big[-\ln F(x)\big]^{m-1}}{\Gamma(m)}f(x). \tag{30}$$

Finally, from (30) and after some algebraic calculations, the mean of L_m arising $BL(a,b,\lambda,\alpha)$ distribution with density (23) is derived as

$$\begin{aligned} E[L_m]^s &= \frac{\lambda^{s+1}}{B(a,b)\Gamma(m)}\sum_{p=0}^{\infty}\sum_{j=0}^{s}(-1)^{j+s}\binom{s}{j} \\ &\quad\times B\left(b-\left(\frac{2+j}{\alpha}\right),a\right)a_{p(m-1)} \\ &\quad\times \mathcal{M}^{(m+p-1)}_{(b-(\frac{2+j}{\alpha}):a+b-(\frac{2+j}{\alpha})-1)}(U). \end{aligned} \tag{31}$$

Where $U \sim CB(b,a)$, $M^{(j)}_{(i:n)}(u)$ and $a_{s(n-1)}$ are given by (23) and (28), respectively. Table (8) shows some instances of $E[L_m]$ for $m = 1, \cdots, 5$ and some selected values of a and b up to three decimals, calculated using (31).

7. Numerical Experiments and Discussions

We perform simulation studies to compare the performance of U_n and L_m for different sample sizes. we generate 10,000 samples each of size n from the BLD and repeat this procedure for several values of U_n and L_m. Figures (4) and (5) show the performance of MSE of U_n and L_m in BLD. From these figures we note that the MSE is always greate when n is large.

TABLE 8. The values of $E[L_m]$ in (31) for some selected values of parameters.

$a=3, b=5, \alpha=5$	m				
λ	1	2	3	4	5
0.5	0.415	0.301	0.216	0.157	0.113
1	0.525	0.390	0.282	0.204	0.148
2	0.641	0.490	0.362	0.266	0.193
$a=3, b=5, \alpha=0.5$	m				
λ	1	2	3	4	5
0.5	0.162	0.121	0.088	0.066	0.048
1	0.205	0.153	0.114	0.084	0.063
2	0.255	0.193	0.143	0.107	0.078
$a=3, b=5, \alpha=2$	m				
λ	1	2	3	4	5
0.5	0.285	0.207	0.152	0.109	0.079
1	0.362	0.268	0.196	0.143	0.104
2	0.452	0.335	0.251	0.183	0.134

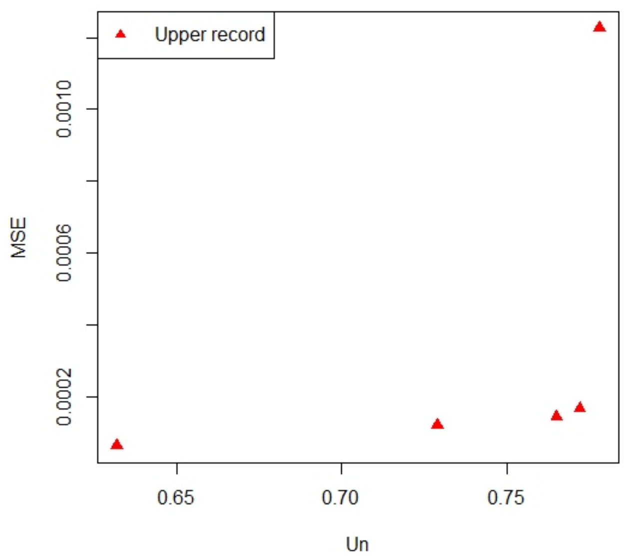

FIGURE 4. The performance of MSE of U_n in BLD.

In Table (9), the estimation of U_n and L_m is compared for different sample sizes n, when $X \sim BL(3,2,3,4)$ and $Y \sim BL(1,2,2,1.5)$ are independent random variables from BLD.
Considering the extremely low Mean Squared Error (MSE) values presented in the graphs and tables, we can conclude that the relationships derived in the preceding sections for U_n and L_m provide an excellent prediction of the observed values. This indicates a high level of efficiency in the predictive models utilized.

