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ERROR QUANTIFICATION FOR APPROXIMATION IN GENERALIZED ZYGMUND CLASSES USING THREE-HARMONIC POISSON INTEGRALS

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Abstract. This paper examines the error in approximation within generalized Zygmund classes using three-harmonic Poisson integrals. The error is measured using two moduli of continuity of order two, within the relevant norm, providing a clear understanding of how the approximation works.

Keywords: Degree of approximation, Three-harmonic Poisson integrals, Generalized Zygmund class, Three-harmonic Poisson kernel, Modulus of continuity of order two.

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1. Introduction

One important area of interest, which has received considerable attention in Fourier analysis, is the study of the convergence rates of various transformations (operators) of the partial sums of a function's Fourier series, with applications in fields such as signal processing, differential equations, and numerical analysis. This area has been shaped by the contributions of mathematicians such as Quade [27], Prössdorf [26], Chandra [2]-[4], Leindler [17]-[19], Das et al. [5], Nayak et al. [23], [24], Kim [11], Nigam and Hadish [25], Değer [6], Singh [28], Singh et al. [29], Lal and Shireen [16], Zhihallo and Kharkevych [32], Borsuk and Khanin [1], Łenski et al. [20] among many others.

In the context of such transformations, special focus is placed on the Abel–Poisson means, the generalized Abel–Poisson means, Picard, Picard-Cauchy and Gauss-Weierstrass singular integrals, and those based on biharmonic and three-harmonic Poisson integrals. These integrals are solutions to elliptic-type integro-differential equations, underscoring the close relationship between their convergence rates and the theory of dynamic games.

Estimating the deviations of functions from certain classes using operators such as those of Fejér, Vallée-Poussin, Riesz, Nörlund, generalized Nörlund, deferred, deferred-Riesz, deferred-Nörlund, Rogozinski, and matrix types is widely discussed in the available literature.

The deviations of Zygmund class Z^{α} functions from their biharmonic Poisson integrals for $1 < \alpha < 2$ have been studied recently in [1].

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In the present work, we estimate the deviations of Zygmund class $Z^{(\omega)}$ functions from their three-harmonic Poisson integrals, a topic that has not yet been studied.

For two positive quantities v_1 and v_2, we denote $v_1 = \mathcal{O}(v_2)$ to indicate that there exists a constant $K > 0$ such that $v_1 \leq K v_2$.

The structure of this paper is as follows: Section 2 recalls some material on three-harmonic Poisson integrals and generalized Zygmund classes, laying the foundation for the subsequent analysis. Section 3 presents a recent known result. Section 4 introduces some helpful lemmas. Section 5 contains the main results, including a new theorem along with its proof. Finally, Section 6 concludes the paper with a summary of the findings and suggestions for future research.

2. Preliminaries

This section is devoted to a short review of the three-harmonic Poisson integrals and the generalized Zygmund spaces.

Let $L_{2\pi}$ denote the space of 2π-periodic functions f that are integrable over period, equipped with the norm

$$\|f\|_L = \int_{-\pi}^{\pi} |f(x)|\, dx.$$

Also, let L_∞ denote the space of 2π-periodic measurable functions f that are essentially bounded, with the norm given by

$$\|f\|_\infty = \operatorname{ess\,sup}_x |f(x)|.$$

For $f(x) \in L_{2\pi}$, the expression

$$A_3(r,x) = \int_{-\pi}^{\pi} f(x+t) P_3(r,t)\, dt,$$

introduced in [32] (p. 1012), is referred to as the *three-harmonic Poisson integral*, where

$$P_3(r,t) = \frac{(1-r)^3(4 - 9r\cos t + 6r^2\cos^2 t - r^3\cos t)}{8\pi(1-2r\cos t + r^2)^3},$$

with $0 \leq r < 1$, denotes the *three-harmonic kernel*.

The three-harmonic Poisson integral can be expressed as a singular integral in the following form (see [9], p. 40):

$$P_3(\delta; f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) K_3(\delta; t)\, dt, \quad \delta > 0,$$

where the three-harmonic kernel is given by

$$K_3(\delta;t) = \frac{1}{2} + \sum_{k=1}^{\infty}\left(1 + \frac{k}{4}(3 - e^{-\frac{2}{\delta}})(1 - e^{-\frac{2}{\delta}}) + \frac{k^2}{8}(1 - e^{-\frac{2}{\delta}})^2\right) e^{-\frac{k}{\delta}} \cos kt.$$

By substituting $\delta = -(\ln\rho)^{-1}$, where $0 < \rho < 1$, into the three-harmonic Poisson integral in its singular form, we obtain

$$P_{3,\rho}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) K_{3,\rho}(t)\, dt,$$

with the three-harmonic kernel

$$K_{3,\rho}(t) = \frac{1}{2} + \sum_{k=1}^{\infty}\left(1 + \frac{k}{4}(3-\rho^2)(1-\rho^2) + \frac{k^2}{8}(1-\rho^2)^2\right)\rho^k \cos kt,$$

where for convenience, we use the shorthand notations $P_{3,\rho}(f;x)$ and $K_{3,\rho}(t)$ to represent $P_3(-(\ln\rho)^{-1};f;x)$ and $K_3(-(\ln\rho)^{-1};t)$, respectively.

