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A NUMERICAL SOLUTION TO NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS BASED ON BELL POLYNOMIALS

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ABSTRACT. This article examines the solutions of high-order nonlinear ordinary differential equations with cubic terms under initial conditions using Bell polynomials, their derivatives, and collocation points. The nonlinear differential equation and the corresponding conditions are transformed into matrix form by means of Bell polynomials and reduced to an algebraic system. From the solution of this system, the unknown Bell coefficients are determined. By substituting these coefficients, the approximate solution of the problem is expressed in terms of Bell polynomials. To illustrate the method, some numerical examples are presented. For these examples, the Bell solutions and the absolute error functions are calculated, and the results are shown in tables and figures for comparison with the exact solutions.

Keywords: Nonlinear differential equation, Bell polynomial, Collocation Method, Absolute Error.

AMS Subject Classification: 34A34,11B73,11B83,65F45.

1. Introduction

Differential equations can be classified as linear or nonlinear. Differential equations that are nonlinear play a significant role in different areas like finance, systems identification, control theory, signal processing, fluid flow, biology, engineering, physics, chemistry, viscoelasticity and fluid mechanics [5],[8],[12],[15],[16],[20],[25],[26],[27]. In science and engineering, nonlinear differential equations are commonly used to simulate a wide range of scientific phenomena. Finding analytical solutions to these equations can be difficult, hence the use of numerical methods is required. In recent years, there has been a focus on nonlinear problems and a variety of numerical methods has been developed. Examples include quasilinearization method for Blasius, Duffing, Lane–Emden and Thomas–Fermi equations [29], Adomian decomposition method [13], [43], Legendre wavelets [36], wavelet analysis method [21], B-spline method [9], He's variational iteration for the Bratu-type equations

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[21], [30], solving Riccati differential equation in He's VIM [2], homotopy perturbation method (HPM)[1], [3], piecewise variational iteration method [17], using cubic B-spline scaling functions and Chebyshev cardinal functions [24], Quasilinearization Methods [29], [34], solving Duffing–Van der Pol's equation in analytical perturbation method [11], [23], differential transform method [33], Taylor collocation method [18], [28], [37], Chebyshev series method [4], Legendre, Bernstein and Bessel [38],[39], Bernoulli collocation method [14] and Lucas polynomial approach for solving nonlinear differential equations [19].

In this paper,we study a numerical method that involves Bell polynomials, their derivatives and collocation points to solve nonlinear ordinary differential equations of the mth-order in the form

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{p,q,r}(x)y^{(p)}y^{(q)}y^{(r)} = g(x) \tag{1}$$

with initial conditions

$$\sum_{k=0}^{m-1}[a_{kj}y^{(k)}(a)] = \lambda_j, j = 0, 1, 2, \dots, m-1. \tag{2}$$

where $y(x)$ are unknown functions. The functions $P_k(x), Q_{(p,q,r)}(x)$ and $g(x)$ are continuous functions in the interval $[a, b]$ and a_{kj}, λ_j are real constants.

1.1. Bell Polynomial Properties and Matrix Relation.

Let n, k be natural numbers and $S(n, k)$ be the Stirling number of second kind [10]

$$S(n,k) = \sum_{j=0}^{k} \frac{(-1)^j}{k!}\binom{k}{j}(k-j)^n$$

and exponential Bell polynomials are defined by [6],[7],[42]

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k.$$

Generating function of the above defined type of Bell polynomials is [22]

$$\sum_{k=0}^{\infty} \frac{B_k(x)}{k!}t^k = e^{(e^t-1)x}.$$

An alternative definition of $B_n(x)$ is given by [32],[35]

$$B_n(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^n x^k}{k!}$$

where $B_0(x) = 1$ and $\binom{n}{k}$ are binomial coefficients.

We consider a nonlinear system of ordinary differential equations defined on the interval $[a,b]$. Let $N \in \mathbf{N}$ denote the truncation level used in the collocation approach. That is, the approximate solution is represented as a truncated series up to degree N.We seek the approximate solution of the nonlinear ordinary differential system $(1),(2)$ in the form of the truncated Bell series

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x) \tag{3}$$

where $a_n, n = 0,1,2,\ldots,N$ are coefficients and $B_n(x)$ are Bell polynomials.

