TWMS J. App. and Eng. Math. V.16, N.1, 2026, pp. 109-122

ON CONTROLLABILITY RESULTS FOR FUZZY CAPUTO-KATUGAMPOLA FRACTIONAL DIFFERENTIAL EQUATIONS

R. HARIHARAN[1], R. UDHAYAKUMAR[1,*], §

Abstract. This article explores the controllability of fuzzy fractional differential equations using the Caputo-Katugampola fractional derivative. First, we prove the existence of a mild solution using fractional calculus, fuzzy set theory, semigroup theory, and the Caputo-Katugampola fractional derivative. The main results are obtained through a fixed-point theorem. Finally, we illustrate our findings with an example.

Keywords: Fuzzy fractional differential equation, Fixed point theorem, Controllability, CK fractional derivative.

AMS Subject Classification: 34A07, 47H08, 47H10.

1. Introduction

Fractional differential equations are an extension of classical differential equations. They use derivatives of non-integer orders. This allows them to represent memory and hereditary effects in systems. The mathematical foundations of fractional calculus, which underpin fractional differential equations, can be traced to works by pioneers such as Leibniz and Riemann. Modern advancements and applications have been detailed in several notable studies and books, such as Podlubny's [1] "Fractional Differential Equations", which provides a comprehensive introduction to the theory and methods of solving fractional differential equations, and for a deeper exploration of current research, articles by Kilbas et al. [2] discuss recent developments and applications of fractional differential equations across scientific disciplines. Iskakova et al. [3] introduced the novel 4D hyperchaotic system and investigate its dynamics using both integer-order and fractional-order derivatives. The fractional-order analysis is conducted using Caputo and Hilfer fractional derivatives, allowing for a deeper understanding of the system's long-term behavior. Fractional differential

[1] Department of Mathematics, School of Advanced Science, Vellore Institute of Technology, Vellore-632 014, Tamil Nadu, India.
e-mail: hariprf99@gmail.com; ORCID: https://orcid.org/0009-0009-3258-1878
e-mail: udhayaram.v@gmail.com; ORCID: https://orcid.org/0000-0002-7020-3466
[*] Corresponding author
§ Manuscript received: December 22, 2024; accepted: April 03, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.1;

equations have been successfully applied in diverse fields such as physics, engineering, biology, and finance, demonstrating their effectiveness in describing memory-dependent and hereditary processes [4, 5, 6].

Fractional calculus began in the late 17th century when Guillaume de l'Hôpital asked about the meaning of taking a derivative with an order of 1/2. This sparked interest, and by 1697, Leibniz explored the idea of fractional (or half-order) derivatives. Later, in the 19th century, Lacroix expanded on these ideas in his calculus book, introducing the concept of derivatives of any fractional order. The first practical application of fractional calculus came in 1823 when Abel used it to solve problems in mechanics, particularly involving the tautochrone problem (a problem related to the motion of a particle along a curve). This marked the beginning of fractional calculus being applied to real-world problems. More recently, there has been significant progress in fractional differential equations, with major contributions from Miller and Ross [7], Lakshmikantham et al. [8], Zhou [9], and Podlubny [1], advancing both theory and applications in science and engineering. Fuzzy numbers, initially introduced by Chang and Zadeh [10], provide a key approach to representing uncertainty in mathematical models.

The concept of fuzzy fractional differential equations was first introduced by Agarwal et al. [11], and since then, it has been widely explored in terms of its foundations, applications, and solution techniques. Arshad [12] examined the characteristics of fuzzy fractional differential equations, focusing on their existence and uniqueness through fuzzy integral equivalent equations. Salahshour [13] and his team took a different approach by applying the fuzzy Laplace transform of the Riemann-Liouville fractional H-derivative to solve these equations. Meanwhile, Allahviranloo et al. [14] studied fuzzy fractional differential equations using the Caputo fractional gH-derivative, with a particular emphasis on their existence and uniqueness properties.

Katugampola [[15, 16]] introduced fractional operators, referred to as the Katugampola fractional integral and derivative. These operators are characterized by an additional parameter $\varrho > 0$. When ϱ tends toward 0_+, the operators become equivalent to the Hadamard fractional operators, and for $\varrho = 1$, they match the Riemann-Liouville fractional operators. This parameter simplifies the theoretical framework, as proving results for the Katugampola derivative also covers the Riemann-Liouville and Hadamard derivatives.

Katugampola [17] investigated the existence and uniqueness of solutions for fractional differential equations involving the Caputo-Katugampola derivative, employing Schauder's second fixed-point theorem. Almeida et al. [18] studied the existence and uniqueness theorem for an initial value problem related to Caputo-Katugampola fractional differential equations and proposed a numerical method to solve it. Zeng et al. [19] introduced a discrete form of the Caputo-Katugampola derivative and presented a numerical approach for solving linear fractional differential equations involving this derivative. Finally, Baleanu et al. [20] explored the chaotic dynamics and stability of fractional differential equations using the Caputo-Katugampola derivative.

Controllability is a fundamental concept in control theory that assesses whether a dynamic system can be steered from one state to another using a set of admissible control inputs within a finite time. It is a critical property for designing effective control strategies and ensuring the desired behavior of systems in various applications, such as robotics, economics, engineering, and biology. Controllability plays a key role in system design, ensuring that desired objectives, such as stabilization or trajectory tracking, can be achieved. This concept is closely related to other properties like observability and stability, forming the foundation for analyzing and designing dynamic systems.

Vikram Singh [21] addressed the challenges of establishing controllability for fractional systems where the domain of the governing operator is non-dense. Ullah et al. [22] eaddressed such uncertainties; fuzzy fractional differential equations provided a powerful mathematical framework by incorporating both fractional calculus and fuzzy set theory. Ahmad et al. [23] focused on finding semi-analytical solutions for third-order fuzzy dispersive partial differential equations under fractional operators. Recently, Zhang et al. [24] studied the controllability of a specific class of systems defined by Sobolev-type structures, Fuzzy logic, and Hilfer fractional derivatives. Ullah et al. [25] introduced the Fuzzy Yang Transform as an analytical technique for solving second-order fuzzy differential equations of both integer and fractional order. The Yang transform, a relatively recent integral transform method, was extended to the fuzzy domain to obtain solutions in a more efficient and systematic manner.

Van Hoa et al. [26] proposed a novel concept of fuzzy fractional derivatives and examined the existence and uniqueness of solutions for an initial value problem related to Caputo-Katugampola fuzzy fractional differential equations.

