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ON ψ- CRITICALITY OF SOME RANDOM GRAPHS

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ABSTRACT. A vertex colouring g of a graph G is said to be pseudocomplete if for any two distinct colours i, j there exists at least one edge $e = (u, v) \in E(G)$ such that $g(u) = i$ and $g(v) = j$. The maximum number of colors used in a pseudocomplete coloring is called the pseudoachromatic number $\psi(G)$ of G. A Graph G is called vertex ψ-critical if $\omega(G) = 2\psi(G) - |V(G)|$. If P^* is a criticality property with respect to ψ then we have obtained some interesting results related to the random graphs as process innovation. We also proved that there is positive probability for the existence of a large collection of family of graphs that are not critical. We also listed a number of open problems.

Keywords: Random Graphs, Colouring, Pseudoachromatic Number, Vertex ψ-Critical Graphs.

AMS Subject Classification: 05C15, 05C80.

1. Introduction

Graphs considered here are finite, simple and undirected [4, 36].

Graph colouring, a subarea of Graph Theory is an interesting part of discrete mathematics. It has its origin in the four colour problem. In Graph colouring, four colour problem was the main challenge in the twentieth century. It asked whether it is possible to colour every planar map by four colours. After this, vertex colouring became one of the vividly pursued topics.

Graph colouring is an elementary problem within graph theory, permitting for a given set of conflicts to be modelled. But, deciding the smallest number of colours such that a proper colouring exists for a given graph is computationally tough and challenging. Graph colouring is one of pertinent set of fundamental NP-complete problems. That is, unless P = Np, it demands an exponential time algorithm to find a solution. It is known that for any $k \geq 3$, it is Np-complete to decide whether a given graph can be coloured with k colours.

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1.1. **Brief History.**

Graph colouring began in the city of Konigsberg of erstwhile Russia. In those days everyone is perplexed when challenged to cross the seven bridges in the city of Konigsberg in one trip without using any bridge more than once. Leonard Euler modelled the task by denoting every landmass as a vertex of a graph, and every bridge as an edge of a graph. He then gave a negative answer to the task of determining a path that contains each edge only once. Euler gave birth to the subject of Graph Theory with his elegant solution and is adored as father of Graph Theory.

1.2. **Motivation.**

A motivation for colouring concepts emanate from its potentiality of applications in several diverse areas such as biology, physics and engineering networks. Graph colouring serves as a model to determine sets that do not conflict. This feature led to new applications in emerging areas of knowledge. For instance, in bioinformatics, graph colouring is being applied to investigate protein-protein interaction networks. These networks that act as models for interaction between cells reap huge benefit from low chromatic number graph representations, as it permit the cells to do multi-tasking concurrently Graph colouring also has several interesting open questions. Even though they are highly hard, a number of innovative methods have emerged in the last few decades.

1.3. **Basics.**

We denote the set $\{1, 2, ..., p\}$ by [p]. The number of elements in a set S is denoted by $|S|$. For a set S with $|S| = p$, the set of all subsets of size k is represented as $\binom{p}{k}$.

A graph G is written as a pair (V, E). Here V stands for vertex set of G and E stands for edge set of G. Graphs considered here are those with $E \subseteq \{\{x, y\} : x \in V, y \in V, x \neq y\}$. Such graphs are termed as simple. We do not discuss and hence do not define hypergraphs, multigraphs or oriented graphs. By a subgraph H of a graph G = (V, E) we mean a graph H = (W, F) where $W \subseteq V$ and $F \subseteq E$. If $G \neq H$, then H is refereed as a proper subgraph. If $F = E \cap \binom{|V|}{2}$, then H is called an induced subgraph. A subset S of V is called a clique if $\binom{|S|}{2}$ is a subset of E. That is, every element of S is adjacent with every other element of S. The size of the biggest clique is termed as the clique number and denoted by $\omega(G)$.

1.4. **Proper colouring.**

A vertex colouring g of a graph G is an onto function $g : V(G) \to \{1, ..., k\}$. g is called a proper vertex colouring if no two adjacent vertices of G are coloured with the same colour. The chromatic number of G, denoted $\chi(G)$, is the smallest integer k used in a proper vertex colouring. A graph G is k-critical with respect to the chromatic number χ if $\chi(G) = k$ but every proper subgraph of G is (k-1)-colourable. But, we define a new notion of criticality in Section 3 with respect to an improper pseudocomplete colouring that is different from proper chromatic colouring. An improper pseudocomplete colouring is defined as follows.

