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PACKING COLORINGS OF THE CORONA PRODUCT OF THE PATH P_n AND THE CYCLE C_n WITH AN EDGE K_2

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Abstract. Given a graph G and a positive integer i, an *i-packing* in G is a subset X of $V(G)$ such that the distance $d_G(u,v)$ between any two distinct vertices $u, v \in X$ is greater than i. The *packing chromatic number* $\chi_\rho(G)$ of a graph G is the smallest integer k such that the vertex set of G can be partitioned into sets V_i, $i \in [k]$, where each V_i is an i-packing. In this paper, we determine the packing chromatic number of the corona products of paths and cycles of small order (at most 11 vertices) with an edge and obtain bounds for the packing chromatic number of corona products of paths and cycles of larger order with an edge.

Keywords: packing chromatic number, corona product of graphs.

AMS Subject Classification: 05C12, 05C15, 05C76.

1. Introduction

The packing chromatic number was first studied, under the name *broadcast chromatic number*, by Goddard, S.M. Hedetniemi, S.T. Hedetniemi, Harris, and Rall [5]. The terms packing coloring and packing chromatic number were coined by Brešar, Klavžar, and Rall [3]. This coloring was introduced because of potential applications in broadcast assignment problems. The development on the packing chromatic number up to 2020 has been summarized in the survey article [2]. Research developments after the survey include [4, 6, 7].

Let $G = (V(G), E(G))$ be a finite undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. The order and the size of G will be denoted with $n(G)$ and $m(G)$, respectively.

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For vertices u and v of a connected graph G, the *distance* $d_G(u, v)$ is the length of a shortest path between u and v in G.

The diameter of G, i.e., $\max\{d_G(u, v) | u, v \in V(G)\}$, will be denoted by diam(G).

Terms and notations not defined in this paper will follow [1].

Given a graph G and a positive integer i, an *i-packing* in G is a subset X of $V(G)$ such that the distance $d_G(u, v)$ between any two distinct vertices $u, v \in X$ is greater than i.

The *i-independence number* of G, denoted by $\alpha_i(G)$, is the maximum cardinality of i-packings of G. In particular, $\alpha_1(G)$ is the independence number $\alpha(G)$ and $\alpha_i(G) = 1$ for $i \geq \text{diam}(G)$.

The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k such that $V(G)$ can be partitioned into sets $V_1, V_2, \ldots, V_k$, where, for each $i \in [k]$, V_i is an i-packing of G, where $[k] = \{1, 2, \ldots, k\}$. Such a partition corresponds to a mapping $c : V(G) \to [k]$ such that $V_i = \{u \in V(G) : c(u) = i\}$. This mapping has the property that $c(u) = c(v) = i$ implies $d_G(u, v) > i$; c is called a *packing k-coloring.*

If an edge or a vertex is removed from a given graph G, then the distances between the (remaining) vertices of G cannot decrease. Hence a packing coloring of G restricted to an arbitrary subgraph H is a packing coloring of H. This implies the following observation.

Observation 1.1. [5] *If H is a subgraph of G, then $\chi_\rho(H) \leq \chi_\rho(G)$.*

Denote by P_n, C_n and K_n, respectively, the path with n vertices, the cycle with n vertices and the complete graph with n vertices.

Proposition 1.1. [5] $\chi_\rho(P_n) = \begin{cases} 2 & \text{if } n \in \{2, 3\}, \\ 3 & \text{if } n \geq 4. \end{cases}$

Proposition 1.2. [5] $\chi_\rho(C_n) = \begin{cases} 3 & \text{if } n = 3 \text{ or } n \equiv 0 \pmod 4, \\ 4 & \text{otherwise.} \end{cases}$

Given two graphs G_1 and G_2 with $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and n disjoint copies $G_2^{(1)}, G_2^{(2)}, \ldots, G_2^{(n)}$ of G_2, the *corona product* of G_1 and G_2, denoted by $G_1 \odot G_2$, is the simple graph obtained from the disjoint union $G_1 \cup (G_2^{(1)} \cup G_2^{(2)} \cup \ldots \cup G_2^{(n)})$ by making the vertex v_i of G_1 adjacent to every vertex of $G_2^{(i)}$, $i \in [n]$.

Theorem 1.1. [8] $\chi_\rho(C_n \odot K_1) = \begin{cases} 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } n \geq 5. \end{cases}$

The packing chromatic number $\chi_\rho(P_n \odot K_1)$ and for $p \geq 2$, the packing chromatic numbers $\chi_\rho(P_n \odot pK_1)$ and $\chi_\rho(C_n \odot pK_1)$ are known, see Section 5.3 of the survey article [2]. In Sections 2 and 3, we consider $\chi_\rho(P_n \odot K_2)$ and $\chi_\rho(C_n \odot K_2)$, respectively.

2. Corona product of P_n and K_2

Let $P_n = v_0v_1v_2 \ldots v_{n-1}$, $K_2^{(i)} = x_iy_i$, $i \in \mathbb{Z}_n$, and $H_n = P_n \odot K_2$. Then, $|V(H_n)| = 3n$, $\alpha_1(H_n) = \alpha_2(H_n) = n$ and $\text{diam}(H_n) = n + 1$. So, $\alpha_{n+1}(H_n) = 1$. Since $H_n \subseteq H_{n+1}$, $\chi_\rho(H_n) \leq \chi_\rho(H_{n+1})$.

Theorem 2.1.
(1)

$$\chi_\rho(P_n \odot K_2) = \begin{cases} 4 & \text{if } n \in \{2, 3\}, \\ 5 & \text{if } n \in \{4, 5\}, \\ 6 & \text{if } n \in \{6, 7, 8, 9, 10, 11\}. \end{cases}$$

(2) For $n \geq 12$,

$$\chi_\rho(P_n \odot K_2) \leq 7.$$

Proof. To prove (1), it is enough if we show that $\chi_\rho(H_{11}) \leq 6$, $\chi_\rho(H_5) \leq 5$, $\chi_\rho(H_3) \leq 4$, $\chi_\rho(H_2) \geq 4$, $\chi_\rho(H_4) \geq 5$ and $\chi_\rho(H_6) \geq 6$.

First, consider H_{11}. Let $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, v_6, y_7, v_8, y_9, v_{10}\}$, $V_2 = \{x_0, x_1, x_2, \ldots, x_{10}\}$, $V_3 = \{y_0, y_2, y_4, y_6, y_8, y_{10}\}$, $V_4 = \{v_1, v_7\}$, $V_5 = \{v_3, v_9\}$, $V_6 = \{v_5\}$. Then, $(V_1, V_2, V_3, V_4, V_5, V_6)$ is a packing 6-coloring of H_{11}, and hence $\chi_\rho(H_{11}) \leq 6$. Next, consider H_5. For $i \in \{1, 2, 3, 4, 5\}$, let $X_i = V_i \cap V(H_5)$. Then, $(X_1, X_2, X_3, X_4, X_5)$ is a packing 5-coloring of H_5, and hence $\chi_\rho(H_5) \leq 5$. Now, consider H_3. For $j \in \{1, 2, 3, 4\}$, let $Y_j = V_j \cap V(H_3)$. Then, (Y_1, Y_2, Y_3, Y_4) is a packing 4-coloring of H_3, and hence $\chi_\rho(H_3) \leq 4$.

For lower bounds, first consider H_2. Clearly, $\chi_\rho(H_2) \geq 4$, since $\alpha_1(H_2) + \alpha_2(H_2) + \alpha_3(H_2) = 5 < 6 = n(H_2)$. Next, consider H_4. Clearly, $\chi_\rho(H_4) \geq 4$, since $\alpha_3(H_4) = \alpha_4(H_4) = 2$. Suppose $\chi_\rho(H_4) = 4$. Let (V_1, V_2, V_3, V_4) be a packing 4-coloring of H_4. Then, $|V_i| = \alpha_i(H_4)$, $i \in [4]$. Assume, by symmetry, that $V_2 = \{x_0, x_1, x_2, x_3\}$ and $V_4 = \{y_0, y_3\}$. Consequently, $V_1 = \{v_0, y_1, y_2, v_3\}$. Now, $V_3 = \{v_1, v_2\}$, a contradiction. Hence, $\chi_\rho(H_4) \geq 5$. Now, consider H_6. It follows, from $\alpha_3(H_6) = 3$ and $\alpha_4(H_6) = \alpha_5(H_6) = 2$, that $\chi_\rho(H_6) \geq 5$. Suppose $\chi_\rho(H_6) = 5$. Let $(V_1, V_2, V_3, V_4, V_5)$ be a packing 5-coloring of H_6. Then, $|V_i| = \alpha_i(H_6)$ for all $i \in \{1, 2, 3, 4, 5\}$ except one i for which $|V_i| = \alpha_i(H_6) - 1$. By symmetry, if necessary, we relabel the vertex x_j by y_j, where $j \in \{0, 1, \ldots, 5\}$. Again, by symmetry, if needed, we relabel the vertex v_k by v_{5-k}, where $k \in \{0, 1, 2\}$. We consider four cases.

Case 1. $|V_1| = 5$ or $|V_4| = 1$.

Then, $V_2 = \{x_0, x_1, \ldots, x_5\}$ and V_5 is $\{y_0, y_4\}$, $\{y_0, v_5\}$, or $\{y_0, y_5\}$. If $V_5 = \{y_0, y_5\}$, then $|V_3| \neq 3$, a contradiction. Hence, V_5 is either $\{y_0, y_4\}$ or $\{y_0, v_5\}$. Then, V_3 is $\{v_0, y_3, y_5\}$ or $\{y_1, y_3, y_5\}$, and therefore, $|V_1| \leq 4$, a contradiction.

Case 2. $|V_3| = 2$.

Then, $V_2 = \{x_0, x_1, \ldots, x_5\}$ and V_5 is $\{y_0, y_4\}$, $\{y_0, v_5\}$, or $\{y_0, y_5\}$. Hence, respectively, we have $\{v_0, y_1, y_3, v_4, y_5\} \subseteq V_1$, $\{v_0, y_1, y_5\} \subseteq V_1$, $\{v_0, y_1, y_4, v_5\} \subseteq V_1$. In any possibility, $|V_4| \leq 1$, a contradiction.

Case 3. $|V_5| = 1$.

Then, $V_2 = \{x_0, x_1, \ldots, x_5\}$. Clearly, by symmetry, one of the following holds:

$\{y_0, y_5\} \subseteq V_1$, $\{v_0, y_5\} \subseteq V_1$, $\{v_0, v_5\} \subseteq V_1$.

If $\{y_0, y_5\} \subseteq V_1$, then $|V_3| \neq 3$, a contradiction.

If $\{v_0, y_5\} \subseteq V_1$, then, in order, $y_1 \in V_1$, $\{y_0, y_2\} \subseteq V_3$, $|V_4| \neq 2$, a contradiction.

If $\{v_0, v_5\} \subseteq V_1$, then $\{y_1, y_4\} \subseteq V_1$. As $\{v_2, v_3\}$ is not a subset of V_1, at least one of y_2, y_3 is in V_1. Assume, by symmetry, $y_2 \in V_1$. Then, $V_3 = \{y_0, y_3, y_5\}$ and so $|V_4| \neq 2$, a contradiction.

Case 4. $|V_2| = 5$.

Then, V_5 is $\{y_0, y_4\}$, $\{y_0, v_5\}$, or $\{y_0, y_5\}$.

Subcase 4.1. $V_5 = \{y_0, y_5\}$.

If $x_0 \in V_4$ (resp. $v_0 \in V_4$), then $v_0 \in V_1$ (resp. $x_0 \in V_1$). So, $V_2 = \{x_1, x_2, x_3, x_4, x_5\}$. Consequently, $|V_3| \neq 3$, a contradiction. By symmetry, if $x_5 \in V_4$ (resp. $v_5 \in V_4$), then we have a contradiction. Hence, $V_4 \cap \{x_0, x_5, v_0, v_5\} = \emptyset$, and therefore, $V_4 = \{y_1, y_4\}$.