8. Real data analysis

This section deals with an example of real data to illustrate the proposed estimation methods. Here, a real life data on 13 and 13 (Data set(I) and Data set(II)) observation

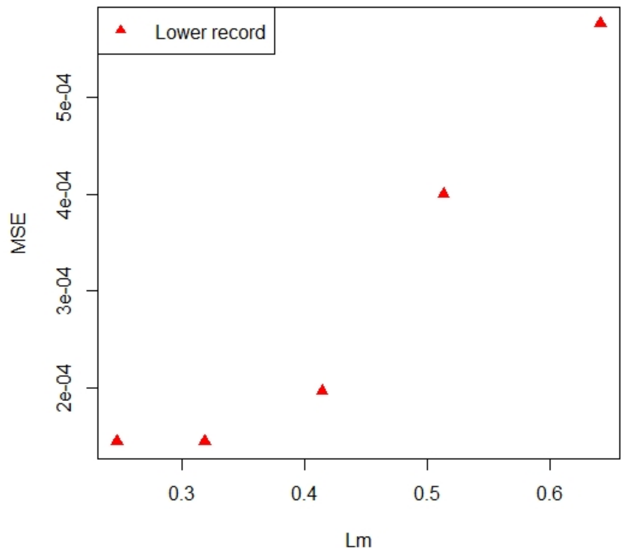

FIGURE 5. The performance of MSE of L_m in BLD.

TABLE 9. Estimation of U_n and L_m.

Beta Lomax	*Results*	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
Upper Record	*Simulated data*	0.667	0.741	0.773	0.783	0.791
	Predicted values	0.632	0.729	0.765	0.772	0.778
	$Bias(U)$	0.035	0.012	0.008	0.011	0.013
	$MSE(U)$	0.00122	0.00014	0.00006	0.00012	0.00016
Lower Record	*Simulated data*	0.653	0.537	0.426	0.338	0.261
	Predicted values	0.641	0.513	0.414	0.318	0.247
	$Bias(L)$	0.012	0.024	0.012	0.020	0.014
	$MSE(L)$	0.00014	0.00057	0.00014	0.0004	0.00019

are considered associated with the failure time of Kevlar 49/epoxy stands with pressure at 90%. The data comes from the studies in [12] based on the recorded failure times in hours. Data for these sample are provided in Tables (10) and (12). The data are fitted by using the BLD. The Kolmogorov–Smirnov (K–S) goodness-of-fit statistic is used for the comparison of the fits. The parameters are estimated by the maximum likelihood technique. The maximum likelihood estimates and the p-values based on the (K–S) goodness-of-fit statistics are given and presented in Table (11). Let us assign the random variable $X \sim f_X(x)$ to Data set(I) and random variable $Y \sim f_Y(y)$ to Data set(II) that have been reproduced in the following tables. According to the figures (6) and (7), and Tables (11) and (13), it is clear that our distributions have a good fit on these data.

TABLE 10. Data set (I)

0.01	0.24	0.80	1.45	0.01	0.24	0.80	1.50	0.02	0.35	0.90	1.53	0.03

In Table (14) and, (15) the observed U_n values for data set(I) and data set(II), and their predicted values are calculated based on the parameters estimated in Table (11) and (13) for different n. Also, the values of bias and MSE have been calculated and included.

TABLE 11. The parameters estimates and goodness of fit criteria for data set(I).