Based on the discussion above, this paper explores the existence of a mild solution and controllability results for a fuzzy fractional differential equation incorporating the Caputo-Katugampola fractional derivative, with the following initial condition:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha;\varrho}\widetilde{y}(t) = A\widetilde{y}(t) + \mathcal{B}\mathtt{u}(t) + f(t,\widetilde{y}(t)), \ t \in [0,\mathcal{P}] = U, \\ \widetilde{y}(0) = \widetilde{y}_0, \end{cases} \tag{1}$$

where ${}^{C}D_{0^+}^{\alpha;\varrho}$ is a Caputo-Katugampola fractional derivative of order $0 < \alpha < 1$. The state variable $\widetilde{y}(\cdot)$ takes on values within the space of all fuzzy numbers $\mathbb{F}$ and A fuzzy number is defined as a fuzzy set $\widetilde{y} : \mathbb{R} \to [0,1]$. $A : \mathrm{Dom}(A) \subseteq \mathbb{B} \to \mathbb{B}$(Banach space) is the infinitesimal generator of a C_0-semigroup consisting of uniformly bounded linear operators $J(t)_{t\geq 0}$ on the space $\mathbb{B}$. $f : U \times \mathbb{F} \to \mathbb{F}$ is a fuzzy valued function. $\mathcal{B} : \mathbb{E} \to \mathbb{B}$ is a bounded linear operator and $\mathtt{u}(\cdot) \in \mathbb{L}^2(U,\mathbb{E})$ is a control fuction.

The key advantage and contribution of this article can be summarized as follows:

(1) In this manuscript, we explore on controllability results for fuzzy fractional differential equations via Caputo-Katugampola derivative.
(2) A mild solution to system (1) is constructed using the Wright function and Laplace transform.
(3) Using Krasnoselskii's fixed point theorem, we proved the existence of a fixed point for a mild solution.
(4) An illustrative example is presented to highlight the proposed results.

The structure of the manuscript is as follows: In section 2, we discuss the fundamental concepts of fuzzy fractional calculus pertinent to our study. Section 3 offers the proof of the controllability. Section 4 presents an example to enhance understanding. Finally, section 5 concludes the paper.

2. Preliminaries

Let $\mathbb{F}$ be a fuzzy number space and $\mathbb{C}(U,\mathbb{F})$ is a space of all fuzzy numbers of all continuous functions form $\widetilde{y} : U \to \mathbb{F}$ with

$$\|\widetilde{y}\|_{\mathbb{C}} = \sup_{t\in U} \|\widetilde{y}(t)\| \text{ for } \widetilde{y} \in \mathbb{C}(U,\mathbb{F}),$$

Consider $\mathbb{E}$ as a closed subspace within $\mathbb{B}$, and $\mathbb{L}^2(U,\mathbb{E})$ denote the set of all Lebesgue square-integrable functions from U to $\mathbb{E}$.

$$\|e\|_{\mathbb{L}^2} = \left(\int_0^u \|e(t)\|^2 dt\right)^{\frac{1}{2}} \text{ for } e \in \mathbb{L}^2(U,\mathbb{E}).$$

It is clear that $\mathbb{L}^2(U,\mathbb{E})$ is also a Banach Space.

Remark 2.1. *Given that $\widetilde{y} \in \mathbb{R}$, $\widetilde{y}$ is regarded as a specific point in $\mathbb{F}$. We define $\mathcal{C}_{\mathbb{R}}$ as the collection of all nonempty, compact, and convex subsets of $\mathbb{R}$. For any elements $\widetilde{y}_i, \widetilde{y}_j \in \mathcal{C}_{\mathbb{R}}$, the Hausdorff distance between $\widetilde{y}_i$ and $\widetilde{y}_j$ is defined as:*

$$d\left(\widetilde{y}_i, \widetilde{y}_j\right) = \max\left\{\sup_{f_1\in\widetilde{y}_i}\inf_{f_2\in\widetilde{y}_j}\|f_1 - f_2\|, \sup_{f_2\in\widetilde{y}_j}\inf_{f_1\in\widetilde{y}_i}\|f_1 - f_2\|\right\}.$$

The fundamental operations are defined as follows:

The operation $\left(\widetilde{y}_i \oplus \widetilde{y}_j\right)(t)$ is determined by:

$$\left(\widetilde{y}_i \oplus \widetilde{y}_j\right)(t) = \sup_{t_i+t_j=t}\min\{\widetilde{y}_i(t_i), \widetilde{y}_j(t_j)\}.$$

The operation $\left[\beta\widetilde{y}\right](t)$ is given by:

$$\left[\beta\widetilde{y}\right](t) = \begin{cases} \widetilde{y}\left(\frac{1}{\beta}t\right), & \text{if } \beta \neq 0; \\ 1, & \text{if } \beta = 0 \text{ and } t = 0; \\ 0, & \text{if } \beta = 0 \text{ and } t \neq 0. \end{cases}$$

Here, β denotes an arbitrary value in $\mathbb{R}$.

Definition 2.1. [27] *We define the mapping $\boldsymbol{H}$ as follows:*

$$\boldsymbol{H}: \mathbb{F}\times\mathbb{F} \to [0,+\infty),\ (\widetilde{y}_1, \widetilde{y}_2) \to \boldsymbol{H}(\widetilde{y}_1, \widetilde{y}_2) = \sup_{t\in U} d\big(\widetilde{y}_1, \widetilde{y}_2\big),$$

for $\widetilde{y} \in \mathbb{F}$. Here, the norm is defined by $\|\cdot\|_{\mathbb{F}}$ on $\mathbb{F}$ as follows:

$$\|\widetilde{y}\|_{\mathbb{F}} = \boldsymbol{H}(\widetilde{y}, 0).$$

The metric space $(\mathbb{F}, \boldsymbol{H})$ is complete (see [27]).

To formulate the Katugampola fractional integral, we need to introduce several special functions that play a crucial role in their definitions and computations.

Definition 2.2. [28] *The katugampola left-sided fractional integral of order α with the lower limit u of $\widetilde{y} \in \mathbb{C}(U,\mathbb{F})$ for $-\infty < 0 < t < \infty$ is defined by*

$$I^{\alpha,\varrho}_{0^+}\widetilde{y}(t) = \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\varrho-1}}{(t^\varrho - s^\varrho)^{1-\alpha}}\widetilde{y}(s)ds,\ t > b;\ \varrho > 0, \alpha > 0.$$

The Katugampola fractional integral is defined with respect to an additional parameter $\varrho > 0$. These operators have special properties based on the value of ϱ.

Remark 2.2. [28] *Specifically, as $\varrho \to 0^+$, the Katugampola fractional integral converge to the Hadamard fractional integral,*

$$\lim_{\varrho\to 0} I^{\alpha,\varrho}_{0^+}\widetilde{y}(t) = \int_0^t \frac{\left(\log\frac{t}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}\widetilde{y}(s)\frac{ds}{s},$$

when the parameter $\varrho = 1$, they coincide with the Riemann-Liouville fractional integral,

$$I^{\alpha,1}_{0^+}\widetilde{y}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\widetilde{y}(s)}{(t-s)^{1-\alpha}}ds.$$

Definition 2.3. [26] *Let order $0 < \alpha < 1$, the Caputo-Katugampola fractional derivative is given by*

$$^{C}D_{0_+}^{\alpha;\varrho}\widetilde{y}(t) = \frac{\varrho^{\alpha}}{\Gamma(1-\alpha)}\int_0^t (t^{\varrho} - s^{\varrho})^{-\alpha}\widetilde{y}^{'}(s)ds.$$

Where $\varrho > 0$ is a constant real number, and $\widetilde{y}^{'}$ represents the generalized Hukuhara derivative of the fuzzy function $\widetilde{y}$.