1.5. **Improper Colouring.**

g is called a pseudocomplete vertex colouring of G if for any two distinct colours i, j there exists at least one edge $e = (u, v) \in E(G)$ such that $g(u) = i$ and $g(v) = j$. g is called a complete vertex colouring if it is both pseudocomplete and proper. The maximum number of colours used in a pseudocomplete(complete) colouring is called the pseudoachromatic(achromatic) number of G, denoted $\psi(G)(\alpha(G))$. There is another way to look at α and ψ of a graph $G = (V, E)$. A partition $V_1, ..., V_t$ of V is said to be complete if, for every i, j, $i \neq j$, there is an edge v_i, v_j such that $v_i \in V_i$ and $v_j \in V_j$. The maximum size of a complete partition of G is called the pseudoachromatic number $\psi(G)$ of G.

It is known that the computation of $\psi(G)$ of any G is Np-hard [5, 12, 24, 25]. The achromatic number $\alpha(G)$ is the maximum size of a complete partition with an additional constraint that the classes in the partition are independent sets [13]. Complete partition problem possess application flavour as it can be applied in network design tasks, especially in clustering. Stephen T. Hedetniemi, a well known name in graph colouring research has declared his top ten favourite conjectures and open problems concerning graphs theory in one of his talks. The list includes pseudoachromatic and achromatic number of graphs. Also see for more in [3, 8, 9, 10, 11, 14, 15, 16, 21, 23, 31, 29, 37, 38, 41, 42]. Pseudo achromatic colourings were coined by Gupta [19] in 1969. It was then well probed. Many authors [17, 26, 30, 46] and [34] for instance investigated this parameter. The complete partition problem was further probed by Bhave [2], Sampathkumar and Bhave [43], Robertson, N. and Seymour [39], Bollobas, Reed and Thomason [6], Thomassen [44], Vetrivel et al. [45], Wood [46, 47], Kostochka [26], Yegnanarayanan [48, 49, 50, 51, 52, 53, 54, 55, 56]. Jianer Chen et.al [11] studied generalizations of this problem and proved that they are parameterized intractable.

2. Random Graphs

Random graphs, well handles the asymptotic attributes of graphs with the help of some probability distributions. For instance, it probes how its structure evolves when the edge set size explodes. Erdos and Renyi introduced the theory of random graphs about 6o years back and since then random graph models have been widely investigated. In the meantime, Graph theory has pierced its way into other branches of sciences as rich models to explain several aspects of a complex phenomenon.

Erdos and Renyi strikingly answered a) the queries concerning the probability of a random graph to be connected, and b) the probability for the biggest component of a random graph to cover almost all vertices. This has led to the evolution of the giant component in a random graph.

2.1. Three famous Random Graph Models.

The following are three famous random graph models created by Erdos-Renyi:

1. The uniform random graph $G(p, q)$: It is a graph picked uniformly from the set of all graphs with vertex set $[p] := \{1, , p\}$ and q edges, for an integer $0 \leq q \leq \binom{p}{2}$ at random.

2. The binomial random graph $G(p, p^*)$: It is a graph with vertex set [p]. Here, every pair of vertices is linked by an edge independently with probability p^*, for a real number $0 \leq p \leq 1$.

3. The Erdos-Renyi process $\{G_p(q) : q = 0, ..., \binom{p}{2}\}$. It starts with $G_p(0)$ with p vertices of degree 0 and no edges, and in every step $1 \leq q \leq \binom{p}{2}$ a new random edge is introduced to an evolving graph $G_p(q)$ to derive a new graph $G_p(q)$. The graph $G_p(q)$ thus built by the Erdos-Renyi process is distributed like the uniform random graph $G(p, q)$. These three models becomes equivalent when the parameters are appropriately chosen. That is, if $q = p\binom{p}{2}$. Random graphs serve as canonical mathematical tool to probe the probabilistic features of social communication networks. The famous Erdos–Renyi model is pertinent to draw useful predictions. If $\mathcal{G}$ is a class of graphs with n vertices with independent probability p for the presence of each edge and (1-p) for its absence, the probability for each vertex to have degree exactly r (participating member to have exactly r connections) with every other member is $p_r = \binom{n-1}{r}p^r(1-p)^{n-1-r}$. In the limiting sense when n is very large in comparison with the edges, a given vertex has then n very large in comparison

to $(n-1)p \approx np \equiv t$ then the binomial distribution tends to poisson for large n so that $p_r = \binom{n-1}{r}(\frac{t}{n-1-t})^r(1-\frac{t}{n-1})^{n-1} = \frac{t^r exp(-t)}{r!}$.