As $|V_3| = 3$, exactly one of x_0, v_0, x_1 is in V_3 and exactly one of x_5, v_5, x_4 is in V_3. If $\{x_0, x_5\}$, $\{x_0, x_4\}$ or $\{x_1, x_5\}$ is contained in V_3, then $|V_2| \neq 5$, a contradiction. If $\{v_0, v_5\}$, $\{v_0, x_4\}$, $\{x_1, v_5\}$ or $\{x_1, x_4\}$ is contained in V_3, then $|V_3| \neq 3$, a contradiction. If $\{x_0, v_5\} \subseteq V_3$, then $V_2 = \{x_1, x_2, x_3, x_4, x_5\}$, and hence $|V_1| \neq 6$, a contradiction. If $\{v_0, x_5\} \subseteq V_3$, then, by symmetry, we have a contradiction.

Subcase 4.2. $V_5 = \{y_0, v_5\}$.

If $x_0 \in V_4$ (resp. $v_0 \in V_4$), then $v_0 \in V_1$ (resp. $x_0 \in V_1$) and so $V_2 = \{x_1, x_2, x_3, x_4, x_5\}$. Consequently, $V_3 = \{y_1, y_3, y_5\}$. Therefore, $|V_1| \neq 6$, a contradiction.

If $x_5 \in V_4$, then $y_5 \in V_1$, and so $V_2 = \{x_0, x_1, x_2, x_3, x_4\}$. Consequently, $|V_3| \neq 3$, a contradiction. Hence, $x_5 \notin V_4$. By symmetry, $y_5 \notin V_4$.

Hence, $V_4 \cap \{x_0, x_5, v_0, y_5\} = \emptyset$, and therefore, $V_4 = \{y_1, y_4\}$.

As $|V_3| = 3$, exactly one of x_0, v_0, x_1 is in V_3 and exactly one of x_5, y_5, x_4 is in V_3. Assume, by symmetry, exactly one of y_5, x_4 is in V_3. If $\{x_1, x_4\} \subseteq V_3$, then $|V_3| \neq 3$, a contradiction. If $\{x_0, x_4\} \subseteq V_3$, then $|V_2| \neq 5$, a contradiction. If $\{v_0, x_4\} \subseteq V_3$, then $x_0 \in V_1 \cap V_2$, a contradiction. If $\{x_0, y_5\} \subseteq V_3$ or $\{x_1, y_5\} \subseteq V_3$, then $x_5 \in V_1 \cap V_2$, a contradiction. If $\{v_0, y_5\} \subseteq V_3$, then $x_0, x_5 \in V_1$, and so, $|V_2| \neq 5$, a contradiction.

Subcase 4.3. $V_5 = \{y_0, y_4\}$.

As $|V_3| = 3$, exactly one of x_0, v_0, x_1, y_1 is in V_3 and exactly one of x_5, y_5, v_5, x_4 is in V_3. By symmetry, assume that exactly one of x_0, v_0, x_1 is in V_3 and exactly one of x_5, v_5, x_4 is in V_3. If $\{v_0, v_5\}$, $\{v_0, x_4\}$, $\{x_1, v_5\}$ or $\{x_1, x_4\}$ is contained in V_3, then $|V_3| \neq 3$, a contradiction. If $\{x_0, x_4\} \subseteq V_3$, then $|V_2| \neq 5$, a contradiction. If $\{x_0, x_5\} \subseteq V_3$, then $\{x_4, y_5\} \subseteq V_2$, and hence $|V_1| \neq 6$, a contradiction. If $\{v_0, x_5\} \subseteq V_3$, then, in order, $x_0 \in V_1$, $\{x_4, y_5\} \subseteq V_2$, $|V_1| \neq 6$, a contradiction. If $\{x_0, v_5\} \subseteq V_3$, then, in order, $V_2 = \{x_1, x_2, x_3, x_4, x_5\}$, $y_2 \in V_3$, $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5\}$, $|V_4| \neq 2$, a contradiction. Hence, $\{x_1, x_5\} \subseteq V_3$. Then, $x_3 \in V_3$. So, $V_2 \subseteq \{x_0, y_1, y_2, y_3, x_4, y_5\}$. As $|V_2| = 5$, either $\{x_0, y_1\} \subseteq V_2$ or $\{x_4, y_5\} \subseteq V_2$. In any possibility, $|V_1| \neq 6$, again a contradiction.

In all cases, we have a contradiction. Hence, $\chi_\rho(H_6) \geq 6$.

To prove (2), let

$V_2 = \{x_0, x_1, x_2, \ldots, x_{n-1}\}$,
$V_4 = \{v_i : i \equiv 1 \text{ or } 7 \pmod{12}\}$,
$V_5 = \{v_i : i \equiv 3 \text{ or } 9 \pmod{12}\}$,
$V_6 = \{v_i : i \equiv 5 \pmod{12}\}$,
$V_7 = \{v_i : i \equiv 11 \pmod{12}\}$.

For even n, let

$V_1 = \{v_0, v_2, v_4, \ldots, v_{n-2}\} \cup \{y_1, y_3, y_5, \ldots, y_{n-1}\}$ and $V_3 = \{y_0, y_2, y_4, \ldots, y_{n-2}\}$.

For odd n, let

$V_1 = \{v_0, v_2, v_4, \ldots, v_{n-1}\} \cup \{y_1, y_3, y_5, \ldots, y_{n-2}\}$ and $V_3 = \{y_0, y_2, y_4, \ldots, y_{n-1}\}$.

In any case, $(V_1, V_2, V_3, V_4, V_5, V_6, V_7)$ is a packing 7-coloring of H_n, and hence $\chi_\rho(H_n) \leq 7$. □

3. Corona product of C_n and K_2

Let $C_n = v_0 v_1 v_2 \ldots v_{n-1} v_0$, $K_2^{(i)} = x_i y_i$, $i \in \mathbb{Z}_n$, the set of integers modulo n, and $G_n = C_n \odot K_2$. Then, $n(G_n) = 3n$, $\text{diam}(G_n) = \lceil \frac{n+3}{2} \rceil$, $\alpha_1(G_n) = \alpha_2(G_n) = n$, and for $i \geq 3$, $\alpha_i(G_n) = \lfloor \frac{n}{i-1} \rfloor$.

Theorem 3.1.
(1)

$$\chi_\rho(C_n \odot K_2) = \begin{cases} 5 & \textit{if } n \in \{3, 4\}, \\ 6 & \textit{if } n \in \{5, 6\}, \\ 7 & \textit{if } n \in \{7, 8, 10\}, \\ 8 & \textit{if } n \in \{9, 11\}. \end{cases}$$

(2) For $n \geq 12$,

$$\chi_\rho(C_n \odot K_2) \leq \begin{cases} 7 & \text{if } n \equiv 0 \pmod 2, \text{ or} \\ & \quad n \equiv 1,\ 3,\ 5,\ 7,\ 9 \pmod{12} \text{ and } n \notin \{13, 15, 21\} \\ 8 & \text{otherwise.} \end{cases}$$

Proof.

(i) $n = 3$.

Since $\alpha_3(G_3) = 1$, we have $\chi_\rho(G_3) \geq 5$. To show equality, take $V_1 = \{y_0, v_1, y_2\}$, $V_2 = \{x_0, x_1, x_2\}$, $V_3 = \{y_1\}$, $V_4 = \{v_0\}$ and $V_5 = \{v_2\}$.

(ii) $n = 4$.

Since $P_4 \odot K_2 \subseteq C_4 \odot K_2$ and $\chi_\rho(P_4 \odot K_2) = 5$, we have $\chi_\rho(G_4) \geq 5$. To show equality, take $V_1 = \{y_0, v_1, y_2, v_3\}$, $V_2 = \{x_0, x_1, x_2, x_3\}$, $V_3 = \{y_1, y_3\}$, $V_4 = \{v_0\}$ and $V_5 = \{v_2\}$.

(iii) $n = 5$.

Since $\alpha_3(G_5) = 2$ and $\alpha_4(G_5) = 1$, we have $\chi_\rho(G_5) \geq 6$. To show equality, take $V_1 = \{y_0, v_1, y_2, v_3, y_4\}$, $V_2 = \{x_0, x_1, x_2, x_3, x_4\}$, $V_3 = \{y_1, y_3\}$, $V_4 = \{v_0\}$, $V_5 = \{v_2\}$ and $V_6 = \{v_4\}$.

(iv) $n = 6$.

Since $P_6 \odot K_2 \subseteq C_6 \odot K_2$ and $\chi_\rho(P_6 \odot K_2) = 6$, we have $\chi_\rho(G_6) \geq 6$. To show equality, take $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5\}$, $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5\}$, $V_3 = \{y_0, y_2, y_4\}$, $V_4 = \{v_1\}$, $V_5 = \{v_3\}$ and $V_6 = \{v_5\}$.

(v) $n = 7$.

Since $P_6 \odot K_2 \subseteq C_7 \odot K_2$ and $\chi_\rho(P_6 \odot K_2) = 6$, we have $\chi_\rho(G_7) \geq 6$. Suppose $\chi_\rho(G_7) = 6$. Let $(V_1, V_2, V_3, V_4, V_5, V_6)$ be any packing 6-coloring of G_7. Since $\alpha_3(G_7) = 3$, $\alpha_4(G_7) = 2$ and $\alpha_5(G_7) = 1$, we have $|V_1| = |V_2| = 7$, $|V_3| = 3$, $|V_4| = 2$ and $|V_5| = |V_6| = 1$. Then, $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$, and therefore $V_3 = \{y_i, y_{i+2}, y_{i+4}\}$ for some $i \in \mathbb{Z}_7$. Assume, by symmetry, that $V_3 = \{y_0, y_2, y_4\}$. Now, $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, y_6\}$. Consequently, V_4, a set of cardinality 2, is contained in $\{v_1, v_3, v_5, v_6\}$, a contradiction. Hence, $\chi_\rho(G_7) \geq 7$. To show equality, take $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, y_6\}$, $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$, $V_3 = \{y_0, y_2, y_4\}$, $V_4 = \{v_1\}$, $V_5 = \{v_3\}$, $V_6 = \{v_5\}$ and $V_7 = \{v_6\}$.

(vi) $n = 8$.

Set $V_1 = \{y_0, v_1, y_2, v_3, y_4, v_5, y_6, v_7\}$, $V_2 = \{x_0, x_1, x_2, \ldots, x_7\}$, $V_3 = \{y_1, y_3, y_5, y_7\}$, $V_4 = \{v_0\}$, $V_5 = \{v_2\}$, $V_6 = \{v_4\}$ and $V_7 = \{v_6\}$. Then, $(V_1, V_2, V_3, V_4, V_5, V_6, V_7)$ is a packing 7-coloring of G_8, and hence $\chi_\rho(G_8) \leq 7$. Note that $\alpha_1(G_8) = \alpha_2(G_8) = 8$, $\alpha_3(G_8) = 4$, $\alpha_4(G_8) = \alpha_5(G_8) = 2$ and $\text{diam}(G_8) = 6$. Suppose $\chi_\rho(G_8) \leq 6$. Let $(V_1, V_2, V_3, V_4, V_5, V_6)$ be any packing 6-coloring of G_8 and let $C = (|V_1|, |V_2|, |V_3|, |V_4|, |V_5|, |V_6|)$. Without loss of generality, assume that $|V_i| \geq 1$, $i \in \{1, 2, 3, 4, 5, 6\}$. Hence, C is $(7, 8, 4, 2, 2, 1)$, $(8, 7, 4, 2, 2, 1)$, $(8, 8, 3, 2, 2, 1)$, $(8, 8, 4, 2, 1, 1)$, or $(8, 8, 4, 1, 2, 1)$.

If $C = (7, 8, 4, 2, 2, 1)$, then, in order, $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $V_3 = \{y_0, y_2, y_4, y_6\}$, V_5 is $\{y_1, y_5\}$ or $\{y_3, y_7\}$. Assume, by symmetry, that $V_5 = \{y_1, y_5\}$. But, then $|V_1| \leq 6$, a contradiction.