The Bell polynomials given by Equation (3) can be expressed in the matrix form

$$\mathbf{B}(x) = \begin{bmatrix} B_0(x) & B_1(x) & \ldots & B_N(x) \end{bmatrix} = \mathbf{X}(x)\mathbf{S}^T$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \ldots & x^N \end{bmatrix}_{1\times(N+1)}$$

and

$$\mathbf{S} = \begin{bmatrix} S(0,0) & 0 & 0 & \ldots & 0 \\ S(1,0) & S(1,1) & 0 & \ldots & 0 \\ S(2,0) & S(2,1) & S(2,2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S(N,0) & S(N,1) & S(N,2) & \ldots & S(N,N) \end{bmatrix}_{(N+1)\times(N+1)} \tag{4}$$

The matrix relation of approximate solution in Equation (3) is in the form

$$y(x) \cong y_N(x) = \mathbf{B}(x)\mathbf{A} = \mathbf{X}(x)\mathbf{S}^T\mathbf{A} \tag{5}$$

and the kth derivative can be written

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \mathbf{B}^{(k)}(x)\mathbf{A} = \mathbf{X}^{(k)}(x)\mathbf{S}^T\mathbf{A} \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \ldots & a_N \end{bmatrix}^T_{1\times(N+1)}.$$

In addition to this in [40], [41] there are relations between $\mathbf{X}(x)$ and its kth derivatives $\mathbf{X}^{(k)}(x)$

$$\begin{array}{rcll} \mathbf{X}^{(1)}(x) & = & \mathbf{X}(x)\mathbf{M} & \\ \mathbf{X}^{(2)}(x) & = & \mathbf{X}^{(1)}(x)\mathbf{M} & = \mathbf{X}(x)\mathbf{M}^2 \\ \mathbf{X}^{(3)}(x) & = & \mathbf{X}^{(2)}(x)\mathbf{M} & = \mathbf{X}(x)\mathbf{M}^3 \\ & \vdots & & \\ \mathbf{X}^{(k)}(x) & = & \mathbf{X}^{(k-1)}(x)\mathbf{M} & = \mathbf{X}(x)\mathbf{M}^k \end{array} \tag{7}$$

where

$$\mathbf{M}=\begin{bmatrix}0&1&0&\dots&0\\0&0&2&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&N\\0&0&0&\dots&0\end{bmatrix}_{(N+1)\times(N+1)},\mathbf{M}^0=\begin{bmatrix}1&0&0&\dots&0\\0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\end{bmatrix}_{(N+1)\times(N+1)}.$$

2. Fundamental Matrix Relation and Bell Collocation Method

In this section we construct the expressions defined in (1), (3) in the matrix form. Firstly the matrix relation (6) is substituted in the linear part of Equation (1) as follows

$$\sum_{k=0}^{m}P_k(x)\mathbf{B}^{(k)}(x)\mathbf{A}+\sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q}Q_{p,q,r}y^{(p)}y^{(q)}y^{(r)}=g(x) \tag{8}$$

Furthermore, the nonlinear part of the matrix relations in Equation (8) can be represented using the relation in (6) as follows;

$$y^3(x)=\mathbf{B}(x)\,\overline{\mathbf{B}}(x)\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{A}}} \tag{9}$$

$$[y(x)]^2\,y'(x)=\mathbf{B}(x)\,\overline{\mathbf{B}}(x)\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{M}}}\,\overline{\overline{\mathbf{A}}} \tag{10}$$

$$[y'(x)]^2\,y(x)=\mathbf{B}(x)\,\mathbf{M}\,\overline{\mathbf{B}}(x)\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{A}}} \tag{11}$$

$$y''(x)\,[y(x)]^2=\mathbf{B}(x)\,\mathbf{M}^2\,\mathbf{B}(x)\,\overline{\mathbf{B}}(x)\,\overline{\overline{\mathbf{A}}} \tag{12}$$

$$[y'(x)]^3=\mathbf{B}(x)\,\mathbf{M}\,\overline{\mathbf{B}}(x)\,\overline{\mathbf{M}}\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{M}}}\,\overline{\overline{\mathbf{A}}} \tag{13}$$

$$y''(x)\,y'(x)\,y(x)=\mathbf{B}(x)\,\mathbf{M}^2\,\overline{\mathbf{B}}(x)\,\overline{\mathbf{M}}\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{A}}} \tag{14}$$