Note:-

When $\varrho = 1$, the Caputo-Katugampola fuzzy fractional derivative reduces to the well-known Caputo fuzzy fractional generalized Hukuhara derivative, and if $\varrho = 0^+$, it becomes the Caputo-Hadamard fuzzy fractional generalized Hukuhara derivative.

Lemma 2.1. [29] *Assume that the linear operator A acts as the infinitesimal generator of a C_0-semigroup if and only if*

- *The set A has the property of being closed and $D(A) = \mathbb{B}$,*
- *The resolvent set $p(A)$ of A includes positive real numbers, $\forall\ \alpha > 0$,*

$$\|\mathscr{R}(\beta, A)\| \leq \frac{1}{\beta},$$

where $\mathscr{R}(\beta, A) = (\beta^q I - A)^{-1}s = \int_0^{\infty} e^{-\beta^q t}J(t)sdt$.

Definition 2.4. [29] *Defined as a Wright-type function*

$$\mathcal{M}_{\alpha}(\delta) = \sum_{u=0}^{\infty}\frac{(-x)^u}{u!\Gamma(-\alpha u + 1 - \alpha)},\ x \in \boldsymbol{C}$$

Proposition 2.1. [29] *The Wright-type function $\mathcal{M}_{\alpha}$ is an entire function with satisfy the succeeding conditions:*

- ❖ $\mathcal{M}_{\alpha}(\delta) \geq 0$ *for* $\theta \geq 0$, $\int_0^{\infty}\mathcal{M}_{\alpha}(\delta)d\delta = 1$;
- ❖ $\int_0^{\infty}\mathcal{M}_{\alpha}(\delta)\delta^u d\delta = \frac{\Gamma(1+u)}{\Gamma(1+\alpha u)}$, *for* $u > -1$;
- ❖ $\int_0^{\infty}\mathcal{M}_{\alpha}(\delta)e^{x\delta}d\delta = E_{\alpha}(-x),\ x \in \boldsymbol{C}$.

Lemma 2.2. [26] *The system (1) is written as the following integral equation:*

$$\widetilde{y}(t) = \widetilde{y}_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\big[A\widetilde{y}(s) + \mathcal{B}\mathrm{u}(t) + f(s, \widetilde{y}(s))\big]ds \tag{2}$$

Proof. □

For any $\widetilde{y} \in \mathbb{F}$, we define the operators $\mathcal{R}_{\alpha}\left(\frac{t^{\varrho}}{\varrho}\right)$ and $\mathcal{H}_{\alpha}\left(\frac{t^{\varrho}}{\varrho}\right)$ by

$$\mathcal{R}_{\alpha}\left(\frac{t^{\varrho}}{\varrho}\right) = \int_0^{\infty}\mathcal{M}_{\alpha}(\delta)J\left(\left(\frac{t^{\varrho}}{\varrho}\right)^{\alpha}\delta\right)d\delta$$

$$\mathcal{H}_{\alpha}\left(\frac{t^{\varrho}}{\varrho}\right) = \alpha\int_0^{\infty}\delta\mathcal{M}_{\alpha}(\delta)J\left(\left(\frac{t^{\varrho}}{\varrho}\right)^{\alpha}\delta\right)d\delta$$

Where $\mathcal{M}_{\alpha}(\delta)$ is a probability density function and is given as follows

$$\mathcal{M}_{\alpha}(\delta) = \frac{1}{\alpha}\delta^{-1-\frac{1}{\alpha}}\widetilde{y}_{\alpha}(\delta^{-\frac{1}{\alpha}}) \leq 0,\ \text{for } 0 < \alpha < 1, \delta \in [0, +\infty),$$

where

$$\widetilde{y}_{\alpha}(\delta) = \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}\delta^{-m\alpha-1}\frac{\Gamma(1+m\alpha)}{m!}\sin(m\pi\alpha).$$

Definition 2.5. [30] *A function* $\widetilde{y} \in \mathbb{C}(U, \mathbb{F})$ *is called mild solution of equation* (1) *if satisfies*

$$\widetilde{y}(t) = \mathcal{R}_\alpha\Big(\frac{t^\varrho}{\varrho}\Big)\widetilde{y}_0 + \int_0^t \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)\mathcal{B}\mathtt{u}(s)ds + \int_0^t \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)f(s, \widetilde{y}(s))ds \qquad (3)$$

Definition 2.6. [30] *The operator* $\mathcal{R}_\alpha(\frac{t^\varrho}{\varrho})$ *and* $\mathcal{H}_\alpha(\frac{t^\varrho}{\varrho})$ *have the following properties:*

- $\{\mathcal{R}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ *and* $\{\mathcal{H}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ *are compact operators, bounded, and linear, therefore we have:*

$$\Big\|\mathcal{R}_\alpha\Big(\frac{t^\varrho}{\varrho}\Big)\widetilde{y}\Big\| \le L\|\widetilde{y}\| \text{ and } \Big\|\mathcal{H}_\alpha\Big(\frac{t^\varrho}{\varrho}\Big)\widetilde{y}\Big\| \le \frac{L}{\Gamma(\alpha)}\|\widetilde{y}\|, \text{ with } L > 0 \text{ and } \widetilde{y} \in \mathbb{C}(U, \mathbb{F}).$$

- $\{\mathcal{R}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ *and* $\{\mathcal{H}_\alpha(\frac{t^\varrho}{\varrho})\}_{t>0}$ *are strongly continuous operators,* $\forall\ t_1, t_2 \in U$, *we have*

$$\Big\|\mathcal{R}_\alpha\Big(\frac{t_2^\varrho}{\varrho}\Big)\widetilde{y} - \mathcal{R}_\alpha\Big(\frac{t_1^\varrho}{\varrho}\Big)\widetilde{y}\Big\| \to 0, \ \Big\|\mathcal{H}_\alpha\Big(\frac{t_2^\varrho}{\varrho}\Big)\widetilde{y} - \mathcal{H}_\alpha\Big(\frac{t_1^\varrho}{\varrho}\Big)\widetilde{y}\Big\| \to 0, \text{ as } t_2^\varrho \to t_1^\varrho.$$

Definition 2.7. [24] *The system* (1) *is said to be controllable on* U, *if* $\forall\ \widetilde{y}_0, \widetilde{y}_1 \in \mathbb{C}(U, \mathbb{F})$, *then there exist a control function* $y(\cdot) \in \mathbb{L}^2(U, \mathbb{E})$ *such that* $\widetilde{y}(\mathcal{P}) = \widetilde{y}_1$, *where* $\widetilde{y}(\cdot)$ *is a mild solution of* (1).