If $C = (8, 7, 4, 2, 2, 1)$, then $V_3 = \{x_0, x_2, x_4, x_6\}$ and $V_5 = \{y_i, y_{i+4}\}$ for some $i \in \mathbb{Z}_8$. Assume, by symmetry, that V_5 is $\{y_0, y_4\}$ or $\{y_1, y_5\}$. If $V_5 = \{y_0, y_4\}$, then, as both the sets $\{x_0, y_0\}$ and $\{x_4, y_4\}$ are contained in $V_3 \cup V_5$, we have $|V_2| \leq 6$, a contradiction. Hence, $V_5 = \{y_1, y_5\}$. Assume, by relabeling the vertices, that $V_5 = \{x_1, x_5\}$. Now, V_4 is $\{y_i, y_{i+3}\}$, $\{y_i, y_{i+4}\}$ or $\{y_i, v_{i+4}\}$ for some $i \in \mathbb{Z}_8$. If V_4 is $\{y_i, y_{i+3}\}$ or $\{y_i, y_{i+4}\}$, then $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \subseteq V_1 \cup V_2 \cup V_6$; since $|V_6| = 1$, we have a path P_7

with seven vertices in $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ which is packing 2-colorable, a contradiction. Hence, $V_4 = \{y_i, v_{i+4}\}$ for some $i \in \mathbb{Z}_8$. Since, both x_0, x_1, x_2 and x_4, x_5, x_6 have color pattern $3, 5, 3$, it is enough if we consider $i \in \{0, 1, 2, 3\}$. However, the partially colored graphs with colors 3, 4 and 5 for $i \in \{0, 2\}$ are isomorphic. Hence, it is enough if we consider $i \in \{0, 1, 3\}$. If $V_4 = \{y_0, v_4\}$ (respectively, $V_4 = \{y_1, v_5\}$), then $V_2 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ (respectively, $V_2 = \{y_0, y_2, y_3, y_4, y_5, y_6, y_7\}$), and hence $|V_1| \leq 7$, a contradiction. If $V_4 = \{y_3, v_7\}$, then $V_1 \cup V_2 \cup V_6$ is $\{x_3, x_7, v_0, v_1, v_2, v_3, v_4, v_5, v_6, y_0, y_1, y_2, y_4, y_5, y_6, y_7\}$. As $|V_6| = 1$, there exist 15 vertices in $V_1 \cup V_2 \cup V_6$ such that the subgraph induced by these 15 vertices is packing 2-colorable. But such a subgraph, clearly, contains a P_4, a path of length 3, a contradiction.

If $C = (8, 8, 3, 2, 2, 1)$, then $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $V_5 = \{y_i, y_{i+4}\}$ for some $i \in \mathbb{Z}_8$. Assume, by symmetry, that $V_5 = \{y_0, y_4\}$. Then, $\{v_0, v_4\} \cup \{y_1, y_3, y_5, y_7\} \subseteq V_1$. Consequently, $|V_3| \leq 2$, a contradiction.

Hence, C is $(8, 8, 4, 2, 1, 1)$ or $(8, 8, 4, 1, 2, 1)$. Then, in order, $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $V_3 = \{y_0, y_2, y_4, y_6\}$, $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, v_6, y_7\}$. Since $V_4 \cup V_5 \cup V_6 = \{v_1, v_3, v_5, v_7\}$, we have $|V_4| = |V_5| = 1$, a contradiction.

(vii) $n = 9$.

Set $V_1 = \{y_0, v_1, y_2, v_3, y_4, v_5, y_6, v_7, y_8\}$, $V_2 = \{x_0, x_1, x_2, \ldots, x_8\}$, $V_3 = \{y_1, y_3, y_5, y_7\}$, $V_4 = \{v_0\}$, $V_5 = \{v_2\}$, $V_6 = \{v_4\}$, $V_7 = \{v_6\}$ and $V_8 = \{v_8\}$. Then, $(V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8)$ is a packing 8-coloring of G_9, and hence $\chi_\rho(G_9) \leq 8$.

Note that $\alpha_1(G_9) = \alpha_2(G_9) = 9$, $\alpha_3(G_9) = 4$, $\alpha_4(G_9) = 3$, $\alpha_5(G_9) = 2$ and $\mathrm{diam}(G_9) = 6$. Hence, $\chi_\rho(G_9) \geq 5$. Suppose $\chi_\rho(G_9) \leq 7$. Let $(V_1, V_2, V_3, V_4, V_5, V_6, V_7)$ be any packing 7-coloring of G_9 and let $C = (|V_1|, |V_2|, |V_3|, |V_4|, |V_5|, |V_6|, |V_7|)$. Without loss of generality, assume that $|V_i| \geq 1$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Hence, C is $(7, 9, 4, 3, 2, 1, 1)$, $(8, 8, 4, 3, 2, 1, 1)$, $(8, 9, 3, 3, 2, 1, 1)$, $(8, 9, 4, 2, 2, 1, 1)$, $(8, 9, 4, 3, 1, 1, 1)$, $(9, 7, 4, 3, 2, 1, 1)$, $(9, 8, 3, 3, 2, 1, 1)$, $(9, 8, 4, 2, 2, 1, 1)$, $(9, 8, 4, 3, 1, 1, 1)$, $(9, 9, 2, 3, 2, 1, 1)$, $(9, 9, 3, 2, 2, 1, 1)$, $(9, 9, 3, 3, 1, 1, 1)$, $(9, 9, 4, 1, 2, 1, 1)$, or $(9, 9, 4, 2, 1, 1, 1)$.

Claim 1. $C \notin \{(7, 9, 4, 3, 2, 1, 1), (8, 9, 3, 3, 2, 1, 1), (8, 9, 4, 3, 1, 1, 1), (9, 9, 2, 3, 2, 1, 1), (9, 9, 3, 3, 1, 1, 1)\}$. I.e., $(|V_2|, |V_4|) \neq (9, 3)$.

Suppose $|V_2| = 9$ and $|V_4| = 3$, then $V_2 = \{x_0, x_1, \ldots, x_8\}$ and $V_4 = \{y_i, y_{i+3}, y_{i+6}\}$ for some $i \in \mathbb{Z}_9$. Assume, by symmetry, that $V_4 = \{y_0, y_3, y_6\}$. Consequently, $|V_3| \neq 4$. Hence, C is neither $(7, 9, 4, 3, 2, 1, 1)$ nor $(8, 9, 4, 3, 1, 1, 1)$,

In addition, if $|V_5| = 2$, then V_5 is $\{y_1, y_5\}$, $\{y_4, y_8\}$, or $\{y_7, y_2\}$. Again, by symmetry, assume that $V_5 = \{y_1, y_5\}$. Consequently, $|V_1| \leq 7$. Hence, C is neither $(8, 9, 3, 3, 2, 1, 1)$ nor $(9, 9, 2, 3, 2, 1, 1)$.

Finally, if $C = (9, 9, 3, 3, 1, 1, 1)$, then $V_2 = \{x_0, x_1, \ldots, x_8\}$ and $V_4 = \{y_0, y_3, y_6\}$. Consequently, $V_1 = \{v_0, y_1, y_2, v_3, y_4, y_5, v_6, y_7, y_8\}$, and therefore, $|V_3| \neq 3$. Hence, $C \neq (9, 9, 3, 3, 1, 1, 1)$.

Claim 2. $C \notin \{(8, 9, 4, 2, 2, 1, 1), (9, 9, 4, 1, 2, 1, 1), (9, 9, 4, 2, 1, 1, 1)\}$.

If $|V_2| = 9$ and $|V_3| = 4$, then $V_2 = \{x_0, x_1, \ldots, x_8\}$ and $V_3 = \{y_i, y_{i+2}, y_{i+4}, y_{i+6}\}$ for some $i \in \mathbb{Z}_9$. Assume, by symmetry, that $V_3 = \{y_0, y_2, y_4, y_6\}$.

In addition, if $|V_1| = 9$, then $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, v_6, y_7, y_8\}$, and therefore, neither $|V_4| = 2$ nor $|V_5| = 2$. In other words, C is neither $(9, 9, 4, 2, 1, 1, 1)$ nor $(9, 9, 4, 1, 2, 1, 1)$.

Finally, if $C = (8, 9, 4, 2, 2, 1, 1)$, then $V_2 = \{x_0, x_1, \ldots, x_8\}$, $V_3 = \{y_0, y_2, y_4, y_6\}$ and $V_5 = \{y_j, y_{j+4}\}$, where $j \in \{1, 3, 8\}$. Assume, by symmetry, that $j \in \{1, 3\}$. Hence, V_5 is $\{y_1, y_5\}$ or $\{y_3, y_7\}$. But, then $|V_1| \leq 7$, a contradiction. Hence, $C \neq (8, 9, 4, 2, 2, 1, 1)$.

Claim 3. $C \notin \{(8, 8, 4, 3, 2, 1, 1), (9, 7, 4, 3, 2, 1, 1), (9, 8, 4, 3, 1, 1, 1)\}$.

If $|V_3| = 4$ and $|V_4| = 3$, then $V_3 = \{x_i, x_{i+2}, x_{i+4}, x_{i+6}\}$ and $V_4 = \{y_j, y_{j+3}, y_{j+6}\}$ for some $i, j \in \mathbb{Z}_9$. Assume, by symmetry, that $V_4 = \{y_0, y_3, y_6\}$. Again, by symmetry, assume that V_3 is $\{x_0, x_2, x_4, x_6\}$, $\{x_1, x_3, x_5, x_7\}$, or $\{x_2, x_4, x_6, x_8\}$. Note that the partially colored graphs with (V_4, V_3) equals $(\{y_0, y_3, y_6\}, \{x_1, x_3, x_5, x_7\})$ and $(\{y_0, y_3, y_6\}, \{x_2, x_4, x_6, x_8\})$ are isomorphic. So, assume that V_3 is $\{x_0, x_2, x_4, x_6\}$ or $\{x_1, x_3, x_5, x_7\}$.

Let C be $(8, 8, 4, 3, 2, 1, 1)$ or $(9, 8, 4, 3, 1, 1, 1)$. If $V_3 = \{x_0, x_2, x_4, x_6\}$, then, since both $\{x_0, y_0\}$ and $\{x_6, y_6\}$ are contained in $V_3 \cup V_4$, we have $|V_2| \leq 7$, a contradiction. If $V_3 = \{x_1, x_3, x_5, x_7\}$, then $V_2 = \{x_0, y_1, x_2, x_4, y_5, x_6, y_7, x_8\}$, and hence $|V_1| \leq 7$, a contradiction.

Let C be $(9, 7, 4, 3, 2, 1, 1)$. If $V_3 = \{x_0, x_2, x_4, x_6\}$, then, in order, $V_2 = \{x_1, y_2, x_3, y_4, x_5, x_7, x_8\}$, $V_5 = \{y_1, y_5\}$, $|V_1| \leq 6$, a contradiction. So, let $V_3 = \{x_1, x_3, x_5, x_7\}$. Then, $\{y_2, v_3, y_4\} \subseteq V_1$ and therefore, V_5 is $\{x_0, x_4\}$, $\{y_1, y_5\}$, $\{x_2, x_6\}$, $\{x_4, x_8\}$, $\{y_5, x_0\}$, $\{x_6, y_1\}$, or $\{y_7, x_2\}$. If V_5 is $\{x_0, x_4\}$, $\{y_1, y_5\}$, $\{x_2, x_6\}$, $\{y_5, x_0\}$, $\{x_6, y_1\}$, or $\{y_7, x_2\}$, then, respectively, $\{y_8, v_0, y_1\} \subseteq V_1$, $\{x_0, v_1, v_5, x_6\} \subseteq V_1$, $\{y_5, v_6, y_7\} \subseteq V_1$, $\{v_0, y_1, v_5, x_6\} \subseteq V_1$, $\{x_0, v_1, y_5, v_6, y_7\} \subseteq V_1$, or $\{x_6, v_7, x_8\} \subseteq V_1$, and hence $|V_2| < 7$, a contradiction. If $V_5 = \{x_4, x_8\}$, then $\{x_0, y_1, x_2, x_6, y_7, y_8\} \subseteq V_2$, and hence $|V_1| \neq 9$, a contradiction.