In essence, the three-harmonic Poisson integrals arise from its connection to the Dirichlet problem for the three-harmonic equation. According to [8] and the references cited therein, the Dirichlet problem for the equation $\Delta^3 u = 0$ is formulated as follows:

To find, in unit disc D, a function

$$u = u(z) = u(\rho e^{i\theta}), \quad 0 \le \rho < 1, \quad 0 \le \theta \le 2\pi,$$

which is a solution of the three-harmonic equation and satisfies the boundary conditions

$$u|_{\partial D} = h, \quad \left.\frac{\partial u}{\partial \rho}\right|_{\partial D} = h_1, \quad \left.\frac{\partial^2 u}{\partial \rho^2}\right|_{\partial D} = h_2,$$

Here, h, h_1, and h_2 are boundary-defined functions possessing specific characteristics that ensure both the existence and uniqueness of the solution to this problem.

It was notably demonstrated that the solution to this Dirichlet problem on the unit circle can be expressed as a three-harmonic Poisson integral.

Additionally, it is important to highlight that the approximation properties of the three-harmonic Poisson integrals have been thoroughly explored by numerous researchers and continue to be a significant area of academic interest. Remaining within the same area of interest, we now turn our attention to the approximation properties of three-harmonic Poisson integrals in approximating periodic functions belonging to the generalized Zygmund classes, as described below.

For this, we proceed by recalling some additional notations and concepts that will be useful throughout the remainder of the discussion. Specifically, we build on the material from [16], [25], and Zygmund's well-known book [33].

Let

$$L^r[0,2\pi] := \left\{ f : [0,2\pi] \to \mathbb{R} : \int_0^{2\pi} |f(x)|^r dx < \infty \right\}, \quad r \ge 1,$$

which represents the space of all 2π-periodic and integrable functions.

The norm $\|f\|_r$ is defined as

$$\|v\|_r = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, & \text{if } 1 \le r < \infty, \\ \operatorname{ess\,sup}_{x\in[0,2\pi]} |f(x)|, & \text{if } r = \infty. \end{cases}$$

As defined by Zygmund [33], let $w : [0,2\pi] \to \mathbb{R}$ be a function such that $w(t) > 0$ for $0 < t \le 2\pi$ and $\lim_{t\to 0^+} w(t) = w(0) = 0$.

The Zygmund class $Z_r^{(w)}$ (see [16], [25]) is defined by

$$Z_r^{(w)} := \left\{ f \in L^r[0,2\pi], r \ge 1 : \sup_{t \ne 0} \frac{\|f(\cdot + t) - 2f(\cdot) + f(\cdot - t)\|_r}{w(t)} < \infty \right\}$$

and the corresponding norm is

$$\|f\|_r^{(w)} := \|f\|_r + \sup_{t\ne 0} \frac{\|f(\cdot + t) - 2f(\cdot) + f(\cdot - t)\|_r}{w(t)}, \quad r \ge 1.$$

The space $Z_r^{(w)}$ is a Banach space under the norm $\|\cdot\|_r^{(w)}$.

The completeness of $Z_r^{(w)}$ can be analyzed by looking at the completeness of L^r for $r \ge 1$.

Using similar principles, the class $Z_r^{(w_1)}$ is introduced.

The functions $w(t)$ and $w_1(t)$ are assumed to be moduli of continuity of order 2, as described in [33]. If $w(t)/w_1(t)$ is non-negative and non-decreasing, it follows that

$$\|f\|_r^{(w_1)} \leq \max\left(1, \frac{w(2\pi)}{w_1(2\pi)}\right) \|f\|_r^{(w)} < \infty.$$

Therefore,

$$Z_r^{(w)} \subset Z_r^{(w_1)} \subset L^r, \quad r \geq 1.$$

Below are specific examples of the Zygmund class $Z_r^{(w)}$, showing some important cases (see [25] for further details):

(i) If we take $r \to \infty$ in $Z_r^{(w)}$ class, it reduces to $Z^{(w)}$ class.
(ii) If we take $w(t) = t^\gamma$ in $Z^{(w)}$ class, it reduces to Z_γ class.
(iii) If we take $w(t) = t^\gamma$ in $Z_r^{(w)}$ class, it reduces to $Z_{\gamma,r}$ class.
(iv) If we take $r \to \infty$ in $Z_{\gamma,r}$ class, it reduces to Z_γ class.