The results available in the combinatorics literature [32, 33] allows us to employ random graphs as models for probing social interactions.

3. Graph Criticality

A graph G is called k-critical if its chromatic number is k, but every proper subgraph is $(k-1)$-colorable. So every graph that is not $(k-1)$ colourable includes in it a k-critical subgraph. This elucidation is pertinent in various colouring problems. A precise explanation than this is unlikely to exist in general, but, when confined to only sparse graphs like the one that can be embedded in a fixed surface it is quite possible. For instance, in [44] it is shown that for all $k \geq 5$, the k-critical graphs that can be embedded in any fixed surface are only finitely many. This means the task of deciding the $(k-1)$-colorability of a graph can be completed in polynomial time. The authors in [35] further improved this result by deriving direct bounds on the sizes of k-critical embedded graphs with reference to list colouring. The lower bounds given by Kostochka and Yancey [27] on the density of critical graphs almost settled a problem of Ore that stands open for a long time. The idea of probability played a key role for the invention of new technique called potential method. Recently entropy compression method has evolved in [18] to probe a random process that could lead to a proper colouring depending on the credibility of its run. It reveals that the number of such runs that could fail is influenced by the count of random choices in any run of a fixed length.

Mostly one would like to establish that for some positive integer k, any graph in an infinite class $\mathcal{G}$ of graphs have $\chi < k$. Note that such a $\mathcal{G}$ will be closed under taking either induced or usual subgraphs. In such instances, a main goal for bounding the χ is to consider the minimal graphs in $\mathcal{G}$ with $\chi = k$. Such graphs possess a distinct attribute that omitting any vertex if it is closed under induced subgraphs or any edge if it is closed under subgraphs decreases the χ from k to k-1. This imposes various constraints on such minimal graphs. Such attributes aid one to show the non-existence of minimal k-chromatic graphs in $\mathcal{G}$. This would then imply that χ of graphs in $\mathcal{G} < k$.

A lot of effort has been made to comprehend vertex-criticality and edge-criticality. The author in [28] conjectured that for every integer $k \geq 4$, there exists a k-chromatic vertex-critical graph G which is not edge-critical. Brown in [7] gave for the first time a procedure to construct a vertex-critical graph with no critical edges. Then, Lattanzio in [28] produced a better procedure that proved Dirac's conjecture in [20] for every integer $k \geq 5$ such that k-1 is not a prime. Also Jensen [22] ended up with a new method to build a k-chromatic vertex-critical graphs with no critical edges for every $k \geq 5$. This essentially means that only the case k=4 of Dirac's conjecture in [28] is open as on date.

In this paper, we study about graph criticality with respect to pseudoachromatic number introduced in [1] and obtain some interesting results that are probabilistic in nature. We also mention a few open problems that are being pursued hotly among the graph colouring community. A Graph G is called vertex ψ-critical if $\omega(G) = 2\psi(G) - |V(G)|$.

Proposition 3.1.

Given a collection of random graphs $\mathscr{G}_{p,q}$ with any of its member say G has q edges, the probability that G is one of the $\binom{\binom{p}{2}}{q}$ graphs with q edges is equally likely.

Proof. Note that the set of all $G \in \mathscr{G}_{p,q}$ is a subset of the set of all those graphs with edge size $|E_{p,q}| = q$. So $\mathscr{P}\{G_{p,q}\} = G|E_{p,q}| = q\} = \frac{\mathscr{P}(G_{p,q}=G),|E_{p,q}|=q)}{\mathscr{P}(|E_{p,q}|=q)} = \frac{\mathscr{P}(G_{p,q})=G)}{\mathscr{P}(|E_{p,q}|)=q)} = \frac{p^q(1-p)^{\binom{p}{2}-q}}{\binom{\binom{p}{2}}{q}p^q(1-p)^{\binom{p}{2}-q}} = \binom{\binom{p}{2}}{q}^{-1}$. □