By Claims 1, 2 and 3, $C \in \{(9, 8, 3, 3, 2, 1, 1), (9, 8, 4, 2, 2, 1, 1), (9, 9, 3, 2, 2, 1, 1)\}$.

If $C = (9, 9, 3, 2, 2, 1, 1)$, then $V_2 = \{x_0, x_1, \ldots, x_8\}$ and $V_5 = \{y_i, y_{i+4}\}$ for some $i \in \mathbb{Z}_9$. Assume, by symmetry, that $V_5 = \{y_0, y_4\}$. Consequently, $\{v_0, y_1, y_3, v_4, y_5, y_8\} \subseteq V_1$, and therefore, $|V_3| \neq 3$. Hence, $C \neq (9, 9, 3, 2, 2, 1, 1)$.

If $C = (9, 8, 3, 3, 2, 1, 1)$, then $V_4 = \{x_i, x_{i+3}, x_{i+6}\}$ and $V_5 = \{y_j, y_{j+4}\}$ for some $i, j \in \mathbb{Z}_9$. Assume, by symmetry, that $V_4 = \{x_0, x_3, x_6\}$. Again, by symmetry, assume that V_5 is $\{y_0, y_4\}$, $\{y_1, y_5\}$, or $\{y_2, y_6\}$. Since $\{y_0, y_4\}$ and $\{y_2, y_6\}$ are similar, we assume that V_5 is $\{y_0, y_4\}$ or $\{y_1, y_5\}$. If $V_5 = \{y_0, y_4\}$, then $V_2 = \{x_1, x_2, y_3, x_4, x_5, y_6, x_7, x_8\}$, and hence $|V_1| \neq 9$, a contradiction. If $V_5 = \{y_1, y_5\}$, then, in order, V_2 is an 8-element subset of $\{y_0, x_1, x_2, y_3, x_4, x_5, y_6, x_7, x_8\}$, and for any 8-element subset, we have $|V_1| \neq 9$, a contradiction.

If $C = (9, 8, 4, 2, 2, 1, 1)$, then $V_3 = \{x_i, x_{i+2}, x_{i+4}, x_{i+6}\}$ and $V_5 = \{y_j, y_{j+4}\}$ for some $i, j \in \mathbb{Z}_9$. Assume, by symmetry, that $V_5 = \{y_0, y_4\}$. Hence, V_3 is $V_3^{(0)} = \{x_0, x_2, x_4, x_6\}$, $V_3^{(1)} = \{x_1, x_3, x_5, x_7\}$, $V_3^{(2)} = \{x_2, x_4, x_6, x_8\}$, $V_3^{(3)} = \{x_3, x_5, x_7, x_0\}$, $V_3^{(4)} = \{x_4, x_6, x_8, x_1\}$, $V_3^{(5)} = \{x_5, x_7, x_0, x_2\}$, $V_3^{(6)} = \{x_6, x_8, x_1, x_3\}$, $V_3^{(7)} = \{x_7, x_0, x_2, x_4\}$, or $V_3^{(8)} = \{x_8, x_1, x_3, x_5\}$. Since $V_3^{(7)}$, $V_3^{(6)}$, $V_3^{(5)}$ and $V_3^{(4)}$ are, respectively, similar to $V_3^{(0)}$, $V_3^{(1)}$, $V_3^{(2)}$, $V_3^{(3)}$, we consider only five possibilities (one in clockwise direction and the other in anticlockwise direction on the cycle). If $V_3 = V_3^{(0)}$, then $|V_2| \leq 7$, a contradiction. If $V_3 = V_3^{(2)}$, then $V_2 = \{x_0, x_1, y_2, x_3, x_5, y_6, x_7, y_8\}$, and hence $|V_1| \leq 8$, a contradiction. If $V_3 = V_3^{(3)}$, then $V_2 = \{x_1, x_2, y_3, x_4, y_5, x_6, y_7, x_8\}$, and hence $|V_1| \leq 8$, a contradiction. If $V_3 = V_3^{(1)}$, then V_2 is an 8-element subset of $\{x_0, y_1, x_2, y_3, x_4, y_5, x_6, y_7, x_8\}$, and hence, for any 8-element subset, we have $|V_1| \leq 8$, a contradiction. If $V_3 = V_3^{(8)}$, then V_2 is an 8-element subset of $\{x_0, y_1, x_2, y_3, x_4, y_5, x_6, y_7, y_8\}$, and hence, for any 8-element subset, we have $|V_1| \leq 8$, a contradiction.

(viii) $n = 10$.

Set $V_1 = \{y_0, v_1, y_2, v_3, y_4, v_5, y_6, v_7, y_8, v_9\}$, $V_2 = \{x_0, x_1, x_2, \ldots, x_9\}$, $V_3 = \{y_1, y_3, v_6, y_9\}$, $V_4 = \{v_2, y_7\}$, $V_5 = \{v_0, y_5\}$, $V_6 = \{v_4\}$ and $V_7 = \{v_8\}$. Then, $(V_1, V_2, V_3, V_4, V_5, V_6, V_7)$ is a packing 7-coloring of G_{10}, and hence $\chi_\rho(G_{10}) \leq 7$.

Note that $\alpha_1(G_{10}) = \alpha_2(G_{10}) = 10$, $\alpha_3(G_{10}) = 5$, $\alpha_4(G_{10}) = 3$, $\alpha_5(G_{10}) = \alpha_6(G_{10}) = 2$ and $\mathrm{diam}(G_{10}) = 7$. Hence, $\chi_\rho(G_{10}) \geq 5$. Suppose $\chi_\rho(G_{10}) \leq 6$. Let $(V_1, V_2, V_3, V_4, V_5, V_6)$ be any packing 6-coloring of G_{10} and let $C = (|V_1|, |V_2|, |V_3|, |V_4|,$

$|V_5|, |V_6|)$. Without loss of generality, assume that $|V_i| \geq 1$, $i \in \{1,2,3,4,5,6\}$. Hence, C is $(8,10,5,3,2,2)$, $(9,9,5,3,2,2)$, $(9,10,4,3,2,2)$, $(9,10,5,2,2,2)$, $(9,10,5,3,1,2)$, $(9,10,5,3,2,1)$, $(10,8,5,3,2,2)$, $(10,9,4,3,2,2)$, $(10,9,5,2,2,2)$, $(10,9,5,3,1,2)$, $(10,9,5,3,2,1)$, $(10,10,3,3,2,2)$, $(10,10,4,2,2,2)$, $(10,10,4,3,1,2)$, $(10,10,4,3,2,1)$, $(10,10,5,1,2,2)$, $(10,10,5,2,1,2)$, $(10,10,5,2,2,1)$, or $(10,10,5,3,1,1)$.

Claim 1. $(|V_2|, |V_3|) \neq (10,5)$.

Otherwise, $(|V_2|, |V_3|) = (10,5)$. Then, $V_2 = \{x_0, x_1, \ldots, x_9\}$ and $V_3 = \{y_0, y_2, y_4, y_6, y_8\}$. Clearly, $|V_6| \neq 2$. (If $|V_6| = 2$, then $V_6 = \{y_i, y_{i+5}\}$ for some $i \in \mathbb{Z}_{10}$.) Hence, $C \notin \{(8,10,5,3,2,2), (9,10,5,2,2,2), (10,10,5,1,2,2), (9,10,5,3,1,2), (10,10,5,2,1,2)\}$. Thus, $|V_6| = 1$. If $|V_5| = 2$, i.e., if $C \in \{(9,10,5,3,2,1), (10,10,5,2,2,1)\}$, then V_5 is $\{y_i, y_{i+4}\}$, $\{y_i, v_{i+5}\}$, or $\{y_i, y_{i+5}\}$ for some $i \in \mathbb{Z}_{10}$. As $V_5 \neq \{y_i, y_{i+5}\}$, V_5 is $\{y_i, y_{i+4}\}$ or $\{y_i, v_{i+5}\}$. By symmetry, V_5 is $\{y_1, y_5\}$ or $\{y_1, v_6\}$. Then, $|V_1| \leq 8$, a contradiction. Hence, C is neither $(9,10,5,3,2,1)$ nor $(10,10,5,2,2,1)$ and $|V_5| = 1$, and therefore $C = (10,10,5,3,1,1)$. Now, $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, v_6, y_7, v_8, y_9\}$, and therefore $V_3 \subseteq \{v_1, v_3, v_5, v_7, v_9\}$, a contradiction to $|V_3| = 3$. Thus, $C \neq (10,10,5,3,1,1)$.

Claim 2. $(|V_2|, |V_6|) \neq (10,2)$.

Otherwise, $(|V_2|, |V_6|) = (10,2)$. Then, $V_2 = \{x_0, x_1, \ldots, x_9\}$ and $V_6 = \{y_0, y_5\}$.

If $|V_5| = 2$, i.e., if $C \in \{(9,10,4,3,2,2), (10,10,3,3,2,2), (10,10,4,2,2,2)\}$, then V_5 is $\{y_i, y_{i+4}\}$, $\{y_j, v_{j+5}\}$, or $\{y_k, y_{k+5}\}$ for some $i, j, k \in \mathbb{Z}_{10}$. Clearly, $i \in \{2,3,4,7,8,9\}$, $j \in \{1,2,3,4,6,7,8,9\}$ and $k \in \{1,2,3,4,6,7,8,9\}$. Assume, by symmetry, $i \in \{2,3\}$ (since any two i's in $\{2,4,7,9\}$ are similar cases and two i's in $\{3,8\}$ are similar cases), $j \in \{1,2\}$ (since any two j's in $\{1,4,6,9\}$ are similar cases and any two j's in $\{2,3,7,8\}$ are similar cases) and $k \in \{1,2\}$ (since two k's in $\{1,6\}$, $\{2,7\}$, $\{3,8\}$ and $\{4,9\}$, are, respectively, equal cases, two k's in $\{1,4\}$ are similar cases and two k's in $\{2,3\}$ are similar cases). Hence, V_5 is $\{y_2, y_6\}$, $\{y_3, y_7\}$, $\{y_1, v_6\}$, $\{y_2, v_7\}$, $\{y_1, y_6\}$, or $\{y_2, y_7\}$. If $V_5 = \{y_1, y_6\}$, then $|V_1| \leq 8$, a contradiction. If V_5 is $\{y_2, y_6\}$ or $\{y_1, v_6\}$, then $|V_1| \leq 9$, and hence $C = (9,10,4,3,2,2)$. If $V_5 = \{y_2, y_6\}$ (respectively, $V_5 = \{y_1, v_6\}$), then $\{v_0, y_1, v_2, y_3, y_9\} \subseteq V_1$ (respectively, $\{y_4, v_5, y_6\} \subseteq V_1$), and hence $|V_4| \leq 2$, a contradiction. Thus, V_5 is $\{y_3, y_7\}$, $\{y_2, v_7\}$, or $\{y_2, y_7\}$.

First, assume that C is $(10,10,3,3,2,2)$ or $(10,10,4,2,2,2)$. If V_5 is $\{y_3, y_7\}$, $\{y_2, v_7\}$, or $\{y_2, y_7\}$, then, respectively, $V_1 = \{v_0, y_1, y_2, v_3, y_4, v_5, y_6, v_7, y_8, y_9\}$, $\{v_0, y_1, v_2, y_3, y_4, v_5, y_6, y_7, y_9\} \subseteq V_1$, $V_1 = \{v_0, y_1, v_2, y_3, y_4, v_5, y_6, v_7, y_8, y_9\}$. Consequently, $|V_3| \leq 2$, a contradiction.