Now, we recall a known result, in which the biharmonic Poisson integrals have been used as a tool for approximating 2π-periodic functions in the Zygmund class.

3. Known Result

The Zygmund classes have been extensively studied over the years, with notable contributions from several authors, including Leindler [17], M'oricz [22], M'oricz and N'emeth [21], Stepanets [30], and others.

Next, we recall a class of 2π-periodic continuous functions Z^α, where $0 < \alpha \leq 2$, which will be important for this section.

For $0 < \alpha \leq 2$, Z^α refers to the class of 2π-periodic continuous functions $f(x)$ that meet the following condition

$$|f(x+t) - 2f(x) + f(x-t)| \leq 2|t|,$$

where $-\pi \leq x \leq \pi$ and $|t| \leq 2\pi$.

Let

$$\mathcal{E}\left(Z^\alpha, B_s(f;x)\right)_C := \sup_{f \in Z^\alpha} \|B_s(f;\cdot) - f(\cdot)\|_C, \tag{1}$$

where

$$\|f\|_C = \max_x |f(x)|.$$

The biharmonic Poisson integral can be expressed as a singular integral in the following form (see [31], p. 344)

$$B_\rho(f;x) := \int_{-\pi}^{\pi} f(x+t) P_2(\rho,t)\, dt,$$

where

$$P_2(\rho,t) := \frac{1}{2} + \sum_{k=1}^{\infty} \left[1 + \frac{k}{2}(1-\rho^2)\right] \rho^k \cos(kt),$$

$0 \leq \rho < 1$, is the biharmonic Poisson kernel.

It comes as a solution of the biharmonic equation

$$\Delta^2 u = \Delta(\Delta u) = 0, \tag{2}$$

where the Laplace operator in polar coordinates is given by

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial x^2}.$$

The solution of the equation (2) is assumed to satisfies the boundary conditions

$$u(\rho,x)\big|_{\rho=1}=f(x),\quad \frac{\partial u(\rho,x)}{\partial\rho}\Big|_{\rho=1}=0,\quad x\in[-\pi,\pi].$$

Many researchers have studied how well biharmonic Poisson integrals can approximate functions, demonstrating their usefulness in solving different problems in mathematics and related areas. For instance, Borsuk and Khanin [1] recognized an upper limit for $\mathcal{E}\left(Z^{\alpha},B_{\rho}(f;x)\right)_{C}$. They established a result that can be restated as follows.

Theorem 3.1 ([1]). *Let $1<\alpha<2$ and $0<\rho<1$. Then, the quantity $\mathcal{E}\left(Z^{\alpha},B_{\rho}(f;x)\right)_{C}$, as defined in (1), can be expressed as follows*

$$\begin{aligned}\mathcal{E}\left(Z^{\alpha},B_{\rho}(f;x)\right)_{C}=&\frac{1}{\pi}\Gamma(\alpha-1)\cos\left(\frac{(\alpha-1)\pi}{2}\right)(1-\rho)\\&\times\sum_{k=0}^{\infty}\frac{\rho^k}{\left(k+\frac{1}{2}\right)^{\alpha}}\left(1+\frac{k}{2}(1-\rho^2)-\frac{1}{2}(1+\rho)\rho\right)\\&+\mathcal{O}\left((1-\rho)\sum_{k=0}^{\infty}\frac{\rho^k}{\left(k+\frac{1}{2}\right)^{2}}\left(1+\frac{k}{2}(1-\rho^2)-\frac{1}{2}(1+\rho)\rho\right)\right),\end{aligned}$$

where $\Gamma(\alpha-1)$ represents the value of the gamma function at $\alpha-1$.

Based on this key result, we now explain our motivation and goals. As seen, the result in Theorem 3.1 is specifically for functions in the Zygmund class Z^{α}, where $1<\alpha<2$. This study has inspired us to extend the use of three-harmonic Poisson integrals to approximate functions from a broader Zygmund class.

It is worth noting that the generalized Hölder class (see [23]) is a subset of the corresponding generalized Zygmund class. However, the norm in the generalized Zygmund class is different from the Hölder norm in the generalized Hölder class (see [28] for details). Because of this, the generalized Zygmund class does not fully generalize the generalized Hölder class.