$$y''(x)\,[y'(x)]^2=\mathbf{B}(x)\,\mathbf{M}^2\,\overline{\mathbf{B}}(x)\,\overline{\mathbf{M}}\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{M}}}\,\overline{\overline{\mathbf{A}}} \tag{15}$$

$$[y''(x)]^2\,y(x)=\mathbf{B}(x)\,\mathbf{M}^2\,\overline{\mathbf{B}}(x)\,\overline{\mathbf{M}^2}\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{A}}} \tag{16}$$

$$[y''(x)]^2\,y'(x)=\mathbf{B}(x)\,\mathbf{M}^2\,\overline{\mathbf{B}}(x)\,\overline{\mathbf{M}^2}\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{M}}}\,\overline{\overline{\mathbf{A}}} \tag{17}$$

$$[y''(x)]^3=\mathbf{B}(x)\,\mathbf{M}^2\,\overline{\mathbf{B}}(x)\,\overline{\mathbf{M}^2}\,\overline{\overline{\mathbf{B}}}(x)\,\overline{\overline{\mathbf{M}^2}}\,\overline{\overline{\mathbf{A}}} \tag{18}$$

where

$$\mathbf{B}(x)=\begin{bmatrix}B_0(x)&B_1(x)&\dots&B_N(x)\end{bmatrix}_{1\times(N+1)}$$

$$\overline{\mathbf{B}}(x)=diag\begin{bmatrix}\mathbf{B}(x)&\mathbf{B}(x)&\dots&\mathbf{B}(x)\end{bmatrix}_{(N+1)\times(N+1)^2}$$

$$\overline{\overline{\mathbf{B}}}(x)=diag\begin{bmatrix}\overline{\mathbf{B}}(x)&\overline{\mathbf{B}}(x)&\dots&\overline{\mathbf{B}}(x)\end{bmatrix}_{(N+1)^2\times(N+1)^3}$$

$$\overline{\mathbf{M}} = diag\begin{bmatrix} \mathbf{M} & \mathbf{M} & \dots & \mathbf{M} \end{bmatrix}_{(N+1)^2\times(N+1)^2}, \overline{\overline{\mathbf{M}}} = diag\begin{bmatrix} \overline{\mathbf{M}} & \overline{\mathbf{M}} & \dots & \overline{\mathbf{M}} \end{bmatrix}_{(N+1)^3\times(N+1)^3}$$

$$\overline{\mathbf{A}} = \begin{bmatrix} a_0\mathbf{A}^T & a_1\mathbf{A}^T & \dots & a_N\mathbf{A}^T \end{bmatrix}^T_{1\times(N+1)^2} \quad \overline{\overline{\mathbf{A}}} = \begin{bmatrix} a_0\overline{\mathbf{A}}^T & a_1\overline{\mathbf{A}}^T & \dots & a_N\overline{\mathbf{A}}^T \end{bmatrix}^T_{1\times(N+1)^3}$$

The collocation points x_i are defined by

$$x_i = a + \frac{b-a}{N} i, i = 0, 1, 2, \dots, N \textit{where } N \textit{ is the truncation level.} \tag{19}$$

These collocation points expressed in relation (19) we substitute in the Equation (8) to become

$$\sum_{k=0}^{m} P_k(x_i)\mathbf{B}^k(x_i)\mathbf{A} + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{p,q,r}(x_i)y^{(p)}(x_i)y^{(q)}(x_i)y^{(r)}(x_i) = g(x_i) \tag{20}$$

Using matrix relation (6) in Equation (20) fundamental matrix relation can be written as

$$\sum_{k=0}^{m} \mathbf{P}_k\mathbf{B}\mathbf{M}^k\mathbf{A} + \sum_{p=0}^{2}\sum_{q=0}^{p}\sum_{r=0}^{q} \mathbf{Q}_{pqr}\mathbf{Y}^{p,q,r}\overline{\overline{\mathbf{A}}} = \mathbf{G} \tag{21}$$

where

$$y^{(p)}(x_i)\,y^{(q)}(x_i)\,y^{(r)}(x_i) = \mathbf{Y}^{p,q,r} = \begin{bmatrix} y^{(p)}(x_0)\,y^{(q)}(x_0)\,y^{(r)}(x_0) \\ y^{(p)}(x_1)\,y^{(q)}(x_1)\,y^{(r)}(x_1) \\ \vdots \\ y^{(p)}(x_N)\,y^{(q)}(x_N)\,y^{(r)}(x_N) \end{bmatrix}_{(N+1)\times(N+1)^3},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{B}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}(x_N) \end{bmatrix}_{(N+1)\times(N+1)}, \qquad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}_{(N+1)\times 1}$$