Next, assume that $C = (9,10,4,3,2,2)$. Since $|V_4| = 3$, $V_4 = \{y_\ell, y_{\ell+3}, y_{\ell+6}\}$ for some $\ell \in \mathbb{Z}_{10}$. As $V_6 = \{y_0, y_5\}$, $\ell \in \{1,3,6,8\}$. Hence, for V_5 equals $\{y_3, y_7\}$, $\{y_2, v_7\}$, $\{y_2, y_7\}$, respectively, we have $\ell \in \{6,8\}$, $\ell \in \{1,3,8\}$, $\ell \in \{3,8\}$. In all the cases, $|V_1| \leq 8$, a contradiction.

Hence, $|V_5| = 1$, and therefore $C = (10,10,4,3,1,2)$. Since $V_6 = \{y_0, y_5\}$, $\{v_0, y_1, y_4, v_5, y_6, y_9\} \subseteq V_1$. As $|V_4| = 3$, $V_4 = \{y_\ell, y_{\ell+3}, y_{\ell+6}\}$ for some $\ell \in \mathbb{Z}_{10}$, a contradiction, since there is no ℓ.

Claim 3. $(|V_3|, |V_6|) \neq (5,2)$.

Otherwise, $(|V_3|, |V_6|) = (5,2)$. Then, $V_3 = \{x_0, x_2, x_4, x_6, x_8\}$, $V_6 = \{y_0, x_5\}$ and $C \in \{(9,9,5,3,2,2), (10,8,5,3,2,2), (10,9,5,2,2,2), (10,9,5,3,1,2)\}$.

First, assume that $|V_2| = 9$. Then, $V_2 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$, and hence $|V_1| \leq 9$. So, $C = (9,9,5,3,2,2)$ and $V_1 = \{v_0, y_1, v_2, y_3, v_4, v_6, y_7, v_8, y_9\}$. Then, $|V_5| \neq 2$, a contradiction.

Next, assume that $|V_2| = 8$. So, $C = (10,8,5,3,2,2)$. Then, $\{v_0, y_1, y_9\} \subseteq V_1$ and $V_4 = \{y_\ell, y_{\ell+3}, y_{\ell+6}\}$ for some $\ell \in \mathbb{Z}_{10}$. Since $y_0 \in V_6$ and $y_1, y_9 \in V_1$, we have $\ell = 2$, i.e., $V_4 = \{y_2, y_5, y_8\}$. Consequently, $V_1 = \{v_0, y_1, v_2, y_3, y_4, v_5, y_6, y_7, v_8, y_9\}$. But, then $|V_2| \leq 4$, a contradiction.

By Claims 1, 2 and 3, $C \in \{(10, 10, 4, 3, 2, 1), (10, 9, 5, 3, 2, 1), (10, 9, 4, 3, 2, 2)\}$. Since $|V_5| = 2$, V_5 is $\{y_i, y_{i+4}\}$, $\{y_j, v_{j+5}\}$, or $\{y_k, y_{k+5}\}$ for some $i, j, k \in \mathbb{Z}_{10}$.

If $C = (10, 10, 4, 3, 2, 1)$, then $V_2 = \{x_0, x_1, \ldots, x_9\}$ and V_5 is $\{y_0, y_4\}$, $\{y_0, v_5\}$, or $\{y_0, y_5\}$. We have, respectively, $\{v_0, y_1, y_3, v_4, y_5, y_9\} \subseteq V_1$. $\{v_0, y_1, y_5, y_9\} \subseteq V_1$. $\{v_0, y_1, y_4, v_5, y_6, y_9\} \subseteq V_1$. Since $|V_4| = 3$, $V_4 = \{y_\ell, y_{\ell+3}, y_{\ell+6}\}$ for some $\ell \in \mathbb{Z}_{10}$, a contradiction, since there is no ℓ.

If $C = (10, 9, 5, 3, 2, 1)$, then $V_3 = \{x_0, x_2, x_4, x_6, x_8\}$ and V_5 is $\{y_0, y_4\}$, $\{y_1, y_5\}$, $\{y_0, v_5\}$, $\{y_1, v_6\}$, $\{y_0, y_5\}$, or $\{y_1, y_6\}$. The cases $\{y_0, y_5\}$ and $\{y_1, y_6\}$ are similar. If $V_5 = \{y_0, y_4\}$, then $|V_2| \leq 8$, a contradiction. If $V_5 = \{y_0, y_5\}$, then $V_2 = \{y_1, y_2, y_3, y_4, x_5, y_6, y_7, y_8, y_9\}$, and therefore $|V_1| \leq 9$, a contradiction. If $V_5 = \{y_0, v_5\}$, then $V_2 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$, and therefore $V_1 = \{v_0, x_1, v_2, x_3, v_4, x_5, v_6, x_7, v_8, x_9\}$, a contradiction to $|V_4| = 3$. If $V_5 = \{y_1, v_6\}$, then interchange x_1 and y_1. So, $V_5 = \{x_1, v_6\}$, and therefore, $y_6 \in V_1$. Consequently, $V_2 = \{y_0, y_1, y_2, y_3, y_4, y_5, y_7, y_8, y_9\}$, a contradiction to $|V_1| = 10$. If $V_5 = \{y_1, y_5\}$, then, for $i \in \{1, 5\}$, interchange x_i and y_i. So, $V_5 = \{x_1, x_5\}$. Since $|V_4| = 3$, $V_4 = \{y_\ell, y_{\ell+3}, y_{\ell+6}\}$ for some $\ell \in \mathbb{Z}_{10}$. Since $|V_2| = 9$, $|\{0, 1, 2, 4, 5, 6, 8\} \cap \{\ell, \ell+3, \ell+6\}| \leq 1$, and hence $\ell \in \{3, 7\}$. If V_4 is $\{y_3, y_6, y_9\}$ or $\{y_7, y_0, y_3\}$, then, respectively, V_2 is $\{y_0, y_1, y_2, x_3, y_4, y_5, y_7, y_8, x_9\}$, $\{y_1, y_2, x_3, y_4, y_5, y_6, x_7, y_8, y_9\}$, and therefore $|V_1| \neq 10$, a contradiction.

If $C = (10, 9, 4, 3, 2, 2)$, then $V_2 = \{x_1, x_2, \ldots, x_9\}$ and $V_6 = \{y_i, y_{i+5}\}$ for some $i \in \mathbb{Z}_{10}$. Assume, by symmetry, that V_6 is $\{x_0, y_5\}$, $\{y_1, y_6\}$, or $\{y_2, y_7\}$. Since $|V_4| = 3$, $V_4 = \{y_\ell, y_{\ell+3}, y_{\ell+6}\}$ for some $\ell \in \mathbb{Z}_{10}$. Clearly, if $V_6 = \{x_0, y_5\}$, then $\ell \notin \{2, 5, 9\}$ (if $V_6 = \{y_1, y_6\}$, then $\ell \notin \{0, 1, 3, 5, 6, 8\}$) (if $V_6 = \{y_2, y_7\}$, then $\ell \notin \{1, 2, 4, 6, 7, 9\}$). As $|V_1| = 10$, if $V_6 = \{x_0, y_5\}$, then $\ell \notin \{0, 1, 3, 4, 6, 8\}$ (if $V_6 = \{y_1, y_6\}$, then $\ell \notin \{2, 4, 7, 9\}$) (if $V_6 = \{y_2, y_7\}$, then $\ell \notin \{0, 3, 5, 8\}$). Hence, $V_6 = \{x_0, y_5\}$ and $V_4 = \{y_7, y_0, y_3\}$. Then, $V_1 = \{v_0, y_1, y_2, v_3, y_4, v_5, y_6, v_7, y_8, y_9\}$, and therefore $|V_5| \neq 2$.

(ix) $n = 11$.

Set $V_1 = \{y_0, v_1, y_2, v_3, y_4, v_5, y_6, v_7, y_8, v_9, y_{10}\}$, $V_2 = \{x_0, x_1, x_2, \ldots, x_{10}\}$, $V_3 = \{y_1, y_3, y_5, y_7, y_9\}$, $V_4 = \{v_0, v_6\}$, $V_5 = \{v_2\}$, $V_6 = \{v_4\}$, $V_7 = \{v_8\}$ and $V_8 = \{v_{10}\}$. Then, $(V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8)$ is a packing 8-coloring of G_{11}, and hence $\chi_\rho(G_{11}) \leq 8$.

Note that $\alpha_1(G_{11}) = \alpha_2(G_{11}) = 11$, $\alpha_3(G_{11}) = 5$, $\alpha_4(G_{11}) = 3$, $\alpha_5(G_{11}) = \alpha_6(G_{11}) = 2$ and $\mathrm{diam}(G_{11}) = 7$. Hence, $\chi_\rho(G_{11}) \geq 6$. Suppose $\chi_\rho(G_{11}) \leq 7$. Let $(V_1, V_2, V_3, V_4, V_5, V_6, V_7)$ be any packing 7-coloring of G_{11} and let $C = (|V_1|, |V_2|, |V_3|, |V_4|, |V_5|, |V_6|, |V_7|)$. Without loss of generality, assume that $|V_i| \geq 1$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Hence, C is $(9, 11, 5, 3, 2, 2, 1)$, $(10, 10, 5, 3, 2, 2, 1)$, $(10, 11, 4, 3, 2, 2, 1)$, $(10, 11, 5, 2, 2, 2, 1)$, $(10, 11, 5, 3, 1, 2, 1)$, $(10, 11, 5, 3, 2, 1, 1)$, $(11, 9, 5, 3, 2, 2, 1)$, $(11, 10, 4, 3, 2, 2, 1)$, $(11, 10, 5, 2, 2, 2, 1)$, $(11, 10, 5, 3, 1, 2, 1)$, $(11, 10, 5, 3, 2, 1, 1)$, $(11, 11, 3, 3, 2, 2, 1)$, $(11, 11, 4, 2, 2, 2, 1)$, $(11, 11, 4, 3, 1, 2, 1)$, $(11, 11, 4, 3, 2, 1, 1)$, $(11, 11, 5, 1, 2, 2, 1)$, $(11, 11, 5, 2, 1, 2, 1)$, $(11, 11, 5, 2, 2, 1, 1)$, or $(11, 11, 5, 3, 1, 1, 1)$.

Claim 1. $(|V_2|, |V_3|) \neq (11, 5)$.

Otherwise, $(|V_2|, |V_3|) = (11, 5)$. Then, $V_2 = \{x_0, x_1, \ldots, x_{10}\}$ and $V_3 = \{y_0, y_2, y_4, y_6, y_8\}$. Clearly, $|V_6| \neq 2$. Suppose $|V_6| = 2$, then $V_6 = \{y_i, y_{i+5}\}$ for some $i \in \mathbb{Z}_{11}$, and therefore $i \in \{5, 7, 9\}$. It follows that $|V_1| \leq 9$, and therefore $C \notin \{(10, 11, 5, 2, 2, 2, 1), (10, 11, 5, 3, 1, 2, 1), (11, 11, 5, 1, 2, 2, 1), (11, 11, 5, 2, 1, 2, 1)\}$ and $C = (9, 11, 5, 3, 2, 2, 1)$. (a) If $V_6 = \{y_5, y_{10}\}$, then $\{y_1, v_2, y_3, v_4, v_6, y_7, v_8, y_9\} \subseteq V_1$. (b) If $V_6 = \{y_7, y_1\}$, then $V_1 = \{v_0, v_2, y_3, v_4, y_5, v_6, v_8, y_9, y_{10}\}$. (c) If $V_6 = \{y_9, y_3\}$, then $\{v_0, y_1, v_2, v_4, y_5, v_6, y_7, y_{10}\} \subseteq V_1$. In any possibility of V_6, $|V_4| \neq 3$, a contradiction. Thus, $|V_6| = 1$, and $C \in \{(10, 11, 5, 3, 2, 1, 1), (11, 11, 5, 2, 2, 1, 1), (11, 11, 5, 3, 1, 1, 1)\}$. If $|V_1| = 11$, then

$V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, v_6, y_7, v_8, y_9, y_{10}\}$, and therefore $|V_4| \leq 2$ and $|V_5| = 1$, a contradiction. Hence, $C = (10, 11, 5, 3, 2, 1, 1)$. As $|V_4| = 3$, V_4 is $\{y_j, y_{j+3}, y_{j+6}\}$, $\{y_j, y_{j+3}, y_{j+7}\}$ or $\{y_j, y_{j+3}, v_{j+7}\}$ for some $j \in \mathbb{Z}_{11}$. It follows that V_4 is $\{y_7, y_{10}, y_3\}$, $\{y_9, y_1, y_5\}$, $\{y_7, y_{10}, v_3\}$, or $\{y_9, y_1, v_5\}$. In any possibility of V_4, $|V_1| \neq 10$, a contradiction.