This brings us to our main goal: estimating the deviation between the three-harmonic Poisson integrals $P_{3,\rho}(f;x)$ and a 2π-periodic function f in the generalized Zygmund class $Z_r^{(w)}$, using the norm $\|\cdot\|_r^{(w_1)}$, where w_1 satisfy the same properties as w.

For readers interested in recent progress, several results that use the classical Abel-Poisson means for approximation in the norm of Hölder class are available in articles [12]-[15]. Moreover, several results closely related to ours are presented in [29].

To reach our goal, we include several key supporting results in the next section to help prove our findings.

4. Supporting Lemmas

First, let's review two key lemmas that are widely known and important to our discussion.

Lemma 4.1 (Generalized Minkowski inequality, [7], [33]). *If $\ell(t_1,t_2)$ is a function in two variables defined for $a\le t_1\le b$, $c\le t_2\le d$, then*

$$\left[\int_a^b\left|\int_c^d\ell(t_1,t_2)dt_2\right|^r dt_1\right]^{1/r}\le\int_c^d\left[\int_a^b|\ell(t_1,t_2)|^r dt_1\right]^{1/r}dt_2,\quad r\ge1.$$

Lemma 4.2 ([25]). *Let $w(t)$ and $w_1(t)$ be moduli of continuity of order two such that $w(t)/w_1(t)$ is nondecreasing and $f\in Z_r^{(w)}$. Then, for $0<t\le\pi$ and $r\ge1$,*

(i) $\|\varphi_{\cdot}(t)\|_r = \mathcal{O}\left(w(t)\right),$

(ii) $\|\varphi_{\cdot+y}(t) - 2\varphi_{\cdot}(t) + \varphi_{\cdot-y}(t)\|_r = \mathcal{O}\left(w_1\left(|y|\right)\frac{w(t)}{w_1(t)}\right).$

Next, we present two transformed forms of the three-harmonic Poisson kernel $K_{3,\rho}(t)$.

Lemma 4.3. *Let $0<\rho<1$ and $0<t\le\pi$. Then, the following identities are valid:*

(i)

$$K_{3,\rho}(t) = \frac{(1-\rho)^3}{8}\sum_{k=0}^{\infty}\big[(1+\rho)^2k^2 + 2(1+\rho)(3+2\rho)k + (3\rho^2+9\rho+8)\big]\rho^k\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}.$$

(ii)

$$K_{3,\rho}(t) = \frac{(1-\rho)^3}{8}\sum_{k=0}^{\infty}\big[(1-\rho)(1+\rho)^2k^2 - 2(1+\rho)(3\rho^2+2\rho-3)k - 2(4\rho^3+9\rho^2+3\rho-4)\big]\rho^k\left(\frac{\sin\frac{(k+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2.$$

Proof. We define $\cos 0 := \frac{1}{2}$ only to express the kernel $K_{3,\rho}(t)$ as a single infinite sum. This is a technical choice to simplify the notation, without affecting the classical meaning of cosine.

(i) We begin by applying Abel's transformation and using the identity

$$\sum_{j=0}^{k}\cos jt = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)},$$

which allows us to rewrite the sum in the following form

$$\begin{aligned}
K_{3,\rho}(t) &= \lim_{m\to\infty}\sum_{k=0}^{m}\left(1+\frac{k}{4}(3-\rho^2)(1-\rho^2)+\frac{k^2}{8}(1-\rho^2)^2\right)\rho^k\cos kt\\
&= \lim_{m\to\infty}\sum_{k=0}^{m-1}\Bigg\{\left(1+\frac{k}{4}(3-\rho^2)(1-\rho^2)+\frac{k^2}{8}(1-\rho^2)^2\right)\\
&\quad-\left(1+\frac{k+1}{4}(3-\rho^2)(1-\rho^2)+\frac{(k+1)^2}{8}(1-\rho^2)^2\right)\rho\Bigg\}\rho^k\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\\
&\quad+\lim_{m\to\infty}\left(1+\frac{m}{4}(3-\rho^2)(1-\rho^2)+\frac{m^2}{8}(1-\rho^2)^2\right)\rho^m\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}.
\end{aligned}$$

Let us denote by E the expression inside the curly brackets. Then, after expanding and factoring, we got

$$E = \frac{(1-\rho)^3}{8}\left(k^2\rho^2+2k^2\rho+k^2+4k\rho^2+10k\rho+6k+3\rho^2+9\rho+8\right).$$