Theorem 2.1. *Let* $M_n(t)$ *be a sequence of functions from [a,b] to* $\mathbb{R}$ *which is uniformly bounded and equicontinuous. Then,* $M_n(t)$ *has a uniformly convergent subsequence.*

Theorem 2.2. [31] *Consider the Banach space* $\mathbb{G}$. *If* $\boldsymbol{M}, \boldsymbol{N} : D \to \mathbb{G}$, *then* D *is a closed, bounded, and convex subset of a Banach space* $\mathbb{G}$ *such that*

(i) $\boldsymbol{M}h + \boldsymbol{N}g \in \mathcal{B}$ $\forall$ *pair of* $h, g \in D$,
(ii) $\boldsymbol{M}$ *is contraction mapping,*
(iii) $\boldsymbol{N}$ *is compact and continuous,*

then $\boldsymbol{M}(h) + \boldsymbol{N}(h) = h$ *has a solution in* D.

3. Controllability

The next results will be based on the following assumptions:

(K1) $\{J(t)\}_{\varrho\ge 0}$ is the C_0-semigroup, such that

$$\sup_{\varrho\in[0,\infty)} \|J(t)\| = L_E.$$

(K2) The function $f : \mathbb{F} \to \mathbb{F}$ is almost continuous $\forall\ \widetilde{y} \in \mathbb{C}(U, \mathbb{F})$, and the function $f : U \to \mathbb{F}$ is strongly measurable $\forall\ t \in U$.

(K3) There exist $\mathcal{Q}_f \in L^\infty(U, \mathbb{R}_+)$ and a function $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ continuous nondecreasing, such that:

$$\big\|f(t, \widetilde{y}(t))\big\| \le \mathcal{Q}_f(t)\Omega\big\|\widetilde{y}(t)\big\|, \text{ for a.e. } t \in U \text{ and } \forall\ \widetilde{y} \in \mathbb{C}(U, \mathbb{F}).$$

(K4) $\mathcal{B} : \mathbb{E} \to \mathbb{B}$ is bounded and $\mathtt{u}(\cdot) \in \mathbb{L}^2(U, \mathbb{E})$ is a control function. The linear operator $\mathcal{W} : \mathbb{L}^2(U, \mathbb{E}) \to \mathbb{B}$ is given as

$$\mathcal{W}\mathtt{u} = \int_0^{\mathcal{P}} \Big(\frac{\mathcal{P}^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\Big(\frac{\mathcal{P}^\varrho - s^\varrho}{\varrho}\Big)\mathcal{B}\mathtt{u}(s)ds.$$

The inverse operator $\mathcal{W}^{-1}$ of $\mathcal{W}$ assumes values in $\mathbb{L}^2(U,\mathbb{E})/\ker\mathcal{W}$ and there exist $L_B, L_W > 0$ such that:

$$\|\mathcal{B}\| \leq L_B, \ \|\mathcal{W}^{-1}\| \leq L_W.$$

For calculative convenience, we consider the following:

$$\Delta = \frac{t^{\alpha\varrho}\mathbf{B}(1,\alpha)}{\Gamma(\alpha)\varrho^{\alpha}}, \ \Im = \sup_{t\in U}\|\widetilde{y}_0\| \text{ and } \mathcal{Q}_f^* = \sup_{t\in U}\mathcal{Q}_f(t).$$

Theorem 3.1. *Let the hypotheses* $(K1)-(K4)$ *are Satisfied. If* $\Delta\mathcal{Q}_f^* < 1$*, then the fuzzy fractional differential equation* (1) *has mild solution and it is controllable on* U*.*

Proof. we define $B_{\mathcal{J}} = \{\widetilde{y} \in \mathbb{C}(U,\mathbb{F}) : \|\widetilde{y}\| \leq \mathcal{J}\}$, it is clear that $B_{\mathcal{J}}$ is closed, bounded and convex subset of $\mathbb{C}(U,\mathbb{F})$ with $\forall\ \mathcal{J} > 0$, such that:

$$L\Bigg[\Delta L_B\widetilde{y}_1 + \bigg(1-\Delta L_B\bigg)\bigg(\Im + \Delta\mathcal{Q}_f^*\Omega(\mathcal{J})\bigg)\Bigg] \leq \mathcal{J}.$$

From definition 2.7 and $(K4)$, We determine the control operator $\mathtt{u}(t)$ as follows:

$$\mathtt{u}(t) = \mathcal{W}^{-1}\Bigg[\widetilde{y}_1 - \mathcal{R}_\alpha\bigg(\frac{t^\varrho}{\varrho}\bigg)\widetilde{y}_0 - \int_0^t\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)^{\alpha-1}s^{\varrho-1}\mathcal{H}_\alpha\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)f(s,\widetilde{y}(s))ds\Bigg]$$

Now, we consider $\Xi : \mathbb{C}(U,\mathbb{F}) \to \mathbb{C}(U,\mathbb{F})$ defined by:

$$\begin{aligned}\Xi\widetilde{y}(t) =& \mathcal{R}_\alpha\bigg(\frac{t^\varrho}{\varrho}\bigg)\widetilde{y}_0 + \int_0^t\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)^{\alpha-1}s^{\varrho-1}\mathcal{H}_\alpha\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)\mathcal{B}\mathtt{u}(s)ds \\ &+ \int_0^t\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)^{\alpha-1}s^{\varrho-1}\mathcal{H}_\alpha\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)f(s,\widetilde{y}(s))ds.\end{aligned}$$

Next, we verify that Ξ contains a fixed point. The operator Ξ can be separated into two parts. (i.e)., $\Xi = \Xi_1 + \Xi_2$.

Where,

$$\begin{aligned}\Xi_1\widetilde{y}(t) =& \mathcal{R}_\alpha\bigg(\frac{t^\varrho}{\varrho}\bigg)\widetilde{y}_0 \\ \Xi_2\widetilde{y}(t) =& \int_0^t\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)^{\alpha-1}s^{\varrho-1}\mathcal{H}_\alpha\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)\mathcal{B}\mathtt{u}(s)ds \\ &+ \int_0^t\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)^{\alpha-1}s^{\varrho-1}\mathcal{H}_\alpha\bigg(\frac{t^\varrho - s^\varrho}{\varrho}\bigg)f(s,\widetilde{y}(s))ds.\end{aligned}$$