Claim 2. $(|V_2|, |V_4|, |V_6|) \neq (11, 3, 2)$.

Otherwise, $(|V_2|, |V_4|, |V_6|) = (11, 3, 2)$. So, $C \in \{(10, 11, 4, 3, 2, 2, 1), (11, 11, 3, 3, 2, 2, 1), (11, 11, 4, 3, 1, 2, 1)\}$. Then, $V_2 = \{x_0, x_1, \ldots, x_{10}\}$ and $V_6 = \{y_0, y_5\}$. Since $|V_4| = 3$, V_4 is $\{y_i, y_{i+3}, y_{i+6}\}$, $\{y_i, y_{i+3}, y_{i+7}\}$ or $\{y_i, y_{i+3}, v_{i+7}\}$ for some $i \in \mathbb{Z}_{11}$. It follows that V_4 is $\{y_i, y_{i+3}, y_{i+6}\}$ with $i \in \{1, 3, 4, 6, 7, 9\}$, or $\{y_j, y_{j+3}, y_{j+7}\}$ with $j \in \{1, 3, 6, 7, 10\}$, or $\{y_k, y_{k+3}, v_{k+7}\}$ with $k \in \{1, 3, 4, 6, 7, 9, 10\}$. As $|V_1|$ is 10 or 11, we have $i \in \{3, 7\}$, $j \in \{6, 7\}$ and $k \in \{3, 6, 7, 10\}$. But, then, in any possibility, $|V_1| = 10$ and $C = (10, 11, 4, 3, 2, 2, 1)$.

(a) If $V_4 = \{y_3, y_6, y_9\}$, then $\{v_0, y_1, y_2, v_3, y_4, y_8, v_9, y_{10}\} \subseteq V_1$.
(b) If $V_4 = \{y_7, y_{10}, y_2\}$, then $\{y_1, v_2, y_3, y_4, v_5, y_6, v_7, y_8\} \subseteq V_1$.
(c) If $V_4 = \{y_6, y_9, y_2\}$, then $\{v_0, y_1, v_2, y_3, y_8, v_9, y_{10}\} \subseteq V_1$.
(d) If $V_4 = \{y_7, y_{10}, y_3\}$, then $\{y_2, v_3, y_4, v_5, y_6, v_7, y_8\} \subseteq V_1$.
(e) If $V_4 = \{y_3, y_6, v_{10}\}$, then $\{v_0, y_1, y_2, v_3, y_4, y_{10}\} \subseteq V_1$.
(f) If $V_4 = \{y_6, y_9, v_2\}$, then $\{v_0, y_1, y_2, y_8, v_9, y_{10}\} \subseteq V_1$.
(g) If $V_4 = \{y_7, y_{10}, v_3\}$, then $\{y_3, y_4, v_5, y_6, v_7, y_8\} \subseteq V_1$.
(h) If $V_4 = \{y_{10}, y_2, v_6\}$, then $\{y_1, v_2, y_3, y_4, v_5, y_6\} \subseteq V_1$.

In any possibility, $|V_3| \neq 4$, a contradiction.

Claim 3. $|V_2| \neq 11$.

If $C = (11, 11, 4, 2, 2, 2, 1)$, then $V_2 = \{x_0, x_1, \ldots, x_{10}\}$ and $V_6 = \{y_0, y_5\}$. As $|V_1| = 11$, $\{v_0, y_1, y_4, v_5, y_6, y_{10}\} \subseteq V_1$. Consequently, $\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{10}, y_2, y_3, y_7, y_8, y_9\} \subseteq V_3$. As $|V_3| = 4$, $|\{y_2, y_3, y_7, y_8, y_9\} \cap V_3| \geq 2$, and so $|\{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_9, v_{10}\} \cap V_3| \leq 2$. If $\{y_2, y_8\} \subseteq V_3$ or $\{y_3, y_8\} \subseteq V_3$, then $|V_3| = 2$, a contradiction. If $\{y_2, y_7\} \subseteq V_3$, $\{y_2, y_9\} \subseteq V_3$, $\{y_3, y_7\} \subseteq V_3$, $\{y_3, y_9\} \subseteq V_3$, or $\{y_7, y_9\} \subseteq V_3$, then $|V_3| \leq 3$, again a contradiction.

If $C = (11, 11, 4, 3, 2, 1, 1)$, then $V_2 = \{x_0, x_1, \ldots, x_{10}\}$ and V_4 is $\{y_0, y_3, y_6\}$, $\{y_0, y_3, y_7\}$ or $\{y_0, y_3, v_7\}$. It follows, respectively, that $\{v_0, y_1, y_2, v_3, y_4, y_5, v_6, y_7, y_{10}\} \subseteq V_1$, $\{v_0, y_1, y_2, v_3, y_4, y_6, v_7, y_8, y_{10}\} \subseteq V_1$, $\{v_0, y_1, y_2, v_3, y_4, y_7, y_{10}\} \subset V_1$. In any possibility, $|V_3| \neq 4$, a contradiction.

Claim 4. $(|V_2|, |V_3|, |V_4|) \neq (10, 5, 3)$ and $(|V_2|, |V_3|, |V_6|) \neq (10, 5, 2)$. In other words, $C \notin \{(10, 10, 5, 3, 2, 2, 1), (11, 10, 5, 3, 1, 2, 1), (11, 10, 5, 2, 2, 2, 1), (11, 10, 5, 3, 2, 1, 1)\}$.

Otherwise, we have $V_2 = \{x_1, x_2, \ldots, x_{10}\}$ and $V_3 = \{y_i, y_{i+2}, y_{i+4}, y_{i+6}, y_{i+8}\}$ for some $i \in \mathbb{Z}_{11}$. Cases $i = 0$ and $i = 3$ are similar. Same for $i = 1$ and $i = 2$; $i = 4$ and $i = 10$; $i = 5$ and $i = 9$; $i = 6$ and $i = 8$. Hence, assume that $i \in \{0, 1, 4, 5, 6, 7\}$. Assume, by symmetry, that V_3 is $\{x_0, y_2, y_4, y_6, y_8\} = V_3^{(1)}$, $\{y_1, y_3, y_5, y_7, y_9\} = V_3^{(2)}$, $\{y_4, y_6, y_8, y_{10}, y_1\} = V_3^{(3)}$, $\{y_5, y_7, y_9, x_0, y_2\} = V_3^{(4)}$, $\{y_6, y_8, y_{10}, y_1, y_3\} = V_3^{(5)}$, or $\{y_7, y_9, x_0, y_2, y_4\} = V_3^{(6)}$.

Case 1. $|V_6| = 2$.

Then, $V_6 = \{y_j, y_{j+5}\}$ for some $j \in \mathbb{Z}_{11}$. If V_3 is $V_3^{(1)}$, $V_3^{(2)}$, $V_3^{(3)}$, $V_3^{(4)}$, $V_3^{(5)}$, or $V_3^{(6)}$, then, respectively, $j \in \{0, 5, 7, 9\}$, $j \in \{6, 8, 10\}$, $j \in \{0, 2, 9\}$, $j \in \{1, 3, 6, 10\}$, $j \in \{0, 2, 4\}$, $j \in \{0, 1, 3, 5\}$. As $|V_1| \geq 10$, we have, respectively, $j \in \{0, 5\}$, $j = 6$, $j = 0$, $j = 6$, $j = 0$, $j = 0$. In any possibility, $|V_1| = 10$. So, $C = (10, 10, 5, 3, 2, 2, 1)$. (a) If $V_3 = V_3^{(1)}$ and $V_6 = \{y_0, y_5\}$, then $V_1 = \{v_0, y_1, v_2, y_3, v_4, v_6, y_7, v_8, y_9, y_{10}\}$. (b) If $V_3 = V_3^{(1)}$ and $V_6 = \{y_5, y_{10}\}$, then $V_1 = \{y_0, y_1, v_2, y_3, v_4, v_6, y_7, v_8, y_9, v_{10}\}$. (c) If $V_3 = V_3^{(2)}$ and $V_6 = \{y_6, y_0\}$, then $V_1 = \{x_0, v_1, y_2, v_3, y_4, v_5, v_7, y_8, v_9, y_{10}\}$. (d)

If $V_3 = V_3^{(3)}$ and $V_6 = \{y_0, y_5\}$, then $V_1 = \{x_0, v_1, y_2, y_3, v_4, v_6, y_7, v_8, y_9, v_{10}\}$. (e) If $V_3 = V_3^{(4)}$ and $V_6 = \{y_6, y_0\}$, then $V_1 = \{v_0, y_1, v_2, y_3, y_4, v_5, v_7, y_8, v_9, y_{10}\}$. (f) If $V_3 = V_3^{(5)}$ and $V_6 = \{y_0, y_5\}$, then $\{x_0, v_1, y_2, v_3, y_4, y_7, v_8, y_9, v_{10}\} \subseteq V_1$. (g) If $V_3 = V_3^{(6)}$ and $V_6 = \{y_0, y_5\}$, then $\{v_0, y_1, v_2, y_3, y_6, v_7, y_8, v_9, y_{10}\} \subseteq V_1$. In any possibility, $|V_5| \neq 2$, a contradiction.

Case 2. $|V_4| = 3$.

It follows from Case 1, $C = (11, 10, 5, 3, 2, 1, 1)$. Then, V_4 is $\{y_k, y_{k+3}, y_{k+6}\}$, $\{y_k, y_{k+3}, y_{k+7}\}$ or $\{y_k, y_{k+3}, v_{k+7}\}$ for some $k \in \mathbb{Z}_{11}$. (a) If $V_3 = V_3^{(1)}$, then V_4 is $\{y_0, y_3, y_7\}$, $\{y_7, y_{10}, y_3\}$, $\{y_9, y_1, y_5\}$, $\{y_0, y_3, v_7\}$, $\{y_7, y_{10}, v_3\}$, or $\{y_9, y_1, v_5\}$. (b) If $V_3 = V_3^{(2)}$, then V_4 is $\{y_8, y_0, y_4\}$, $\{y_{10}, y_2, y_6\}$, $\{y_8, y_0, v_4\}$, or $\{y_{10}, y_2, v_6\}$. (c) If $V_3 = V_3^{(3)}$, then V_4 is $\{y_0, y_3, y_7\}$, $\{y_2, y_5, y_9\}$, $\{y_0, y_3, v_7\}$, or $\{y_2, y_5, v_9\}$. (d) If $V_3 = V_3^{(4)}$, then V_4 is $\{y_0, y_3, y_6\}$, $\{y_8, y_0, y_3\}$, $\{y_1, y_4, y_8\}$, $\{y_3, y_6, y_{10}\}$, $\{y_8, y_0, y_4\}$, $\{y_0, y_3, v_7\}$, $\{y_1, y_4, v_8\}$, $\{y_3, y_6, v_{10}\}$, or $\{y_8, y_0, v_4\}$. (e) If $V_3 = V_3^{(5)}$, then V_4 is $\{y_2, y_5, y_9\}$, $\{y_4, y_7, y_0\}$, $\{y_2, y_5, v_9\}$, or $\{y_4, y_7, v_0\}$. (f) If $V_3 = V_3^{(6)}$, then V_4 is $\{y_0, y_3, y_6\}$, $\{y_5, y_8, y_0\}$, $\{y_8, y_0, y_3\}$, $\{y_3, y_6, y_{10}\}$, $\{y_5, y_8, y_1\}$, $\{y_0, y_3, v_7\}$, $\{y_3, y_6, v_{10}\}$, $\{y_5, y_8, v_1\}$, or $\{y_8, y_0, v_4\}$. In any possibility, $|V_1| \neq 11$, a contradiction.