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}_{(N+1)\times(N+1)}$$

$$\mathbf{Q}_{pqr} = \begin{bmatrix} Q_{pqr}(x_0) & 0 & \cdots & 0 \\ 0 & Q_{pqr}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{pqr}(x_N) \end{bmatrix}_{(N+1)\times(N+1)}$$

The fundamental matrix equation (21) can be briefly expressed in the form

$$\mathbf{WA} + \mathbf{V}\overline{\overline{\mathbf{A}}} = \mathbf{G} \tag{22}$$

where the linear part of the relation (21) is

$$\mathbf{W} = [w_{ij}] = \sum_{k=0}^{m} \mathbf{P}_k \mathbf{B} \mathbf{M}^k, i, j = 0, 1, 2, \ldots, N$$

and the nonlinear part is

$$\mathbf{V} = \sum_{p=0}^{2} \sum_{q=0}^{p} \sum_{r=0}^{q} \mathbf{Q}_{pqr} \mathbf{Y}^{p,q,r}$$

Briefly,the augmented matrix form of the relation (22) is written as follows

$$[\mathbf{W}; \mathbf{V}; \mathbf{G}] = \begin{bmatrix} w_{0,0} & \cdots & w_{0,N} & ; & v_{0,0} & v_{0,1} & \cdots & v_{0,(N+1)^3} & ; & g(x_0) \\ w_{1,0} & \cdots & w_{1,N} & ; & v_{1,0} & v_{1,1} & \cdots & v_{1,(N+1)^3} & ; & g(x_1) \\ \vdots & \ddots & \vdots & ; & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{N,0} & \cdots & w_{N,N} & ; & v_{N,0} & v_{N,1} & \cdots & v_{N,(N+1)^3} & ; & g(x_N) \end{bmatrix} \tag{23}$$

Now, a matrix representation of the initial conditions in Equation (2) can be found. Using the initial condition (2) of problem (1) by matrix relation

$$\sum_{k=0}^{m-1} \left[a_{kj} \mathbf{B}(a) \mathbf{M}^k \right] \mathbf{A} = \lambda_j \Rightarrow \mathbf{UA} + \mathbf{0}^{**}\overline{\overline{\mathbf{A}}} = \lambda \tag{24}$$

or

$$[\mathbf{U}; \mathbf{0}^{**}; \lambda] = \begin{bmatrix} u_{0,0} & u_{0,1} & \cdots & u_{0,N} & ; & 0 & 0 & \cdots & 0 & ; & \lambda_0 \\ u_{1,0} & u_{1,1} & \cdots & u_{1,N} & ; & 0 & 0 & \cdots & 0 & ; & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & 0 & 0 & \cdots & 0 & ; & \lambda_{m-1} \end{bmatrix}$$

where

$$\mathbf{U}_j = \begin{bmatrix} u_{j,0} & u_{j,1} & \cdots & u_{j,N} \end{bmatrix}_{1\times(N+1)} \quad \text{for } j = 0, 1, 2, \ldots, m-1,$$

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{m-1} \end{bmatrix}_{m\times 1}, \quad \mathbf{0}^{**} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times(N+1)^3}$$

Since the conditions are linear, they were incorporated into the linear part of the formulation. A zero column vector of size $m \times 1$ is explicitly included in the nonlinear part of the formulation to ensure dimensional consistency with Equation (24). After reviewing the literature on all polynomial- and ordering-based methods, we found no explicit rule for this step. Therefore, we prefer to remove the rows corresponding to the more complex values as a heuristic choice aligned with common practice in similar approaches. By switching the order and replacing the m rows of the augmented matrix (23) by the row matrices (24), we obtain the new augmented matrix

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{V}}; \tilde{\mathbf{G}}] \tag{25}$$

Consequently, solving this nonlinear algebraic system Equation (20) we obtain unknown Bell coefficients. Then, these coefficients are substituted in the solution (3) and the truncated Bell series form of the equation (1) is found.