Step-1 The operator Ξ maps the set $B_{\mathcal{J}}$ into itself. Let $\widetilde{y}(t) \in B_{\mathcal{J}}$, $\forall t \in U$ we have:

we know that

$$
\begin{aligned}
\|\Xi\widetilde{y}(t)\|_{\mathbb{C}} =& \bigg\|\mathcal{R}_\alpha\Big(\frac{t^\varrho}{\varrho}\Big)\widetilde{y}_0 + \int_0^t \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)\mathcal{B}\mathtt{u}(s)ds \\
&+ \int_0^t \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big) f(s,\widetilde{y}(s))ds\bigg\| \\
\le& \bigg\|\mathcal{R}_\alpha\Big(\frac{t^\varrho}{\varrho}\Big)\widetilde{y}_0\bigg\| + \bigg\|\int_0^t \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)\mathcal{B}\mathtt{u}(s)ds\bigg\| \\
&+ \bigg\|\int_0^t \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big) f(s,\widetilde{y}(s))ds\bigg\| \\
\le& L\bigg[\|\widetilde{y}_0\| + \int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\Gamma(\alpha)\varrho^{\alpha-1}} s^{\varrho-1}\|\mathcal{B}\mathtt{u}(s)\|ds \\
&+ \int_0^t \frac{(t^\varrho - s^\varrho)^{\alpha-1}}{\Gamma(\alpha)\varrho^{\alpha-1}} s^{\varrho-1}\|f(s,\widetilde{y}(s))\|ds\bigg] \\
\le& L\|\widetilde{y}_0\| + L\Delta L_B\|\mathtt{u}(s)\| + L\Delta\|f(s,\widetilde{y}(s))\| \\
\le& L\bigg[\|\widetilde{y}_0\| + \Delta L_B\Big(\widetilde{y}_1 - L\|\widetilde{y}_0\| - L\Delta\|f(s,\widetilde{y}(s))\|\Big) + \Delta\|f(s,\widetilde{y}(s))\|\bigg]
\end{aligned}
$$

by using $(K3)$ $\forall t \in U$, we have

$$\big\|f(t,\widetilde{y}(t))\big\| \le \mathcal{Q}_f(t)\Omega\big\|\widetilde{y}(t)\big\|.$$

Hence,

$$\|\Xi\widetilde{y}(t)\|_{\mathbb{C}} \le L\bigg[\Delta L_B\widetilde{y}_1 + \Big(1 - \Delta L_B\Big)\Big(\Im + \Delta\mathcal{Q}_f^*\Omega(\mathcal{J})\Big)\bigg] \le \mathcal{J}.$$

This implies that:

$$\|\Xi\widetilde{y}(t)\|_{\mathbb{C}} \le \mathcal{J}.$$

This demonstrates that Ξ maps the ball $B_{\mathcal{J}}$ onto itself, implying that Ξ is bounded.

Step-2 In this case we prove that Ξ_1 satisfied a contraction on $B_{\mathcal{J}}$. We assume $\widetilde{y}^*$, $\widetilde{y}^{**} \in B_{\mathcal{J}}$.

If $t \in U$, we get

$$
\begin{aligned}
\|\Xi_1\widetilde{y}^*(t) - \Xi_1\widetilde{y}^{**}(t)\|_{\mathbb{C}} &= \bigg\|\mathcal{R}_\alpha\Big(\frac{t^\varrho}{\varrho}\Big)\widetilde{y}_0^* - \mathcal{R}_\alpha\Big(\frac{t^\varrho}{\varrho}\Big)\widetilde{y}_0^{**}\bigg\| \\
&\le L\bigg\|\frac{1}{\Gamma(\alpha)}\widetilde{y}_0^* - \frac{1}{\Gamma(\alpha)}\widetilde{y}_0^{**}\bigg\| \\
&\le \frac{L}{\Gamma(\alpha)}\|\widetilde{y}_0^* - \widetilde{y}_0^{**}\| \\
&\le \hat{\Phi}\|\widetilde{y}_0^* - \widetilde{y}_0^{**}\|, \text{ where } \hat{\Phi} = \frac{L}{\Gamma(\alpha)}.
\end{aligned}
$$

We see that $\hat{\Phi} < 1$. Thus, Ξ_1 is a contraction.

Step-3 Next, we demonstrate that Ξ_2 is continuous. We assume that $\widetilde{y}_n \in B_{\mathcal{J}}$, such that $\widetilde{y}_n \to \widetilde{y}$ as $n \to \infty$. We need to prove that $\|\Xi_2\widetilde{y}_n - \Xi_2\widetilde{y}\| \to 0$ as $n \to \infty$. By using $(K2)$, It is clear that

$$f(s,\widetilde{y}_n(s)) \to f(s,\widetilde{y}(s)) \text{ as } n \to \infty. \tag{4}$$

If the equation (4) is satisfied, the control term automatically becomes zero as $n \to \infty$. (i.e).,

$$\mathrm{u}_n(s) \to \mathrm{u}(s) \text{ as } n \to \infty. \tag{5}$$

On the other hand, taking $(K3)$ into consideration, we get the following inequality:

$$\|f(s, \widetilde{y}_n(s)) - f(s, \widetilde{y}(s))\| \le 2\mathcal{Q}_f \Omega(\mathcal{J}).$$

By the Lebesgue dominated convergence theorem implies that:

$$\begin{aligned}
\|\Xi_2\widetilde{y}_n - \Xi_2\widetilde{y}\|_{\mathbb{C}} = & \bigg\| \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)\mathcal{B}\mathrm{u}_n(s)ds \\
& + \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) f_n(s, \widetilde{y}(s))ds \\
& - \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)\mathcal{B}\mathrm{u}(s)ds \\
& - \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) f(s, \widetilde{y}(s))ds \bigg\| \\
\le & \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right) L_B\|\mathrm{u}_n(s) - \mathrm{u}(s)\|ds \\
& + \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\mathcal{H}_\alpha\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)\|f_n(s, \widetilde{y}(s)) - f(s, \widetilde{y}(s))\|ds \\
\le & \frac{LL_B}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\|\mathrm{u}_n(s) - \mathrm{u}(s)\|ds \\
& + \frac{L}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\|f_n(s, \widetilde{y}(s)) - f(s, \widetilde{y}(s))\|ds \\
\le & L\Delta L_B\|\mathrm{u}_n(s) - \mathrm{u}(s)\| + L\Delta\|f_n(s, \widetilde{y}(s)) - f(s, \widetilde{y}(s))\|.
\end{aligned}$$

Applying the equations (4) and (5) to the above inequality, we get $\|\Xi_2\widetilde{y}_n - \Xi_2\widetilde{y}\| \to 0$ as $n \to 0$. Thus the operator Ξ_2 is continuous.

Step-4 We now proceed to demonstrate that Ξ_2 is equicontinuous for each $t \in U$.