Let $\mathscr{G}_{p,q}$ be the family of all labeled graphs with $|V| = [p] = \{1, 2, ..., p\}$ and with q edges $0 \leq q \leq \binom{p}{2}$. Here the labels to the vertices are assigned as per the requirement of a pseudocomplete colouring. To each graph $G \in \mathscr{G}_{p,q}$ we allot a probability $\mathscr{P}(G) = \binom{\binom{p}{2}}{q}^{-1}$. We assign such a random graph, a symbol $G^*_{p,q} = ([p], E_{p,q})$. One can also assign for $p_0 \in [0, 1]$, a probability $\mathscr{P}(G) = p_0^q(1 - p_0)^{\binom{p}{2}-q}$. Allot a similar symbol for this way of allotting probabilities say, $G^*_{p,p_0} = ([p], E_{p,p_0})$. It is known by Proposition 1 that a random graph $\mathscr{G}_{p,p_0}$ given that its number of edges is q is equally likely to be one of the $\binom{\binom{p}{2}}{q}$ graphs that have q edges. The main difference between $G^*_{p,q}$ and G^*_{p,p_0} is that in the former we select it number of edges whenever in the latter the number of edges is the binomial random variable with parameter $\binom{p}{2}$ and p_0. For large p, $G_{p,q}$ and G_{p,p_0} be have in he same manner when the number of edges q in $G_{p,q}$ is equal or almost close to the expected number of edges of G_{p,p_0}. That is, when $q = \binom{p}{2}p_0 \approx \frac{p^2p_0}{2}$ or the edge probability is $p_0 \approx \frac{2q}{p^2}$. Now let P^* be the criticality property with respect to the pseudoachromatic number ψ and $p_0 = \frac{q}{\binom{p}{2}}$ where $q = q(p) \to \infty$, $\binom{p}{2} - q \to \infty$. Then it is known that for large p, $\mathscr{P}(G_{p,q} \in P^*) \leq 10q^{\frac{1}{2}}\mathscr{P}(G_{p,p_0} \in P^*)$. Now we call P^* monotonically increasing if $G \in P^*$ implies $G + e \in P^*$. Here we restrict our attention to all these family of graphs for which $\omega(G) = \omega(G+e)$. It it easy to see that $G \in P^* \to G + e \in P^*$. So the property of criticality with respect to ψ is monotonically increasing. It is known that for large p and $p_0 = 0(1)$. If $\binom{p}{2}p_0, \frac{\binom{p}{2}(1-p_0)}{\sqrt{\binom{p}{2}p_0}} \to \infty$ then $P(G_{p,q} \in P^*) \leq P(G_{p,p_0} \in P^*)$. Further the asymptotic equivalence of random graphs $G_{p,q}$ and G_{p,p_0} was established by Luczak. Hence, we get

Theorem 3.1.
Let $p_0 \in [0, 1]$, $s(p) = p\sqrt{p_0(1 - p_0)} \to \infty$ and $t(p) \to \infty$ arbitrarily slowly as $p \to \infty$. If P^* is the criticality property such that $P(G_{p,q} \in P^*) \to p_0$ for all $q \in [\binom{p}{2}p_0 - t(p)s(p)$, $\binom{p}{2}p_0 + t(p)s(p)]$, then $P(G_{p,p_0} \in P^*) \to p_0$ as $p \to \infty$.

For $k \geq 3$, let G be a graph with $|V(G)| = 2^{\frac{k}{2}-1}$. Flip a coin for every possible edge to ascertain its presence or absence. Then there is a positive probability that the independence number $\alpha^*(G)$ to be strictly less than k and clique number $\omega(G)$ to be strictly less than k. So that a G will have the value $\frac{\omega(G)+|V(G)|}{2} \leq \frac{(k-1)+2^{\frac{k}{2}-1}}{2}$. Clearly such a graph will have $|E(G)| \leq \binom{2^{\frac{k}{2}-1}}{2}$. So if ψ is any pseudocomplete coloring of G then $\binom{\psi}{2} \leq |E(G)| \leq \binom{2^{\frac{k}{2}-1}}{2}$. That is, $\psi^2 - \psi - 2\binom{2^{\frac{k}{2}-1}}{2} \leq 0 \implies \psi^2 - \psi - (2^{\frac{k}{2}-1})(2^{\frac{k}{2}-1} - 1) \leq 0$ $\implies \psi \leq \frac{1+\sqrt{1+4(2^{\frac{k}{2}-1})(2^{\frac{k}{2}-1}-1)}}{2} \implies \psi \approx 2^{k-1}$. So $\psi \approx 2^{k-1} > \frac{2^{\frac{k-2}{2}}+(k-1)}{2} = \frac{\omega(G)+|V(G)|}{2}$ $\implies G$ is not critical.