3. Error functions and Numerical Results

In this section, three examples are given in to demonstrate the accuracy and reliability of the study. In the first example, the applicability of the solution is shown with the exact solution. Numerical solutions and error estimation are included in the second and third example. The exact solution of the $y(x)$ problem, the approximate solution $y_N(x)$ of the problem, the error function E_N is defined as follows

$$E_N(x_t) = \mid \sum_{k=0}^{m} P_k(x_t) y_N^{(k)}(x_t) + \sum_{p=0}^{2} \sum_{q=0}^{p} \sum_{r=0}^{p} Q_{p,q,r} y_N^{(p)}(x_t) y_N^{(q)}(x_t) y_N^{(r)}(x_t) - g(x_t) \cong 0 \mid \tag{26}$$

and the absolute error function

$$e_N(x) = \mid y(x) - y_N(x) \mid .$$

Example 1. Consider the exact solution to first order differential equation with a cubic nonlinearity

$$y''(x) + y'(x) - x^2 y(x) + y^3(x) = 3x - 2x^2 \tag{27}$$

with the initial condition $y(0) = -1$ and $y'(0) = 1$ where $P_0(x) = -x^2$,$P_1(x) = 1$,$P_2(x) = 1$,$Q_{000}(x) = 1$, and $g(x) = 3x - 2x^2$. The solution to the problem can be represented by a Bell series for $N = 2$

$$y(x) \cong y_2(x) = \sum_{n=0}^{2} a_n B_n(x) = a_0 B_0(x) + a_1 B_1(x) + a_2 B_2(x) \tag{28}$$

and the collocation points using (19)

$$\{x_0 = 0, x_1 = 1/2, x_2 = 1\}$$

From Equation (20), the fundamental matrix equation of the problem is

$$(\mathbf{P}_0\mathbf{B} + \mathbf{P}_1\mathbf{B}\mathbf{M} + \mathbf{P}_2\mathbf{B}\mathbf{M}^2)\mathbf{A} + (\mathbf{Q}_{000}\mathbf{Y}^{0,0,0})\overline{\overline{\mathbf{A}}} = \mathbf{G} \tag{29}$$

where

$$\mathbf{P}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_{000} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{3}{4} \\ 1 & 1 & 2 \end{bmatrix}, \mathbf{Y}^{0,0,0} = \begin{bmatrix} \mathbf{B}(0)\overline{\mathbf{B}}(0)\overline{\overline{\mathbf{B}}}(0) \\ \mathbf{B}(\frac{1}{2})\overline{\mathbf{B}}(\frac{1}{2})\overline{\overline{\mathbf{B}}}(\frac{1}{2}) \\ \mathbf{B}(1)\overline{\mathbf{B}}(1)\overline{\overline{\mathbf{B}}}(1) \end{bmatrix},$$

$$\mathbf{B}(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \overline{\mathbf{B}}(0) = diag\begin{bmatrix} \mathbf{B}(0) & \mathbf{B}(0) & \mathbf{B}(0) \end{bmatrix}$$

$$\overline{\overline{\mathbf{B}}}(0) = diag\begin{bmatrix} \overline{\mathbf{B}}(0) & \overline{\mathbf{B}}(0) & \overline{\mathbf{B}}(0) \end{bmatrix}$$

$$\mathbf{B}(\frac{1}{2}) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{4} \end{bmatrix}, \overline{\mathbf{B}}(\frac{1}{2}) = diag\begin{bmatrix} \mathbf{B}(\frac{1}{2}) & \mathbf{B}(\frac{1}{2}) & \mathbf{B}(\frac{1}{2}) \end{bmatrix}$$

$$\overline{\overline{\mathbf{B}}}(\frac{1}{2}) = diag\begin{bmatrix} \overline{\mathbf{B}}(\frac{1}{2}) & \overline{\mathbf{B}}(\frac{1}{2}) & \overline{\mathbf{B}}(\frac{1}{2}) \end{bmatrix}$$

$$\mathbf{B}(1) = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}, \overline{\mathbf{B}}(1) = diag\begin{bmatrix} \mathbf{B}(1) & \mathbf{B}(1) & \mathbf{B}(1) \end{bmatrix}$$