For any $\widetilde{y} \in B_{\mathcal{J}}$ and $0 \leq t_1 \leq t_2 \leq \mathcal{P}$,

$$\begin{aligned}
\|\Xi_2\widetilde{y}(t_2) - \Xi_2\widetilde{y}(t_1)\|_{\mathbb{C}} \leq & \frac{LL_B}{\Gamma(\alpha)}\Big\|\int_0^{t_2} \frac{(t_2^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}\mathtt{u}(s)ds - \int_0^{t_1} \frac{(t_1^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}\mathtt{u}(s)ds\Big\| \\
& + \frac{L}{\Gamma(\alpha)}\Big\|\int_0^{t_2} \frac{(t_2^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1} f(s,\widetilde{y}(s))ds \\
& - \int_0^{t_1} \frac{(t_1^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1} f(s,\widetilde{y}(s))ds\Big\| \\
\leq & \frac{LL_B}{\Gamma(\alpha)}\int_0^{t_1} \frac{[(t_2^\varrho - s^\varrho)^{\alpha-1} - (t_1^\varrho - s^\varrho)^{\alpha-1}]}{\varrho^{\alpha-1}} s^{\varrho-1}\|\mathtt{u}(s)\|ds \\
& + \frac{LL_B}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{(t_2^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}\|\mathtt{u}(s)\|ds \\
& + \frac{L}{\Gamma(\alpha)}\int_0^{t_1} \frac{[(t_2^\varrho - s^\varrho)^{\alpha-1} - (t_1^\varrho - s^\varrho)^{\alpha-1}]}{\varrho^{\alpha-1}} s^{\varrho-1}\|f(s,\widetilde{y}(s))\|ds \\
& + \frac{L}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{(t_2^\varrho - s^\varrho)^{\alpha-1}}{\varrho^{\alpha-1}} s^{\varrho-1}\|f(s,\widetilde{y}(s))\|ds \\
\leq & \frac{L\big[\mathcal{Q}_f^*\Omega(\mathcal{J}) + L_B\|\mathtt{u}(s)\|\big]}{\Gamma(\alpha)}\Big[\frac{t_2^{\alpha\varrho}}{\varrho^{\alpha-1}}\int_0^{\frac{t_1}{t_2}}(1-y^\varrho)^{\alpha-1}y^{\varrho-1}ds \\
& - \frac{t_1^{\alpha\varrho}}{\varrho^{\alpha-1}}\int_0^1 (1-y^\varrho)^{\alpha-1}y^{\varrho-1}ds + \frac{t_2^{\alpha\varrho}}{\varrho^{\alpha-1}}\int_{\frac{t_1}{t_2}}^1 (1-y^\varrho)^{\alpha-1}y^{\varrho-1}ds\Big]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{L\big[\mathcal{Q}_f^*\Omega(\mathcal{J}) + L_B\|\mathtt{u}(s)\|\big]}{\Gamma(\alpha)}\int_0^1 (1-y^\varrho)^{\alpha-1}y^{\varrho-1}ds\Big[\frac{t_2^{\alpha\varrho}}{\varrho^{\alpha-1}} - \frac{t_1^{\alpha\varrho}}{\varrho^{\alpha-1}}\Big] \\
&\leq \frac{L\big[L_B\mathbf{Q}L_W + \mathcal{Q}_f^*\Omega(\mathcal{J})\big]\mathbf{B}(1,\alpha)}{\Gamma(\alpha)\varrho^\alpha}[t_2^{\varrho\alpha} - t_1^{\varrho\alpha}].
\end{aligned}$$

Where,

$$\mathbf{Q} = \Big(\|\widetilde{y}_1\| - L\|\widetilde{y}_0\| - \frac{L\mathcal{Q}_f^*\Omega(\mathcal{J})\mathbf{B}(1,\alpha)}{\Gamma(\alpha)\varrho^\alpha}[t_2^{\varrho\alpha} - t_1^{\varrho\alpha}]\Big).$$

Therefore, $\|\Xi_2\widetilde{y}(t_2) - \Xi_2\widetilde{y}(t_1)\|_{\mathbb{C}} \to 0$ as $t_2^\varrho \to t_1^\varrho$. Hence, we conclude that $\Xi_2(B_{\mathcal{J}}) \subseteq \mathbb{C}(U,\mathbb{F})$ is bounded and equicontinuous.

Step-5 We have to show that, for any $t \in U$, $\mathbb{V}(t) = \{(\Xi_2\widetilde{y})(t) : \widetilde{y} \in B_{\mathcal{J}}\}$ is relatively compact in $\mathbb{F}$, take $t \in U$, then $\forall\ \theta > 0$ and $\varepsilon > 0$, let $\widetilde{y} \in B_{\mathcal{J}}$ and introduce the operator $\Xi_2^{\theta,\varepsilon}$ on $B_{\mathcal{J}}$ by

$$\begin{aligned}
(\Xi_2^{\theta,\varepsilon}\widetilde{y})(t) =& \int_0^{t-\theta}\int_\varepsilon^\infty \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\alpha\delta\mathcal{M}_\alpha(\delta)J\Big[\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^\omega\delta\Big] \\
& \times [\mathcal{B}\mathtt{u}(s) + f(s,\widetilde{y}(s))]d\delta ds \\
=& \alpha J\Big[\Big(\frac{\theta^\varrho}{\varrho}\Big)^\alpha\varepsilon\Big]\int_0^{t-\theta}\int_\varepsilon^\infty \Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} s^{\varrho-1}\delta\mathcal{M}_\alpha(\delta)J\Big[\Big(\frac{t^\varrho - s^\varrho}{\varrho}\Big)^\omega\delta - \Big(\frac{\theta^\varrho}{\varrho}\Big)^\alpha\varepsilon\Big] \\
& \times [\mathcal{B}\mathtt{u}(s) + f(s,\widetilde{y}(s))]d\delta ds.
\end{aligned}$$

Since $J\left[\left(\frac{\theta^\varrho}{\varrho}\right)^\alpha \varepsilon\right]$ is compact for $\left(\frac{\theta^\varrho}{\varrho}\right)^\alpha \varepsilon > 0$, then the set $(\mathbb{V}^{\theta,\varepsilon}\widetilde{y})(t) = \{(\Xi_2^{\theta,\varepsilon}\widetilde{y})(t) : \widetilde{y} \in B_{\mathcal{J}}\}$ is relatively compact in $\mathbb{F}$ for every $\theta \in (0,t)$, $\varepsilon > 0$, and we get that