Finally, we show that $C \notin \{(11, 10, 4, 3, 2, 2, 1), (11, 9, 5, 3, 2, 2, 1)\}$.

If $C = (11, 10, 4, 3, 2, 2, 1)$, then $V_2 = \{x_1, x_2, \ldots, x_{10}\}$ and $V_6 = \{y_i, y_{i+5}\}$ for some $i \in \mathbb{Z}_{11}$, By symmetry, assume that $i \in \{0, 1, 2, 3, 7, 8\}$. Hence, V_6 is $\{x_0, y_5\} = V_6^{(1)}$, $\{y_1, y_6\} = V_6^{(2)}$, $\{y_2, y_7\} = V_6^{(3)}$, $\{y_3, y_8\} = V_6^{(4)}$, $\{y_7, y_1\} = V_6^{(5)}$, or $\{y_8, y_2\} = V_6^{(6)}$. As $|V_4| = 3$, V_4 is $\{y_j, y_{j+3}, y_{j+6}\}$, $\{y_k, y_{k+3}, y_{k+7}\}$ or $\{y_\ell, y_{\ell+3}, v_{\ell+7}\}$, where $j, k, \ell \in \mathbb{Z}_{11}$. Clearly, for $V_6^{(1)}$, $j \notin \{2, 5, 10\}$, $k \notin \{2, 5, 9\}$ and $\ell \notin \{2, 5\}$; for $V_6^{(2)}$, $j \notin \{0, 1, 3, 6, 9\}$, $k \notin \{1, 3, 5, 6, 9, 10\}$ and $\ell \notin \{1, 3, 6, 9\}$; for $V_6^{(3)}$, $j \notin \{1, 2, 4, 7, 10\}$, $k \notin \{0, 2, 4, 6, 7, 10\}$ and $\ell \notin \{2, 4, 7, 10\}$; for $V_6^{(4)}$, $j \notin \{0, 2, 3, 5, 8\}$, $k \notin \{0, 1, 3, 5, 7, 8\}$ and $\ell \notin \{0, 3, 5, 8\}$; for $V_6^{(5)}$, $j \notin \{1, 4, 6, 7, 9\}$, $k \notin \{0, 1, 4, 5, 7, 9\}$ and $\ell \notin \{1, 4, 7, 9\}$; for $V_6^{(6)}$, $j \notin \{2, 5, 7, 8, 10\}$, $k \notin \{1, 2, 5, 6, 8, 10\}$ and $\ell \notin \{2, 5, 8, 10\}$. As $|V_1| = 11$, we have: for $V_6^{(1)}$, $j \notin \{0, 1, 3, 4, 6, 9\}$, $k \notin \{1, 3, 4, 6, 8, 10\}$ and $\ell \notin \{1, 3, 4, 6, 9\}$; for $V_6^{(2)}$, $j \notin \{2, 4, 5, 7, 10\}$, $k \notin \{0, 2, 4, 7\}$ and $\ell \notin \{2, 4, 5, 7, 10\}$; for $V_6^{(3)}$, $j \notin \{0, 3, 5, 6, 8, 9\}$, $k \notin \{1, 3, 5, 8, 9\}$ and $\ell \notin \{0, 1, 3, 5, 6, 8, 9\}$; for $V_6^{(4)}$, $j \notin \{1, 4, 6, 7, 9, 10\}$, $k \notin \{2, 4, 6, 9, 10\}$ and $\ell \notin \{1, 2, 4, 6, 7, 9, 10\}$; for $V_6^{(5)}$, $j \notin \{0, 2, 3, 5, 8, 10\}$, $k \notin \{2, 3, 6, 8, 10\}$ and $\ell \notin \{0, 2, 3, 5, 6, 8, 10\}$; for $V_6^{(6)}$, $j \notin \{0, 1, 3, 4, 6, 9\}$, $k \notin \{0, 3, 4, 7, 9\}$ and $\ell \notin \{0, 1, 3, 4, 6, 7, 9\}$. Hence, V_6 is $V_6^{(1)}$ or $V_6^{(2)}$.

First, consider $V_6^{(1)}$.

(a) If $V_4 = \{y_7, y_{10}, y_2\}$, then $V_1 = \{y_0, y_1, v_2, y_3, y_4, v_5, y_6, v_7, y_8, y_9, v_{10}\}$.
(b) If $V_4 = \{y_8, y_0, y_3\}$, then $V_1 = \{v_0, y_1, y_2, v_3, y_4, v_5, y_6, y_7, v_8, y_9, y_{10}\}$.
(c) If $V_4 = \{y_0, y_3, y_7\}$, then $\{v_0, y_1, y_2, v_3, y_4, v_5, y_6, v_7, y_8, y_{10}\} \subseteq V_1$.
(d) If $V_4 = \{y_7, y_{10}, y_3\}$, then $\{y_0, y_2, v_3, y_4, v_5, y_6, v_7, y_8, y_9, v_{10}\} \subseteq V_1$.
(e) If $V_4 = \{y_0, y_3, v_7\}$, then $\{v_0, y_1, y_2, v_3, y_4, v_5, y_6, y_7, y_{10}\} \subseteq V_1$.
(f) If $V_4 = \{y_7, y_{10}, v_3\}$, then $\{y_0, y_3, y_4, v_5, y_6, v_7, y_8, y_9, v_{10}\} \subseteq V_1$.
(g) If $V_4 = \{y_8, y_0, v_4\}$, then $\{v_0, y_1, y_4, v_5, y_6, y_7, v_8, y_9, y_{10}\} \subseteq V_1$.
(h) If $V_4 = \{y_{10}, y_2, v_6\}$, then $\{y_0, y_1, v_2, y_3, y_4, v_5, y_6, y_9, v_{10}\} \subseteq V_1$.

Next, consider $V_6^{(2)}$.

(i) If $V_4 = \{y_8, y_0, y_3\}$, then $\{x_0, v_1, y_2, v_3, y_4, y_5, v_6, y_7, v_8, y_9\} \subseteq V_1$.
(j) If $V_4 = \{y_8, y_0, y_4\}$, then $\{x_0, v_1, y_2, y_3, v_4, y_5, v_6, y_7, v_8, y_9\} \subseteq V_1$.
(k) If $V_4 = \{y_0, y_3, v_7\}$, then $\{x_0, v_1, y_2, v_3, y_4, y_5, v_6, y_7\} \subseteq V_1$.

(l) If $V_4 = \{y_8, y_0, v_4\}$, then $\{x_0, v_1, y_2, y_4, y_5, v_6, y_7, v_8, y_9\} \subseteq V_1$.
In any possibility, $|V_3| \neq 4$, a contradiction.

If $C = (11, 9, 5, 3, 2, 2, 1)$, then, since $|V_2| = 9$, we consider six cases. As $|V_3| = 5$, $V_3 = \{y_i, y_{i+2}, y_{i+4}, y_{i+6}, y_{i+8}\}$ for some $i \in \mathbb{Z}_{11}$. As $|V_4| = 3$, V_4 is $\{y_j, y_{j+3}, y_{j+6}\}$, $\{y_k, y_{k+3}, y_{k+7}\}$ or $\{y_\ell, y_{\ell+3}, v_{\ell+7}\}$, where $j, k, \ell \in \mathbb{Z}_{11}$.

Case A. $V_2 = \{v_0, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

Cases $i = 0$ and $i = 3$ are similar. Same for $i = 1$ and $i = 2$; $i = 4$ and $i = 10$; $i = 5$ and $i = 9$; $i = 6$ and $i = 8$. Hence, by symmetry, V_3 is $\{x_0, y_2, y_4, y_6, y_8\} = V_3^{(0)}$, $\{x_1, y_3, y_5, y_7, y_9\} = V_3^{(1)}$, $\{y_4, y_6, y_8, x_{10}, x_1\} = V_3^{(4)}$, $\{y_5, y_7, y_9, x_0, y_2\} = V_3^{(5)}$, $\{y_6, y_8, x_{10}, x_1, y_3\} = V_3^{(6)}$, or $\{y_7, y_9, x_0, y_2, y_4\} = V_3^{(7)}$. Clearly, for $V_3^{(0)}$, there is no j and $k, \ell \in \{0, 7, 9\}$; for $V_3^{(1)}$, there is no j and $k, \ell \in \{1, 8, 10\}$; for $V_3^{(4)}$, $j \in \{7, 10\}$, $k \in \{0, 2, 7, 9\}$ and $\ell \in \{0, 2, 7, 9, 10\}$; for $V_3^{(5)}$, $j \in \{0, 8\}$, $k \in \{1, 3, 8\}$ and $\ell \in \{0, 1, 3, 8\}$; for $V_3^{(6)}$, $j \in \{1, 4, 7, 9, 10\}$, $k \in \{2, 4, 9\}$ and $\ell \in \{1, 2, 4, 7, 9, 10\}$; for $V_3^{(7)}$, $j \in \{0, 5, 8\}$, $k \in \{3, 5\}$ and $\ell \in \{0, 3, 5, 8\}$. In any possibility, $|V_1| \neq 11$, a contradiction.

Case B. $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

For each $i \in \{1, 2, 3, 4, 5\}$, cases i and $11 - i$ are similar. Hence, by symmetry, V_3 is $\{y_0, y_2, y_4, y_6, y_8\} = V_3^{(0)}$, $\{y_1, y_3, y_5, y_7, x_9\} = V_3^{(1)}$, $\{y_2, y_4, y_6, y_8, x_{10}\} = V_3^{(2)}$, $\{y_3, y_5, y_7, x_9, y_0\} = V_3^{(3)}$, $\{y_4, y_6, y_8, x_{10}, y_1\} = V_3^{(4)}$, or $\{y_5, y_7, x_9, y_0, y_2\} = V_3^{(5)}$. Clearly, for $V_3^{(0)}$, there is no j and $k, \ell \in \{7, 9\}$; for $V_3^{(1)}$, there is no j and $k, \ell \in \{6, 8, 10\}$; for $V_3^{(2)}$, there is no j and $k, \ell \in \{0, 7, 9\}$; for $V_3^{(3)}$, $j \in \{6, 9\}$, $k \in \{1, 6, 8, 10\}$ and $\ell \in \{1, 6, 8, 9, 10\}$; for $V_3^{(4)}$, $j \in \{7, 10\}$, $k \in \{0, 2, 7\}$ and $\ell \in \{0, 2, 7, 10\}$; for $V_3^{(5)}$, $j \in \{3, 6, 9\}$, $k \in \{1, 3\}$ and $\ell \in \{1, 3, 6, 9\}$. In any possibility, $|V_1| \neq 11$, a contradiction.

Case C. $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9\}$.

For each $i \in \{0, 1, 2, 3, 4\}$, cases i and $10 - i$ are similar. Hence, by symmetry, V_3 is $\{y_0, y_2, y_4, y_6, x_8\} = V_3^{(0)}$, $\{y_1, y_3, y_5, y_7, y_9\} = V_3^{(1)}$, $\{y_2, y_4, y_6, x_8, x_{10}\} = V_3^{(2)}$, $\{y_3, y_5, y_7, y_9, y_0\} = V_3^{(3)}$, $\{y_4, y_6, x_8, x_{10}, y_1\} = V_3^{(4)}$, or $\{y_5, y_7, y_9, y_0, y_2\} = V_3^{(5)}$. Clearly, for $V_3^{(0)}$, there is no j and $k, \ell \in \{5, 7, 9\}$; for $V_3^{(1)}$, there is no j and $k, \ell \in \{8, 10\}$; for $V_3^{(2)}$, $j \in \{5, 8\}$, $k \in \{0, 5, 7, 9\}$ and $\ell \in \{0, 5, 7, 8, 9\}$; for $V_3^{(3)}$, there is no j and $k, \ell \in \{1, 10\}$; for $V_3^{(4)}$, $j \in \{2, 5, 7, 8, 10\}$, $k \in \{0, 2, 7\}$ and $\ell \in \{0, 2, 5, 7, 8, 10\}$; for $V_3^{(5)}$, there is no j and $k, \ell \in \{1, 3\}$; In any possibility, $|V_1| \neq 11$, a contradiction.