$$\overline{\overline{\mathbf{B}}}(1) = diag\begin{bmatrix} \overline{\mathbf{B}}(1) & \overline{\mathbf{B}}(1) & \overline{\mathbf{B}}(1) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}, \quad \overline{\mathbf{A}} = \begin{bmatrix} a_0\mathbf{A} \\ a_1\mathbf{A} \\ a_2\mathbf{A} \end{bmatrix}, \quad \overline{\overline{\mathbf{A}}} = \begin{bmatrix} a_0\overline{\mathbf{A}} \\ a_1\overline{\mathbf{A}} \\ a_2\overline{\mathbf{A}} \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 \\ \frac{5}{2} \\ 3 \end{bmatrix}.$$

From Equation (24) , the matrix form for initial conditions is

$$\mathbf{U}_0\mathbf{A} + \mathbf{0}^{**}\overline{\overline{\mathbf{A}}} = \lambda_\mathbf{0} \quad and \quad \mathbf{U}_1\mathbf{A} + \mathbf{0}^{**}\overline{\overline{\mathbf{A}}} = \lambda_\mathbf{1} \tag{30}$$

where

$$[\mathbf{U}_0; \lambda_\mathbf{0}] = \begin{bmatrix} 1 & 0 & 0 & ; & -1 \end{bmatrix}, \quad [\mathbf{U}_1; \lambda_\mathbf{1}] = \begin{bmatrix} 0 & 1 & 1 & ; & 1 \end{bmatrix} \quad and$$

$$\mathbf{0}^{**} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}_{2\times 27}$$

The matrix equation which is obtained by replacing the last row of (29) with the condition matrix (30) is solved for the unknown Bell coefficients and we obtanined

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T$$

The Bell coefficients are substituted in (28), the approximate solution coincides with the exact solution which is $y(x) \cong y_2(x) = x - 1$.

Example 2. As the second example, consider the second order differential equation with a cubic nonlinearity

$$y''(x) + 2y'(x) + y(x) + 8y^3(x) = e^{-3x} \tag{31}$$

with initial conditions $y(0) = \frac{1}{2}$ and $y'(0) = \frac{-1}{2}$. The exact solution to (31)is $y(x) = \frac{-1}{2}e^{-x}$ [31]. For $N = 3, 7$ and 8, absolute error functions of problem (31)is calculated. The absolute error functions are presented in Table 1. The exact and approximate solutions are given in Figure 1, and the absolute error functions are shown in Figure 2.

x	$\mid e_3(x) \mid$	$\mid e_7(x) \mid$	$\mid e_8(x) \mid$
0	0	0	0
0.2	$5.7690e^{-05}$	$6.7307e^{-10}$	$2.6700e^{-11}$
0.4	$2.2451e^{-04}$	$1.0685e^{-09}$	$4.3417e^{-11}$
0.6	$1.6978e^{-05}$	$1.1824e^{-09}$	$4.9505e^{-11}$
0.8	$1.5882e^{-03}$	$1.0680e^{-09}$	$4.0439e^{-11}$
1	$6.0564e^{-03}$	$8.9996e^{-08}$	$4.0643e^{-09}$

TABLE 1. The absolute errors of $y(x)$ for $N = 3, 7, 8$.

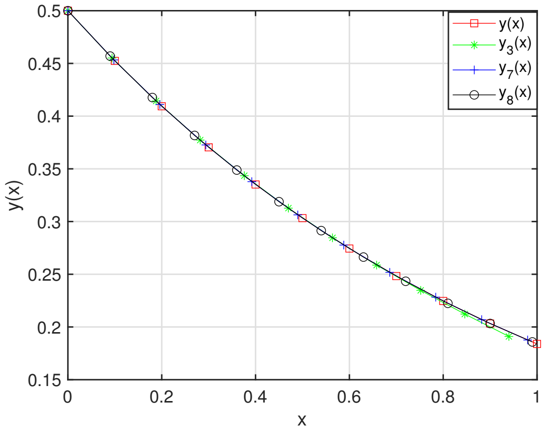

FIGURE 1. Exact solution and numerical solutions for Example 2.