Proposition 3.2.

Let P^* be the criticality property with respect to the pseudoachromatic number ψ and $p_0 = \frac{q}{\binom{p}{2}}$, where $q = q(p) \to \infty, \binom{p}{2} - q \to \infty$. Then, one can claim for large p that $\mathscr{P}(\mathscr{G}_{p,q} \in P^*)$ is bounded above by $10\sqrt{q}(\mathscr{G}_{p,p_0} \in P^*)$.

Proof. Note that $\mathscr{P}(\mathscr{G}_{p,p_0} \in P^*) = \sum_{r=0}^{\binom{p}{2}} \mathscr{P}(\mathscr{G}_{p,p_0} \in P^* || E_{p,p_0}| = r)\mathscr{P}(|E_{p,p_0}| = r)$ by invoking the law of total probability. So we can write $P(G_{p,p_0} \in P^*) = \sum_{r=0}^{\binom{p}{2}} \mathscr{P}(G_{p,r} \in P^*)\mathscr{P}(|E_{p,p_0}| = r) \geq \mathscr{P}(G_{p,q} \in P^*)\mathscr{P}(|E_{p,p_0}| = q)$. To see the equality observe that, $\mathscr{P}(G_{p,p_0} \in P^* |E_{p,p_0}| = r) = \frac{\mathscr{P}(G_{p,p_0} \in P^*) \cap (|E_{p,p_0}| = r)}{\mathscr{P}(|E_{p,p_0}| = r)} = \sum_{G \in P^*, |E(G)| = r)} \frac{p^r(1-p)^{N-r}}{\binom{N}{r} p^{r(1-p)^{N-r}}}$ $= \sum_{G \in P^*, |E(G)| = r} \binom{N}{r}^{-1} = P(G_{p,r} \in P^*)$. Now $|E_{p,p_0}|$ is a random variable following binomial distribution with parameters $\binom{p}{2}$ and p_0. So by using a formula of approximation due to Stirling, namely, $r! = (1 + 0(1))(\frac{r}{e})^r \sqrt{2\pi r}$ and taking $N = \binom{p}{2}$, one can derive that $P(|E_{p,p_0}| = q = \binom{N}{q} p^q (1-p)^{\binom{p}{2} - q} = (1+0(1)) \frac{N^M \sqrt{2\pi N} p^q (1-p)^{N-q}}{q^q (N-q)^{N-q} 2\pi \sqrt{q(N-q)}} = (1+0(1)) \sqrt{\frac{N}{2\pi q(N-q)}}$. So $\mathscr{P}(|(E_{p,p_0}| = q) \geq (10\sqrt{q})^{-1}$. That is, $\mathscr{P}(G_{p,p_0} \in P^*) \leq 10\sqrt{q}\mathscr{P}(G_{p,p_0} \in P^*)$. □

Theorem 3.2.

There is positive probability for the existence of a large collection of families of graphs that are not critical.