Case D. $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9\}$.

For each $i \in \{0, 1, 2, 3, 4\}$, cases i and $9 - i$ are similar. Hence, by symmetry, V_3 is $\{y_0, y_2, y_4, y_6, y_8\} = V_3^{(0)}$, $\{y_1, y_3, y_5, x_7, y_9\} = V_3^{(1)}$, $\{y_2, y_4, y_6, y_8, x_{10}\} = V_3^{(2)}$, $\{y_3, y_5, x_7, y_9, y_0\} = V_3^{(3)}$, $\{y_4, y_6, y_8, x_{10}, y_1\} = V_3^{(4)}$, or $\{x_{10}, y_1, y_3, y_5, x_7\} = V_3^{(10)}$. Clearly, for $V_3^{(0)}$, there is no j and $k, \ell \in \{7, 9\}$; for $V_3^{(1)}$, $j \in \{4, 7\}$, $k \in \{4, 8, 10\}$ and $\ell \in \{4, 7, 8, 10\}$; for $V_3^{(2)}$, there is no j and $k, \ell \in \{0, 7, 9\}$; for $V_3^{(3)}$, $j \in \{1, 4, 7\}$, $k \in \{1, 10\}$ and $\ell \in \{1, 4, 7, 10\}$; for $V_3^{(4)}$, $j \in \{7, 10\}$, $k \in \{0, 2, 7\}$ and $\ell \in \{0, 2, 7, 10\}$; for $V_3^{(10)}$, $j \in \{4, 7\}$, $k \in \{4, 6, 8, 10\}$ and $\ell \in \{4, 6, 7, 8, 10\}$. Except the following two possibilities, $|V_1| \neq 11$, a contradiction.

(a) $V_3 = \{y_0, y_2, y_4, y_6, y_8\} = V_3^{(0)}$ and $V_4 = \{y_7, y_{10}, v_3\}$.
Then, $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, v_6, x_7, v_8, y_9, x_{10}\}$.

(b) $V_3 = \{y_4, y_6, y_8, x_{10}, y_1\} = V_3^{(4)}$ and $V_4 = \{y_7, y_{10}, v_3\}$.

Then, $V_1 = \{y_0, v_1, y_2, y_3, v_4, y_5, v_6, x_7, v_8, y_9, v_{10}\}$.
In the above two possibilities, $|V_6| \neq 2$, a contradiction.
Case E. $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9\}$.

For each $i \in \{0,1,2,3\}$, cases i and $8-i$ are similar. Also, cases $i = 9$ and $i = 10$ are similar. Hence, by symmetry, V_3 is $\{y_0, y_2, y_4, x_6, y_8\} = V_3^{(0)}$, $\{y_1, y_3, y_5, y_7, y_9\} = V_3^{(1)}$, $\{y_2, y_4, x_6, y_8, x_{10}\} = V_3^{(2)}$, $\{y_3, y_5, y_7, y_9, y_0\} = V_3^{(3)}$, $\{y_4, x_6, y_8, x_{10}, y_1\} = V_3^{(4)}$, or $\{y_9, y_0, y_2, y_4, x_6\} = V_3^{(9)}$. Clearly, for $V_3^{(0)}$, $j \in \{3,6\}$, $k \in \{3,7,9\}$ and $\ell \in \{3,6,7,9\}$; for $V_3^{(1)}$, there is no j and $k, \ell \in \{8,10\}$; for $V_3^{(2)}$, $j \in \{0,3,6\}$, $k \in \{0,3,7,9\}$ and $\ell \in \{0,3,6,7,9\}$; for $V_3^{(3)}$, there is no j and $k,\ell \in \{1,10\}$; for $V_3^{(4)}$, $j \in \{0,3,7,10\}$ and $k,\ell \in \{0,2,3,6,7,10\}$; for $V_3^{(9)}$, there is no j and $k,\ell \in \{3,5,7\}$. Except the following possibility, $|V_1| \neq 11$, a contradiction.

$V_3 = \{y_9, y_0, y_2, y_4, x_6\} = V_3^{(9)}$ and $V_4 = \{y_7, y_{10}, v_3\}$.
Then, $V_1 = \{v_0, y_1, v_2, y_3, v_4, y_5, y_6, v_7, y_8, v_9, x_{10}\}$.
It follows, in this possibility, that $|V_6| \neq 2$, a contradiction.
Case F. $V_2 = \{x_0, x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_9\}$.

For each $i \in \{0,1,2,3\}$, cases i and $7-i$ are similar. Also, cases $i = 8$ and $i = 10$ are similar. Hence, by symmetry, V_3 is $\{y_0, y_2, y_4, y_6, y_8\} = V_3^{(0)}$, $\{y_1, y_3, x_5, y_7, y_9\} = V_3^{(1)}$, $\{y_2, y_4, y_6, y_8, x_{10}\} = V_3^{(2)}$, $\{y_3, x_5, y_7, y_9, y_0\} = V_3^{(3)}$, $\{y_8, x_{10}, y_1, y_3, x_5\} = V_3^{(8)}$, or $\{y_9, y_0, y_2, y_4, y_6\} = V_3^{(9)}$. Clearly, for $V_3^{(0)}$, there is no j and $k,\ell \in \{7,9\}$; for $V_3^{(1)}$, $j \in \{2,5,10\}$, $k \in \{8,10\}$ and $\ell \in \{2,5,8,10\}$; for $V_3^{(2)}$, there is no j and $k,\ell \in \{0,7,9\}$; for $V_3^{(3)}$, $j \in \{2,10\}$, $k \in \{1,5,10\}$ and $\ell \in \{1,2,5,10\}$; for $V_3^{(8)}$, $j \in \{4,7,10\}$, $k \in \{2,4,6,10\}$ and $\ell \in \{2,4,6,7,10\}$; for $V_3^{(9)}$, there is no j and $k,\ell \in \{5,7\}$. In any possibility, $|V_1| \neq 11$, a contradiction.

(ix) $n \geq 12$.

First, we find a packing 7-coloring for G_n. Let

$$V_1 = \begin{cases} \{y_0, v_1, y_2, v_3, y_4, v_5, \ldots, v_{n-2}, y_{n-1}\} & \text{if } n \text{ is odd,} \\ \{y_0, v_1, y_2, v_3, y_4, v_5, \ldots, y_{n-2}, v_{n-1}\} & \text{if } n \text{ is even,} \end{cases}$$

$V_2 = \{x_0, x_1, x_2, \ldots, x_{n-1}\}$
and

$$V_3 = \begin{cases} \{y_1, y_3, y_5, \ldots, y_{n-2}\} & \text{if } n \text{ is odd,} \\ \{y_1, y_3, y_5, \ldots, y_{n-1}\} & \text{if } n \text{ is even.} \end{cases}$$

Now, we have to color the vertices $v_0, v_2, v_4, \ldots, v_{n-1}$ if n is odd and $v_0, v_2, v_4, \ldots, v_{n-2}$ if n is even. Color these vertices with the following sequence of colors:
$4,5,6,7,\ 4,5,6,7,\ \ldots,\ 4,5,6,7$, if $n \equiv 0 \pmod 8$;
$4,5,6,4,5,7,\ 4,5,6,7,\ 4,5,6,7,\ \ldots,\ 4,5,6,7$, if $n \equiv 4 \pmod 8$;
$5,4,6,5,7,\ 4,5,6,7,\ 4,5,6,7,\ \ldots,\ 4,5,6,7$, if $n \equiv 2 \pmod 8$;
$4,5,6,4,5,7,\ 4,5,6,4,5,7,\ \ldots,\ 4,5,6,4,5,7,\ 4,5,6,7$, if $n \equiv 7 \pmod{12}$,
$4,5,6,4,5,7,\ 4,5,6,4,5,7,\ \ldots,\ 4,5,6,4,5,7,\ 4,5,6,7,\ 4,5,6,7$,
if $n \equiv 3 \pmod{12}$ and $n \neq 15$.
For $n \equiv 6 \pmod 8$, reset V_3 as $V_3 = \{y_1, y_3, y_5, \ldots, y_{n-5}\}$. Let $y_{n-3} \in V_6$, $y_{n-1} \in V_7$ and color the vertices
$v_0, v_2, v_4, v_6,\ v_8, v_{10}, v_{12}, v_{14},\ \ldots,\ v_{n-14}, v_{n-12}, v_{n-10}, v_{n-8},\ v_{n-6}, v_{n-4}, v_{n-2}$
with the sequence $4,5,6,7,\ 4,5,6,7,\ \ldots,\ 4,5,6,7,\ 4,5,3$.
For $n \equiv 1 \pmod 8$, color the vertices
$v_0, v_2, v_4, v_6,\ v_8, v_{10}, v_{12}, v_{14},\ \ldots,\ v_{n-17}, v_{n-15}, v_{n-13}, v_{n-11},\ v_{n-9}$

with the sequence $7, 5, 4, 6,\ 7, 5, 4, 6,\ \ldots,\ 7, 5, 4, 6,\ 7$;
$v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}$ with $4, 1, 5, 1, 6, 1, 4, 1$;
reset V_1 as $\{y_0, v_1, y_2, v_3, y_4, v_5, \ldots, y_{n-11}, v_{n-10}, y_{n-9}\} \cup \{y_{n-8}, y_{n-6}, y_{n-4}, y_{n-2}\}$ and V_3 as $\{y_1, y_3, y_5, \ldots, y_{n-10}\} \cup \{y_{n-7}, y_{n-5}, y_{n-3}, y_{n-1}\}$.
For $n \equiv 5 \pmod 8$ and $n \geq 29$, color the vertices
$v_0, v_2, v_4, v_6,\ v_8, v_{10}, v_{12}, v_{14},\ \ldots,\ v_{n-29}, v_{n-27}, v_{n-25}, v_{n-23}$,
$v_{n-21}, v_{n-19}, v_{n-17}, v_{n-15}, v_{n-13}, v_{n-11}, v_{n-9}, v_{n-7}, v_{n-5}, v_{n-3}, v_{n-1}$
with the sequence $7, 5, 4, 6,\ 7, 5, 4, 6,\ \ldots,\ 7, 5, 4, 6,\ 7, 4, 5, 6, 4, 5, 7, 4, 5, 6, 4$.

Next, in the remaining cases ($n \in \{13, 15, 21\}$, $n \equiv 11 \pmod{12}$), we find a packing 8-coloring for G_n. Take V_1, V_2, V_3 as above and color the vertices $v_0, v_2, v_4, \ldots, v_{n-1}$ with the following sequence of colors:
$4, 5, 6, 4, 5, 7, 8$, if $n = 13$;
$4, 5, 6, 7, 4, 5, 6, 8$, if $n = 15$;
$4, 5, 6, 4, 5, 7, 4, 5, 6, 7, 8$, if $n = 21$;
$4, 5, 6, 4, 5, 7,\ 4, 5, 6, 4, 5, 7,\ \ldots,\ 4, 5, 6, 4, 5, 7,\ 4, 5, 6, 4, 7, 8$, if $n \equiv 11 \pmod{12}$. □

4. Conclusion

We propose the following:
Conjecture. $\chi_\rho(P_{12} \odot K_2) \geq 7$.
The validity of this conjecture shows that $\chi_\rho(P_n \odot K_2) = 7$ for $n \geq 12$.
Problem. Compute $\chi_\rho(C_n \odot K_2)$ for $n \geq 12$.

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