$$\begin{aligned}
\|(\Xi_2\widetilde{y})(t) - (\Xi_2^{\theta,\varepsilon}\widetilde{y})(t)\| =& \Bigg\|\alpha \int_0^t \int_0^\infty \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\delta\mathcal{M}_\alpha(\delta) J\left[\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^\omega \delta\right] \\
&\times [\mathcal{B}\mathtt{u}(s) + f(s,\widetilde{y}(s))]d\delta ds \\
&- \alpha J\left[\left(\frac{\theta^\varrho}{\varrho}\right)^\alpha \varepsilon\right] \int_0^{t-\theta} \int_\varepsilon^\infty \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\delta\mathcal{M}_\alpha(\delta) \\
&\times J\left[\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^\omega \delta - \left(\frac{\theta^\varrho}{\varrho}\right)^\alpha \varepsilon\right][\mathcal{B}\mathtt{u}(s) + f(s,\widetilde{y}(s))]d\delta ds\Bigg\| \\
\leq& \alpha L_E\left[\int_0^{t-\theta} \int_0^\varepsilon \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\delta\mathcal{M}_\alpha(\delta) d\delta ds\right] \\
&\times \|\mathcal{B}\mathtt{u}(s) + f(s,\widetilde{y}(s))\| \\
&+ \alpha L_E\left[\int_{t-\theta}^t \int_0^\infty \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} s^{\varrho-1}\delta\mathcal{M}_\alpha(\delta) d\delta ds\right] \\
&\times \|\mathcal{B}\mathtt{u}(s) + f(s,\widetilde{y}(s))\| \\
\leq& \frac{\alpha L_E t^{\varrho\alpha}}{\varrho^\alpha}\mathbf{B}_{\left(\frac{\vartheta-\epsilon}{\vartheta}\right)^\varrho}(1,\omega)\left[\frac{1}{\Gamma(1+\alpha)} + \int_0^\varepsilon \delta\mathcal{M}_\alpha(\delta)d\delta\right]\mathcal{Q}_f^*\Omega(\mathcal{J}) \\
&+ \frac{\alpha L_E L_B t^{\varrho\alpha}}{\varrho^\alpha}\mathbf{B}_{\left(\frac{\vartheta-\epsilon}{\vartheta}\right)^\varrho}(1,\omega)\left[\frac{1}{\Gamma(1+\alpha)} + \int_0^\varepsilon \delta\mathcal{M}_\alpha(\delta)d\delta\right]\|\mathtt{u}\|.
\end{aligned}$$

Where $\int_0^\infty \delta\mathcal{M}_\alpha(\delta)d\delta = \frac{1}{\alpha+1}$. From the absolute continuity of the Lebesgue integral, we obtain

$$\|(\Xi_2\widetilde{y})(t) - (\Xi_2^{\theta,\varepsilon}\widetilde{y})(t)\| \to 0 \text{ as } \theta, \varepsilon \to 0.$$

Therefore, we demonstrate that $\mathbb{V}^{\theta,\varepsilon}(t)$ is relatively compact in $\mathbb{F}$, $\forall\, t \in U$. Therefore, the using Ascoli-Arzela theorem, we can get that Ξ_2 is relatively compact.

Hence, the Krasnoselskii fixed point theorem (2.2) Ξ has a fixed point in $B_{\mathcal{J}}$, the system (1) has a mild solution. Thus equation (1) is controllable on U. The proof of the theorem is now concluded. □

4. Example

We consider the equation problem as follows:

$$\begin{cases} {}^\varrho D_{0_+}^{\frac{1}{2}}\widetilde{y}(t,\mu) = \Delta\widetilde{y}(t,\mu) + \mathcal{B}\mathtt{u}(t,\mu) + \frac{1}{5}e^{-t}\widetilde{y}(t,\mu), \ t \in (0,1] = U_1, \\ \widetilde{y}(t,0) = \widetilde{y}(t,\pi) = 0, \ t \in U_1, \\ \widetilde{y}(0,\mu) = \widetilde{y}_0(\mu), \ \mu \in [0,\pi], \end{cases} \tag{6}$$

where ${}^\varrho D_{0_+}^{\frac{1}{2}}$ is the Caputo-Katugampola fuzzy fractional derivative of order $\alpha = \frac{1}{2}$ and ϱ is additional parameter, which is greaterthan zero. Assume that $\mathbb{B} = \mathbb{E} = \mathbb{L}^2([0,\pi])$, the function $\widetilde{y}(t)(\mu) = \widetilde{y}(t,\mu)$. $f : U_1 \times \mathbb{B} \to \mathbb{F}$ is a fuzzy mapping and the continuous function $f(t,\widetilde{y}(t))$ is given by

$$\|f(t,\widetilde{y}(t))\| \leq \frac{e^{-t}}{5}\|\widetilde{y}\| \text{ and } \mathtt{u}(t,\mu) = \mathtt{u}(t)(\mu) \in \mathbb{L}^2((0,1],\mathbb{E}).$$

The linear operators $\Delta : D(\Delta) \subset \mathbb{B} \to \mathbb{B}$ is given by

$$D(A) = \{\widetilde{y} \in \mathbb{C}^2(0,1) | \widetilde{y}, \Delta \text{ are absolutely continuous } \widetilde{y}(0) = \widetilde{y}(\pi) = 0\}.$$

First we consider $\widetilde{y} = \sum_{n=1}^{\infty} \langle \widetilde{y}, \mathtt{h}_n \rangle \mathtt{h}_n$ for $\widetilde{y} \in \mathbb{F}$, where $\mathtt{h}_n(\mu) = \sqrt{\frac{2}{\pi}} \sin(n\mu),\ n = 1, 2, ...,$ $\mathbb{C}^2(0,1)$ is the set of all continuous partial derivatives with respect to the norm

$$\|\widetilde{y}\|_{\mathbb{C}^2} = \left(\sum_{n=1}^{\infty} \langle \widetilde{y}, \mathtt{h}_n \rangle^2 \right)^{\frac{1}{2}}.$$

Then A can be given as

$$A\widetilde{y} = \sum_{n=1}^{\infty} (-n^2) \langle \widetilde{y}, \mathtt{h}_n \rangle \mathtt{h}_n, \text{ for } \widetilde{y} \in D(A).$$

Therefore, for any $0 \leq t \leq 1$, we have

$$\|f(t, \widetilde{y}) - f(t, \widetilde{y}^{'})\| \leq \|\widetilde{y} - \widetilde{y}^{'}\|, \text{ for } \widetilde{y}, \widetilde{y}^{'} \in B(0, \mathcal{J}).$$

Consequently, system (6) can be rewritten as system (1). The function f clearly satisfies the assumptions $(K1) - (K4)$. This implies that Theorem 3.1, the equation (6) is controllable on U_1.

5. Conclution

This paper focused on proving the existence and controllability results of mild solution for a fuzzy fractional differential equation by employing the Caputo-Katugampola fractional derivative, which generalizes the Caputo and Caputo-Hadamard derivatives. The necessary conditions for these solutions were derived using fixed point theorem. Furthermore, a particular example was included to demonstrate the practical implementation of the theoretical findings. In future research, we plan to focus on order 1 to 2 of these solutions under various conditions.

Acknowledgments. The authors would like to thank the editor and reviewer for their valuable feedback and suggestions on enhancing the quality of the article.