For the benefit of the interested reader, we proceed to verify this identity in full detail. Specifically, first we have:

$$\begin{aligned}
E &= \Bigg\{ \left(1+\frac{k}{4}(3-\rho^2)(1-\rho^2)+\frac{k^2}{8}(1-\rho^2)^2\right) \\
&\quad - \left(1+\frac{k+1}{4}(3-\rho^2)(1-\rho^2)+\frac{(k+1)^2}{8}(1-\rho^2)^2\right)\rho \Bigg\} \\
&= \Bigg\{ 1+\frac{k}{4}(3-\rho^2)(1-\rho^2)+\frac{k^2}{8}(1-\rho^2)^2 \\
&\quad -\rho-\frac{k+1}{4}(3-\rho^2)(1-\rho^2)\rho-\frac{(k+1)^2}{8}(1-\rho^2)^2\rho \Bigg\} \\
&= \Bigg\{ 1-\rho+\frac{k}{4}(3-\rho^2)(1-\rho^2)+\frac{k^2}{8}(1-\rho^2)^2 \\
&\quad -\frac{k+1}{4}(3-\rho^2)(1-\rho^2)\rho-\frac{(k+1)^2}{8}(1-\rho^2)^2\rho \Bigg\} \\
&= (1-\rho)\Bigg\{ \underbrace{1}_{=:C} + \underbrace{\frac{k}{4}(3-\rho^2)(1+\rho)-\frac{k+1}{4}(3-\rho^2)(1+\rho)\rho}_{=:A} \\
&\quad + \underbrace{\frac{k^2}{8}(1+\rho)^2(1-\rho)-\frac{(k+1)^2}{8}(1+\rho)^2(1-\rho)\rho}_{=:B} \Bigg\}.
\end{aligned}$$

Next, we transform each of the components A, B, and C separately. First, we express the term A as:

$$\begin{aligned}
A &= \frac{k}{4}(3-\rho^2)(1+\rho)-\frac{k+1}{4}(3-\rho^2)(1+\rho)\rho \\
&= \frac{k}{4}(3-\rho^2)(1+\rho)-\frac{k}{4}(3-\rho^2)(1+\rho)\rho-\frac{1}{4}(3-\rho^2)(1+\rho)\rho \\
&= (1-\rho)\frac{k}{4}(3-\rho^2)(1+\rho)-\frac{1}{4}(3-\rho^2)(1+\rho)\rho,
\end{aligned}$$

We now turn to term B:

$$\begin{aligned}
B &= \frac{k^2}{8}(1+\rho)^2(1-\rho)-\frac{(k+1)^2}{8}(1+\rho)^2(1-\rho)\rho \\
&= (1-\rho)\left(\frac{k^2}{8}(1+\rho)^2-\frac{(k+1)^2}{8}(1+\rho)^2\rho\right) \\
&= (1-\rho)\left(\frac{k^2}{8}(1+\rho)^2-\frac{k^2}{8}(1+\rho)^2\rho-\frac{2k+1}{8}(1+\rho)^2\rho\right) \\
&= (1-\rho)\left[(1-\rho)\frac{k^2}{8}(1+\rho)^2-\frac{k}{4}(1+\rho)^2\rho-\frac{1}{8}(1+\rho)^2\rho\right] \\
&= (1-\rho)^2\frac{k^2}{8}(1+\rho)^2-\frac{k}{4}(1+\rho)^2\rho(1-\rho)-\frac{1}{8}(1+\rho)^2\rho(1-\rho).
\end{aligned}$$

At this step, we have nothing to do with $C = 1$. We now proceed to add the three terms:

$$\begin{aligned}
A+B+C &= (1-\rho)\frac{k}{4}(3-\rho^2)(1+\rho) - \frac{1}{4}(3-\rho^2)(1+\rho)\rho \\
&\quad + (1-\rho)^2\frac{k^2}{8}(1+\rho)^2 - \frac{k}{4}(1+\rho)^2\rho(1-\rho) - \frac{1}{8}(1+\rho)^2\rho(1-\rho) + 1 \\
&= (1-\rho)^2\frac{k^2}{8}(1+\rho)^2 \\
&\quad + (1-\rho)\frac{k}{4}(3-\rho^2)(1+\rho) - \frac{k}{4}(1+\rho)^2\rho(1-\rho) \\
&\quad - \frac{1}{4}(3-\rho^2)(1+\rho)\rho - \frac{1}{8}(1+\rho)^2\rho(1-\rho) + 1 \\
&= (1-\rho)^2\frac{k^2}{8}(1+\rho)^2 \\
&\quad + (1-\rho)\frac{k}{4}\left[(3-\rho^2)(1+\rho) - (1+\rho)^2\rho\right] \\
&\quad - \frac{1}{4}(3-\rho^2)(1+\rho)\rho - \frac{1}{8}(1+\rho)^2\rho(1-\rho) + 1 \\
&= (1-\rho)^2\frac{k^2}{8}(1+\rho)^2 + (1-\rho)\frac{k}{4}\left(3+2\rho-3\rho^2-2\rho^3\right) \\
&\quad - \frac{1}{4}(3-\rho^2)(1+\rho)\rho - \frac{1}{8}(1+\rho)^2\rho(1-\rho) + 1 \\
&= (1-\rho)^2\frac{k^2}{8}(1+\rho)^2 + (1-\rho)^2\frac{k}{4}\left(2\rho^2+5\rho+3\right) \\
&\quad - \frac{1}{4}(3-\rho^2)(1+\rho)\rho - \frac{1}{8}(1+\rho)^2\rho(1-\rho) + 1 \\
&= (1-\rho)^2\frac{k^2}{8}(1+\rho)^2 + (1-\rho)^2\frac{2k}{8}\left(2\rho+3\right)\left(1+\rho\right) \\
&\quad \underbrace{-\frac{1}{4}(3-\rho^2)(1+\rho)\rho - \frac{1}{8}(1+\rho)^2\rho(1-\rho) + 1}_{=:D}.
\end{aligned}$$

We now rewrite the remaining term D as:

$$\begin{aligned}
D &= -\frac{1}{4}(3-\rho^2)(1+\rho)\rho - \frac{1}{8}(1+\rho)^2\rho(1-\rho) + 1 \\
&= -\frac{1}{8}\left[2(3-\rho^2)(1+\rho)\rho + (1+\rho)^2\rho(1-\rho) - 8\right] \\
&= \frac{1}{8}\left(3\rho^4+3\rho^3-7\rho^2-7\rho+8\right) \\
&= -\frac{1}{8}(1-\rho)\left(3\rho^3+6\rho^2-\rho-8\right) \\
&= \frac{1}{8}(1-\rho)^2\left(3\rho^2+9\rho+8\right).
\end{aligned}$$

Therefore, the total expression $A+B+C$ becomes

$$\begin{aligned}
A+B+C &= (1-\rho)^2\frac{k^2}{8}(1+\rho)^2 + (1-\rho)^2\frac{2k}{8}\left(2\rho+3\right)\left(1+\rho\right) \\
&\quad + \frac{1}{8}(1-\rho)^2\left(3\rho^2+9\rho+8\right).
\end{aligned}$$

Finally, by substituting $A+B+C$ into the expression for E, we obtain:

$$E = \frac{(1-\rho)^3}{8}\left(k^2(1+\rho)^2 + 2k\,(2\rho+3)\,(1+\rho) + \left(3\rho^2+9\rho+8\right)\right).$$

Further, given that $0<\rho<1$, we observe that

$$\lim_{m\to\infty}\rho^m = 0, \quad \lim_{m\to\infty} m\rho^m = 0, \quad \text{and} \quad \lim_{m\to\infty} m^2\rho^m = 0. \tag{3}$$

Thus, after taking the limit, we get (i) for $0<t\le\pi$.

(ii) For simplicity in calculations, we denote the second factor of E, multiplied by ρ^k, as

$$\lambda_k^{(2)}(\rho) := [(\rho^2+2\rho+1)k^2 + (4\rho^2+10\rho+6)k + (3\rho^2+9\rho+8)]\rho^k.$$

Using the result from part (i), we apply Abel's transformation and utilize the identity

$$\sum_{i=0}^{k}\frac{\sin\left(i+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} = \left(\frac{\sin\frac{(k+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2$$

to derive

$$\begin{aligned}
K_{3,\rho}(t) &= \frac{(1-\rho)^3}{8}\sum_{k=0}^{\infty}\lambda_k(\rho)\rho^k\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \\
&= \frac{(1-\rho)^3}{8}\lim_{m\to\infty}\sum_{k=0}^{m-1}\left[\lambda_k^{(2)}(\rho)-\lambda_{k+1}^{(2)}(\rho)\right]\sum_{i=0}^{k}\frac{\sin\left(i+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \\
&\quad + \lim_{m\to\infty}\lambda_m^{(2)}(\rho)\sum_{i=0}^{m}\frac{\sin\left(i+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \\
&= \frac{(1-\rho)^3}{8}\lim_{m\to\infty}\sum_{k=0}^{m-1}\left[\lambda_k^{(2)}(\rho)-\lambda_{k+1}^{(2)}(\rho)\right]\left(\frac{\sin\frac{(k+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2 \\
&\quad + \lim_{m\to\infty}\lambda_m^{(2)}(\rho)\left(\frac{\sin\frac{(m+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2.
\end{aligned}$$