Distribution	MLE(SRS)	$(K-S)$ statistics	p-value
$Beta-Lomax$	$\hat{a}=2.25$ $\hat{b}=2.38$ $\hat{\lambda}=0.156$ $\hat{\alpha}=5.49$	0. 9754	0.0694

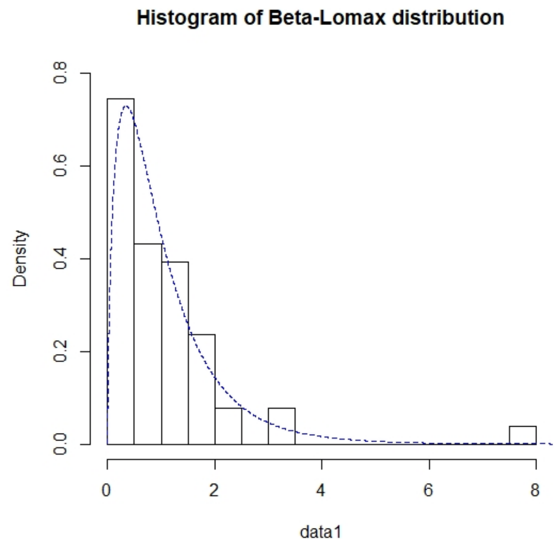

FIGURE 6. Plot of the PDF for BLD based on data set(I).

TABLE 12. Data set (II).

0.02	0.29	0.83	1.51	0.02	0.03	0.85	1.52	0.03	0.38	0.95	1.54	0.04

TABLE 13. The parameters estimates and goodness of fit criteria for data set (II).

Distribution	MLE(SRS)	$(K-S)$ statistics	p-value
$Beta-Lomax$	$\hat{a}=1.61$ $\hat{b}=1.97$ $\hat{\lambda}=0.14$ $\hat{\alpha}=5.49$	1.1434	0.0714

Figures (8) and (9) show the MSE and $Bias$ of Predicted U_n for data set(I) and data set(II).

Figures (10) shows the MSE and $Bias$ of Predicted U_n for data set(I) and data set(II). Based on Figures (10), it is evident that the predicted values for U_n closely align with the observed values for the real data. This demonstrates a significant level of efficiency in the predictive models and relationships established in this research.

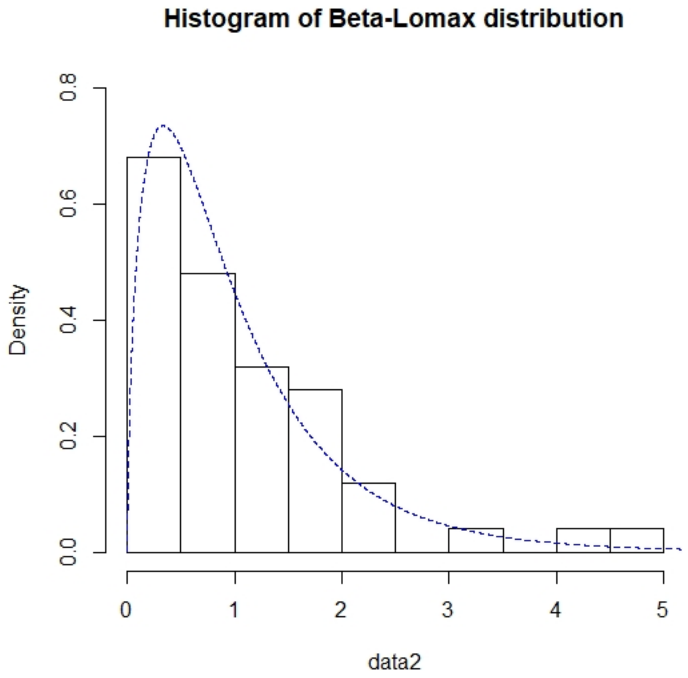

FIGURE 7. Plot of the PDF for BLD based on data set(II).

TABLE 14. Observed U_n and their predicted values for data set(I).

$Results$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$Observed\ U_n$	0.01	0.24	0.80	1.45	1.50	1.53
$Predicted\ U_n$	0.013	0.234	0.792	1.397	1.517	1.529
$Bias$	0.004	0.025	0.053	0.078	0.039	0.024
MSE	0.0000	0.0000	0.0002	0.0006	0.0001	0.0005

TABLE 15. Observed U_n and their predicted values for data set(II).