Proof. For $k \geq 3$, let G be a graph with $|V(G)| = 2^{\frac{k}{2}-1}$. Flip a coin to decide the presence or absence of an edge. We claim that there is a positive probability that $w(G) < k$. We know that for any set of k vertices there are at most $\binom{k}{2}$ edges between them, Every coin flip for those $\binom{k}{2}$ edges will have a probability of $\frac{1}{2}^{\binom{k}{2}}$. But there are many k-vertex sets to form a clique. This means as an upper bound there are $|V(G)|^k = 2^{k(\frac{k}{2}-1)}$ cliques. We know that if there are N events and each of them has probability p, then the probability of any of them happens is at most Np. These we have $2^{k(\frac{k}{2}-1)}$ events and each of them happens with probability $\frac{1}{2}^{\frac{k(k-1)}{2}}$ or$\frac{1}{2}^{\binom{k}{2}}$. So the probability of any of them happens is at most $2^{k(\frac{k}{2}-1)} - 2^{-\binom{k}{2}} = 2^{\frac{k(k-2)}{2}} 2^{-\frac{k(k-1)}{2}} = 2^{-\frac{k}{2}} = \frac{1}{2^{\frac{k}{2}}}$. This is an upper bound for the probability that G has a k-vertex clique, Now consider such a collection of graphs $\mathcal{G}$. Then for any $G \in \mathcal{G}$, we have $W(G) < k$, so $\frac{W(G)+|V(G)|}{2} < \frac{k+2^{\frac{k}{2}-1}}{2}$. Also $|E(G)| \leq \binom{2^{\frac{k}{2}-1}}{2}$. So, if ψ is any pseudo complete ψ- coloring of G, then $\binom{\psi}{2} \leq \binom{2^{\frac{k}{2}-1}}{2} \implies \psi \leq \frac{1+\sqrt{1+2^2(2^{\frac{k}{2}-1})(2^{\frac{k}{2}-1}-1))}}{2}$ $\approx 2^{k-1}$. So $\psi \approx 2^{k-1} > \frac{2^{\binom{k-2}{2}}+k}{2} = \frac{W(G)+|V(G)|}{2} \geq [\frac{W(G)+|V(G)|}{2}] \implies G$is a collection of family of graphs that are not critical. □

4. Notion of Colouring in Graph Minors and Expanders

By a minor of a graph G, we mean a graph built from G by performing operations such as: a) edge contraction, b) edge deletion, and c) deletion of vertices of degree zero. By a proper minor of a graph G, we mean any minor of G other than G itself. For instance, the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are both minors of the Peterson graph. Wagner revised the statement of classical Kuratowski's Theorem : A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph into: A graph G is planar if and only if K_5 and $K_{3,3}$ are not minors of G. Wagner's research has led to a serious probe about families of graphs with forbidden minors.

4.1. **Forbidden Minors.**

A graph H is termed as a forbidden minor for a set $\mathscr{X}$ of graphs if H is not a minor of any member in $\mathscr{X}$. A forbidden minor H of $\mathscr{X}$ is said to be minimal if no proper minor of H is a forbidden minor. One can see that any family $\mathscr{X}$ of graphs with at least one forbidden minor is minor-closed. Every minor of a graph in $\mathscr{X}$ also belongs to $\mathscr{X}$. Conversely, if a minor-closed family of graphs avoids a graph H, it must also avoid any graph for which H is a minor. So, every minor-closed family of graphs has at least one forbidden minor. Neil Robertson and Paul Seymour in the middle of 1980 produced a proof of one of the deepest theorems in Combinatorics. They published the details of their findings in a series of 21 papers that runs to a several number of hundred pages each in the next twenty years. They completely settled a conjecture raised by Wagner in 1930.

Theorem 4.1. *(The Graph Minor Theorem (Robertson and Seymour). In any infinite set of graphs, at least one graph is a proper minor of another.* [38].

Also there is an algorithm to determine whether a graph is a minor of another.

Theorem 4.2. *([Robertson and Seymour* [39]*). For any fixed graph H, there is an algorithm to determine whether a given n-node graph has H a a minor in $O(n^3)$ time.*

Archdeacon [40] *showed that graphs embeddable on a fixed surface can be 3-coloured so that each colour class induces a subgraph of bounded maximum degree. Edwards et.al in* [41] *showed that graphs with no K_{r+1}-minor can be r-coloured so that each colour class induces a subgraph of bounded maximum degree.*

4.2. **Graph Expanders.**

Expander graphs are some families of graphs that becomes larger and larger, possess two interesting attributes.

a) Fairly sparse in terms of edge set size when compared to vertex set size and b) Highly connected.

If graphs are considered as objects that could be thought of as communication network with some cost associated with every physical edge then increasing the vertex set size will increase the cost linearly. So sparsity is pertinet if one built a tramway network. The second attribute mean that one can go to any vertex from a given vertex by a short path. But this is not enough to define an expander. This is because, there may be some small subset Y of edges such that the graph derived by omitting Y becomes disconnected. So we require a subset $V \subset V_p$ of vertices should have several links with its complement $V^c = V_p – V$. Hence, expanders are formulated by the condition that, for some constant $k > 0$, independent of p, the number of such edges should be at least k $\min(|V|, |V^c|)$ for all subsets $V \subset V_p$, and for all p, $V \neq \emptyset$.