Additionally, after performing the necessary calculations, we find that

$$\begin{aligned}
\lambda_k^{(2)}(\rho)-\lambda_{k+1}^{(2)}(\rho) &= \big\{[(\rho^2+2\rho+1)k^2 + (4\rho^2+10\rho+6)k + (3\rho^2+9\rho+8)] \\
&\quad - [(\rho^2+2\rho+1)(k+1)^2 + (4\rho^2+10\rho+6)(k+1) \\
&\quad + (3\rho^2+9\rho+8)]\rho\big\}\rho^k \\
&= \big[-(\rho^3+\rho^2-\rho-1)k^2 \\
&\quad - (6\rho^3+10\rho^2-2\rho-6)k - (8\rho^3+18\rho^2+6\rho-8)\big]\rho^k \\
&= \big[(1-\rho)(1+\rho)^2k^2 \\
&\quad - 2(1+\rho)(3\rho^2+2\rho-3)k - 2(4\rho^3+9\rho^2+3\rho-4)\big]\rho^k.
\end{aligned}$$

Using this equality, the limits in (3) from the proof in part (i), along with the condition $0 < t \leq \pi$, we deduce that

$$K_{3,\rho}(t) = \frac{(1-\rho)^3}{8} \sum_{k=0}^{\infty} \big[(1-\rho)(1+\rho)^2k^2 - 2(1+\rho)(3\rho^2+2\rho-3)k - 2(4\rho^3+9\rho^2+3\rho-4)\big]\rho^k \left(\frac{\sin\frac{(k+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2,$$

which completes the proof of part (ii).

This completes the proof of the lemma. □

At this point, we present the main result of this paper, the key focus of our study.

5. Main Result

Our main result is represented by the following statement.

Theorem 5.1. *Let $w(t)$ and $w_1(t)$ be moduli of continuity of order two such that $w(t)/w_1(t)$ is non–decreasing, $w(t)/(tw_1(t))$ is non–increasing, and $f \in Z_r^{(w)}$. Then, for $0 < \rho < 1$ and $r \geq 1$, the deviation of the three-harmonic Poisson integral $P_{3,\rho}(f;x)$ from f measured in the norm $\|\cdot\|^{(w_1)}$, is given by*

$$\begin{aligned}&\|P_{3,\rho}(f)(\cdot) - f(\cdot)\|_r^{(w_1)} \qquad (4)\\ &= \mathcal{O}(1)\Big\{\Big[2\rho(1+\rho)^2(1+4\rho+\rho^2) + \rho(13+9\rho)(1+\rho)^2(1-\rho)\\ &\quad + 2\rho(11+14\rho+5\rho^2)(1-\rho)^2 + (3\rho^2+9\rho+8)(1-\rho)^3\Big]\\ &\quad + 2\left[\rho(7+6\rho+5\rho^2)(1+\rho) + 2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)\right]\ln\left(\frac{\pi}{1-\rho}\right)\Big\}\frac{w(1-\rho)}{w_1(1-\rho)}.\end{aligned}$$

Proof. We use the three-harmonic Poisson integral in its singular form

$$P_{3,\rho}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_{3,\rho}(t)\,dt,$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_{3,\rho}(t) = 1,$$

to derive

$$P_{3,\rho}(f)(x) := P_{3,\rho}(f;x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+t)-f(x)]K_{3,\rho}(t)\,dt.$$

Utilizing the symmetry property $K_{3,\rho}(-t) = K_{3,\rho}(t)$, we rewrite the above as

$$P_{3,\rho}(f)(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)K_{3,\rho}(t)\,dt, \qquad (5)$$

where

$$\varphi_x(t) := f(x+t) - 2f(x) + f(x-t).$$

Our objective is to establish an upper bound for the quantity

$$\|P_{3,\rho}(f)\|_r^{(w_1)} = \|P_{3,\rho}(f)\|_r + \sup_{y\neq 0}\frac{\|P_{3,\rho}(f)(\cdot+y) - 2P_{3,\rho}(f)(\cdot) + P_{3,\rho}(f)(\cdot-y)\|_r}{w_1(|y|)}, \qquad (6)$$

where $r \geq 1$.