$Results$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$Observed\ U_n$	0.02	0.29	0.83	1.51	1.52	1.54
$Predicted\ U_n$	0.023	0.287	0.826	1.509	1.513	1.542
$Bias$	0.003	0.003	0.004	0.001	0.007	0.002
MSE	0.000009	0.000009	0.000016	0.000001	0.000049	0.000004

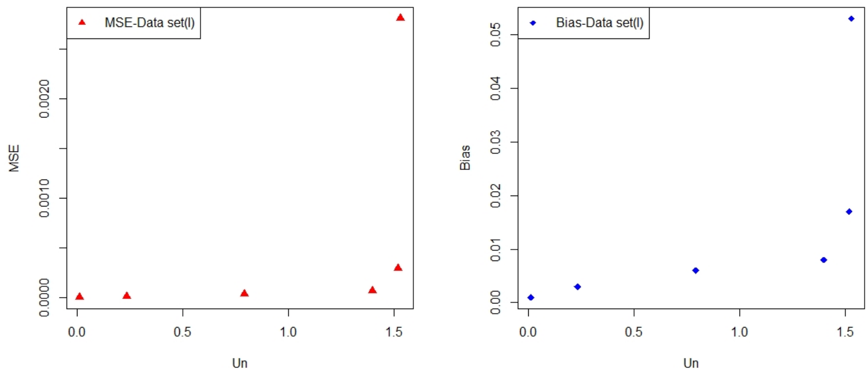

FIGURE 8. The performance of MSE and $Bias$ of U_n for data set(I).

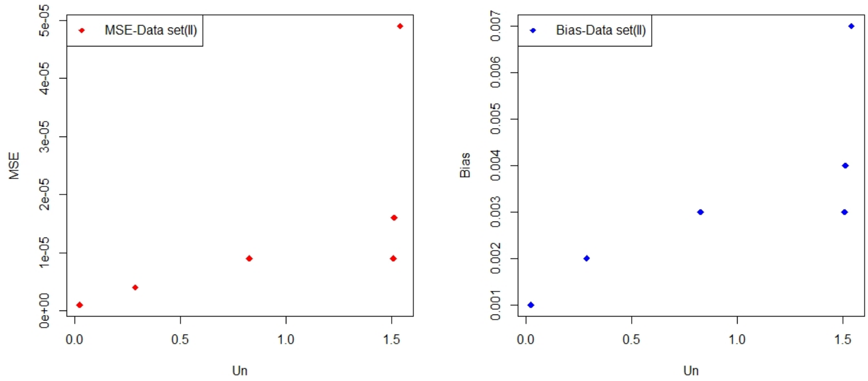

FIGURE 9. The performance of MSE and $Bias$ of U_n for data set (II).

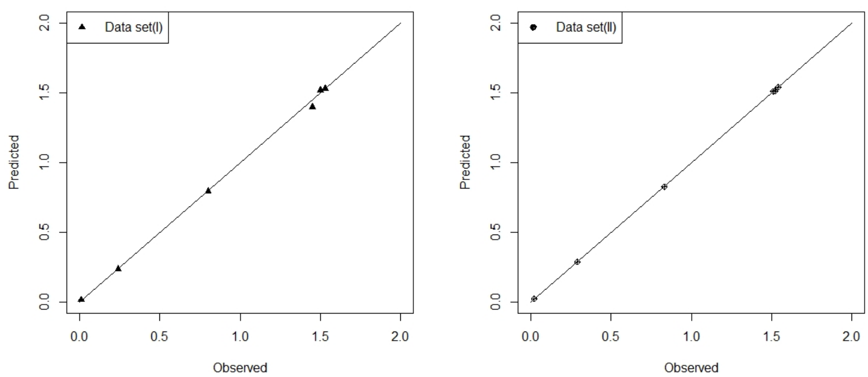

FIGURE 10. Plot of Predicted U_n versus the Observed U_n given by Tables (14) and (15).

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