Graph G^* is a minor of graph G if for every vertex $v \in G^*$ there is a connected subgraph $G_v \subset G$ such that $V(G_v) \cap V(G_{v'}) = \emptyset$ for all $v \neq v'$ and for every edge (v, v') of G^* there is an edge from G_v to $G_{v'}$.

Problem 4.1.

Determine a sufficient condition for graph G to contain a given G^* as a minor.

Problem 4.2.

Hadwiger Conjecture (1943): Any graph with $\chi \geq k$ contains a clique on k vertices as a minor.

Problem 4.3.

Kostochka-Thomason Theorem (1980): Any graph with average degree d contains a clique-minor of order $\frac{cd}{\sqrt{\log d}}$.

Theorem 4.3.

Thomassen, Diestel-Rompel, Kuhn-Osthus-Theorem: If G has girth 2k+1 and average degree d, then it contains a minor with average degree $\Omega(d^{\frac{k+1}{2}})$.

Problem 4.4.

Do we need the girth presumption or Is it suffice to forbid cycle C_{2k}*?*

A separator of a graph G of order n is a set of vertices whose removal separates G into connected components of size $\leq \frac{2n}{3}$.

Theorem 4.4.

Plotkin-Rao-Smith Theorem: If an n-vertex graph G has no clique-minor on h vertices, then it has a separator of order $O(h\sqrt{n \log n})$.

Problem 4.5.

What size clique-minors can one find in a graph G on n vertices, whose expansion factor $t \to \infty$ *together with n ? What if edges of G are distributed like in the random graph?*

5. Applications of Colouring Parameters

Graph colouring is used in several real time tasks in computer science such as data mining, clustering, image capturing, image segmentation, networking etc. For instance, a tree with its vertices and edges serve as a data structure.

Resource Optimization:
Graph colouring offers prudent allocation of available resources by allotting colours to vertices that stands for resources so that vertices identified as conflicts receive distinct colours. This results in optimal (minimal) use of valuable resources that guarantees smooth operation of the overall system. This fact can be observed in hardware register allocation in computer systems, assignment of channels in wireless communication networks.

Problem Solving:
Colouring procedures such as genetic algorithms, backtracking, heuristic-methods leads to near-optimal solutions or solutions for hard optimization challenges.

Visualization:
Graph colouring improves network visibility prospects among its members. This culminates in better pattern recognition and data analysis to comprehend complex systems and provides for decision-making.

Quantum Colouring:
Lately it has captured attention and its algorithms provides for exponential speedup when compared to traditional algorithms.

6. Conclusion and Open Problems

Arriving at the best colouring by making use of the currently available highly efficient procedures for very huge real life graphs is still a very hard task. Researchers in graph colouring continue their best effort to to determine the chromatic number, achromatic number, pseudo achromatic number to improve algorithmic efficiency, adopt to dynamic networks and quantum computing techniques to invent new methodologies to resolve challenging tasks in several domains. We obtained some interesting results concerning ψ-criticality of random graphs. χ-bounded graph classes are the ones whose chromatic number is bounded by a function of their clique number ω. A number of questions asked in [5] about χ-bounded graph classes were settled in [6,7].

6.1. **Open problem 1.**

Prove or Disprove: For every hereditary χ-bounded class $\mathcal{G}$, there is a constant r such that $\chi(G) \leq \omega(G)^r$ for all graphs $G \in \mathcal{G}$?

An affirmative answer to the above problem could result in an interesting sequel that: The class $\mathcal{G}$ of graphs avoiding a given graph H as an induced subgraph is χ-bounded and hence every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq \omega(G)^r$.

The authors in [9] showed that graphs of rank-width at most k form a $\chi-$bounded class for any integer k. Suppose that for any class of graphs $\mathcal{G}$ with hereditary property, let $g_{\mathcal{G}}$ be the optimal $\chi-$ bounding function for $\mathcal{G}$ defined by $g_{\mathcal{G}}(t) = max\{\chi(G)$ for all graph $G \in \mathcal{G}$ such that $\omega(G) = t\}$.

6.2. **Open Problem 2.**

Prove or Disprove: The class of χ-bounded graphs of rank-width at most k, has a polynomial dependence of its χ on ω for any integer $k \geq 1$.

6.3. **Open Problem 3.**

Prove or Disprove: There exists a class G with hereditary property such that $g_{\mathscr{G}}$= g for a given function g that is non-decreasing.

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