References

[1] Podlubny, I., (1999), Fractional Differential Equations, Academic Press, San Diego.
[2] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., (2006), Theory and applications of fractional differential equations, elsevier, 204.
[3] Iskakova, K., Alam, M. M., Ahmad, S., Saifullah, S., Akgül, A. and Yılmaz, G., (2023), Dynamical study of a novel 4D hyperchaotic system: An integer and fractional order analysis, Math. Comput. Simul., 208, pp. 219-245.
[4] Hanif, A., Butt, A. I. K., Ahmad, S., Din, R. U. and Inc, M., (2021), A new fuzzy fractional order model of transmission of Covid-19 with quarantine class, Eur. Phys. J. Plus, 136 (11), pp. 1-28.
[5] Alqahtani, R. T., Ahmad, S. and Akgül, A., (2021), Dynamical analysis of bio-ethanol production model under generalized nonlocal operator in Caputo sense, Mathematics, 9 (19), p. 2370.
[6] Xuan, L., Ahmad, S., Ullah, A., Saifullah, S., Akgül, A. and Qu, H., (2022), Bifurcations, stability analysis and complex dynamics of Caputo fractal-fractional cancer model, Chaos Solitons Fractals, 159, p. 112113.
[7] Miller K. S. and Ross B., (1993), An Introduction to the Fractional Calculus and Differential Equations, John Wiley & Sons.
[8] Lakshmikantham, V., Leela, S. and Devi, J. V., (2009), Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers.
[9] Zhou, Y., (2013), Basic theory of fractional differential equations, World scientific.

[10] Chang, S. S. and Zadeh, L. A., (1972), On fuzzy mapping and control, IEEE Trans. Syst. Man Cybern., (1), pp. 30-34.

[11] Agarwal, R. P., Lakshmikantham, V. and Nieto, J. J., (2010), On the concept of solution for fractional differential equations with uncertainty, Nonlinear Anal. Theory Methods Appl., 72 (6), pp. 2859-2862.

[12] Arshad, S., (2013), On existence and uniqueness of solution of fuzzy fractional differential equations, Iran. J. Fuzzy Syst., 10, pp. 137-151.

[13] Salahshour, S., Allahviranloo, T. and Abbasbandy, S., (2012), Solving fuzzy fractional differential equations by fuzzy Laplace transforms, Commun. Nonlinear Sci. Numer. Simul., 17 (3), pp. 1372-1381.

[14] Allahviranloo, T., Armand, A. and Gouyandeh, Z., (2014), Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, J. Intell. Fuzzy Syst., 26 (3), pp. 1481-1490.

[15] Katugampola, U. N., (2011), New approach to a generalized fractional integral, Appl. Math. Comput., 218 (3), pp. 860-865.

[16] Katugampola, U. N., (2014), A new approach to generalized fractional derivatives, Bull. Math. Anal. Appl., (6), pp. 1-15.

[17] Katugampola, U. N., (2014), Existence and uniqueness results for a class of generalized fractional differential equations, arXiv:1411.5229 [math-ph].

[18] Almeida, R., Malinowska, A. B. and Odzijewicz, T., (2016), Fractional differential equations with dependence on the Caputo-Katugampola derivative, J. Comput. Nonlinear Dyn., 11 (6), 061017.

[19] Zeng, S., Baleanu, D., Bai, Y. and Wu, G., (2017), Fractional differential equations of Caputo-Katugampola type and numerical solutions, Appl. Math. Comput., 315, pp. 549-554.

[20] Baleanu, D., Wu, G. C. and Zeng, S. D., (2017), Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, Chaos Solitons Fractals, 102, pp. 99-105.

[21] Singh, V., (2019), Controllability of Hilfer fractional differential systems with non-dense domain. Numer. Funct. Anal. Optim., 40 (13), pp. 1572-1592.

[22] Ullah, A., Ullah, A., Ahmad, S., Ahmad, I. and Akgül, A., (2023), On solutions of fuzzy fractional order complex population dynamical model, Numer. Methods Partial Differ. Equ., 39 (6), pp. 4595-4615.

[23] Ahmad, S., Ullah, A., Akgül, A. and Abdeljawad, T., (2021), Semi-analytical solutions of the 3rd order fuzzy dispersive partial differential equations under fractional operators, Alex. Eng. J., 60 (6), pp. 5861-5878.

[24] Zhang, C., Ye, G., Liu, W. and Liu, X., (2024), On controllability for Sobolev-type fuzzy Hilfer fractional integro-differential inclusions with Clarke subdifferential, Chaos Solitons Fractals, 183, 114907.

[25] Ullah, A., Ullah, A., Ahmad, S. and Van Hoa, N., (2023), Fuzzy Yang transform for second order fuzzy differential equations of integer and fractional order, Phys. Scr., 98 (4), p. 044003.

[26] Van Hoa, N., Vu, H. and Duc, T. M., (2019), Fuzzy fractional differential equations under Caputo-Katugampola fractional derivative approach, Fuzzy Sets Syst., 375, pp. 70-99.

[27] Puri, M. L. and Ralescu, D. A., (1983), Differentials of fuzzy functions, J. Math. Anal. Appl., 91 (2), pp. 552-558.

[28] Hariharan, R. and Udhayakumar, R., (2025), Existence of mild solution for fuzzy fractional differential equation utilizing the Hilfer-Katugampola fractional derivative, Int. J. Optim. Control Theor. Appl., 15 (1), pp. 80-89.

[29] Varun Bose, C. S. and Udhayakumar, R., Velmurugan, S., Saradha, M. and Almarri, B., (2023), Approximate controllability of ψ-Hilfer fractional neutral differential equation with infinite delay, Fractal Fractional, 7 (7), 537.

[30] Zhou, Y., Zhang, L. and Shen, X. H., (2013), Existence of mild solutions for fractional evolution equations, J. Integr. Equ. Appl., 25 (4), pp. 557-586.

[31] Hariharan, R. and Udhayakumar, R., (2024), Approximate Controllability for Fuzzy Fractional Evolution Equations of Order $\ell \in (1, 2)$, Contemp. Math., pp. 3287-3312.

R. Hariharan obtained his B.Sc., and M.Sc. degrees in Mathematics from Bishop Heber College affiliated under Bharathidasan University, Tiruchirappalli, Tamil Nadu, India in 2020 and 2022, respectively. Now, he is pursuing his Ph.D. degree under the guidance of Prof. R. Udhayakumar at the Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu, India. His current research interests include Fractional Calculus, Control Theory, Fuzzy Fractional Differential Systems.

Dr. R. Udhayakuamr received B.Sc., M.Sc., M.Phil., and Ph.D. degrees in Mathematics from the Periyar University, Salem, Tamil Nadu, India, in 2006, 2008, 2009, and 2014, respectively. In 2016, he was a postdoctoral fellow at the Institute of Technology Bandung, Indonesia. He was an Assistant Professor at the Department of Mathematics, Bannari Amman Institute of Technology, Sathyamangalam, from 2015 to 2018. Since April 2018, he has been an Assistant Professor (Senior) of Mathematics at Vellore Institute of Technology, Tamil Nadu, India. His research interests include Fractional Calculus, Control Theory, Stochastic Differential Systems, and Algebra. He has published more than 131 papers in reputed scientific journals to his credit. He is an active researcher serving as an editorial board member/reviewer in many reputed scientific journals.