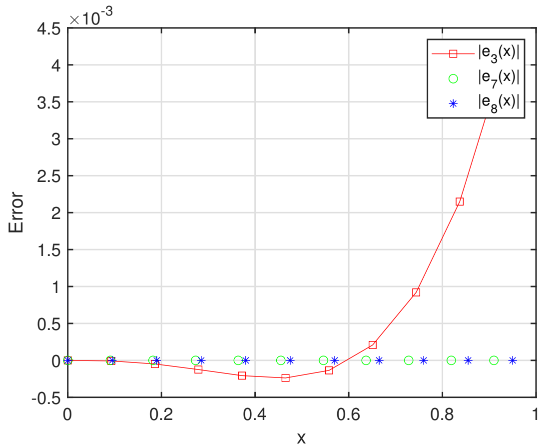

FIGURE 2. Comparison of absolute error functions for Example 2.

Example 3. Finally, consider the Abel differential equation

$$y'(x)y(x) + xy(x) + y^2(x) + x^2y^3(x) = xe^{-x} + x^2e^{-3x} \tag{32}$$

with the initial condition $y(0) = 1$, for $0 \leq x \leq 1$, and the exact solution to the problem is $y(x) = e^{-x}$ [18]. For $N = 6, 8$ and 9, approximate solution found by the Bell polynomial method are presented. These errors are shown in Table 2. The results are illustrated in Figures 3 and 4.

x	$\mid e_6(x) \mid$	$\mid e_8(x) \mid$	$\mid e_9(x) \mid$
0	0	0	0
0.2	$3.1456e^{-08}$	$3.4062e^{-11}$	$1.0587e^{-12}$
0.4	$1.7311e^{-08}$	$2.6415e^{-11}$	$7.9803e^{-13}$
0.6	$6.4375e^{-09}$	$1.8837e^{-11}$	$5.1270e^{-13}$
0.8	$8.5865e^{-08}$	$2.0391e^{-11}$	$4.4048e^{-13}$
1	$3.9523e^{-07}$	$7.4277e^{-10}$	$2.6891e^{-11}$

TABLE 2. The absolute errors of $y(x)$ for $N = 6, 8, 9$.

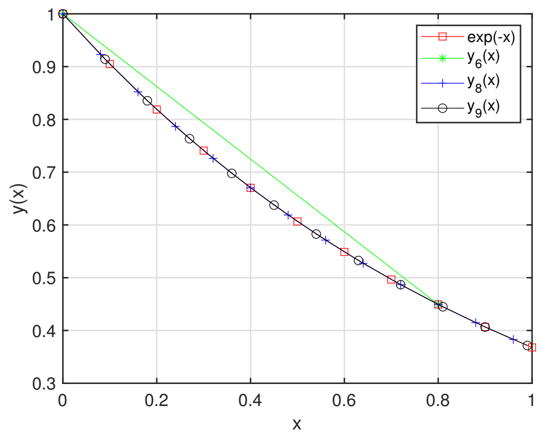

FIGURE 3. Exact solution and approximate solutions for Example 3.

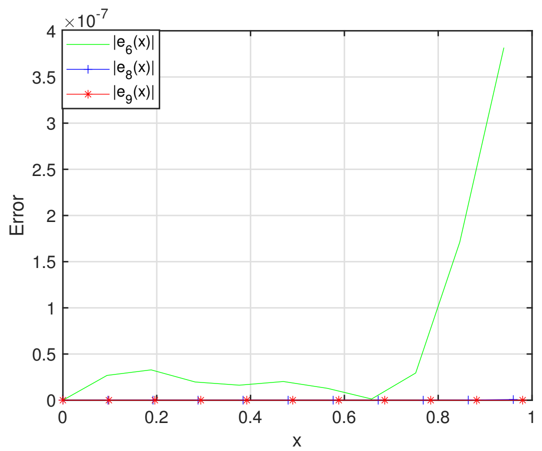

FIGURE 4. Comparison of the absolute error functions for Example 3.

Conclusions

This study introduces a new approach that uses Bell polynomials and their derivatives to solve nonlinear ordinary differential equations. The method's efficiency is illustrated with numerical examples, with the solutions and errors shown in tables and figures. An increase in the value of N resulted in an improvement in the solution's accuracy. The calculations are performed using the MATLAB program, and this technique can be extended to other types of nonlinear differential equations.

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