By applying Lemma 4.1 to the equality derived from (5),

$$\begin{aligned}P_{3,\rho}(f)(x+y) &- 2P_{3,\rho}(f)(x) + P_{3,\rho}(f)(x-y)\\ &= \frac{1}{\pi}\int_0^{\pi} [\varphi_{x+y}(t) - 2\varphi_x(t) + \varphi_{x-y}(t)]K_{3,\rho}(t)\,dt,\end{aligned}$$

we estimate

$$\begin{aligned}\|P_{3,\rho}(f)(\cdot+y) &- 2P_{3,\rho}(f)(\cdot) + P_{3,\rho}(f)(\cdot-y)\|_r\\ &\leq \frac{1}{\pi}\int_0^{\pi} \left\|\varphi_{\cdot+y}(t) - 2\varphi_{\cdot}(t) + \varphi_{\cdot-y}(t)\right\|_r |K_{3,\rho}(t)|\,dt.\end{aligned}$$

Under assumption that $f \in Z_r^{(w)}$ with $r \geq 1$, and invoking Lemma 4.2 part (ii), it follows that

$$\begin{aligned}\|P_{3,\rho}(f)(\cdot+y) - 2P_{3,\rho}(f)(\cdot) &+ P_{3,\rho}(f)(\cdot-y)\|_r\\ &= \mathcal{O}\left(w_1\left(|y|\right)\right)\int_0^{\pi} \frac{w(t)}{w_1(t)}|K_{3,\rho}(t)|\,dt.\end{aligned}$$

By splitting this integral, we write

$$\begin{aligned}\|P_{3,\rho}(f)(\cdot+y) &- 2P_{3,\rho}(f)(\cdot) + P_{3,\rho}(f)(\cdot-y)\|_r\\ &= \mathcal{O}\left(w_1\left(|y|\right)\right)\Bigg(\underbrace{\int_0^{1-\rho} \frac{w(t)}{w_1(t)}|K_{3,\rho}(t)|\,dt}_{=:R_{11}(\rho)} + \underbrace{\int_{1-\rho}^{\pi} \frac{w(t)}{w_1(t)}|K_{3,\rho}(t)|\,dt}_{=:R_{12}(\rho)}\Bigg). \qquad (7)\end{aligned}$$

Let's find an upper bound for the absolute value of the kernel $K_{3,\rho}(t)$, as shown in Lemma 4.3 (i). Using basic trigonometric inequalities, such as $\pi\sin\varpi \geq 2\varpi$ for $\varpi \in [0,\pi/2]$, and $|\sin\varpi| \leq |\varpi|$ for any ϖ, we get the following result

$$\begin{aligned}|K_{3,\rho}(t)| \leq \frac{\pi(1-\rho)^3}{16}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)\big[&(1+\rho)^2k^2 + 2(1+\rho)(3+2\rho)k\\ &+ (3\rho^2+9\rho+8)\big]\rho^k. \qquad (8)\end{aligned}$$

Moreover, by successively differentiating both sides of the geometric series

$$\sum_{k=0}^{\infty}\rho^k = \frac{1}{1-\rho} \quad \text{for } |\rho| < 1, \qquad (9)$$

and substituting the resulting equalities

$$\sum_{k=0}^{\infty}k\rho^{k-1} = \frac{1}{(1-\rho)^2} \overset{\cdot\rho}{\Longrightarrow} \sum_{k=0}^{\infty}k\rho^k = \frac{\rho}{(1-\rho)^2} \quad \text{for } |\rho| < 1, \qquad (10)$$

$$\sum_{k=0}^{\infty}k^2\rho^{k-1} = \frac{1+\rho}{(1-\rho)^3} \overset{\cdot\rho}{\Longrightarrow} \sum_{k=0}^{\infty}k^2\rho^k = \frac{\rho(1+\rho)}{(1-\rho)^3}, \quad \text{for } |\rho| < 1, \qquad (11)$$

and

$$\sum_{k=0}^{\infty}k^3\rho^{k-1} = \frac{1+4\rho+\rho^2}{(1-\rho)^4} \overset{\cdot\rho}{\Longrightarrow} \sum_{k=0}^{\infty}k^3\rho^k = \frac{\rho(1+4\rho+\rho^2)}{(1-\rho)^4}, \quad \text{for } |\rho| < 1, \qquad (12)$$

into (8) yields

$$\begin{aligned}
|K_{3,\rho}(t)| \leq& \frac{\pi(1-\rho)^3}{16}\Big\{\sum_{k=0}^{\infty}\big[(1+\rho)^2k^3\rho^k+2(1+\rho)(3+2\rho)k^2\rho^k+(3\rho^2+9\rho+8)k\rho^k\big] \\
&+\frac{1}{2}\sum_{k=0}^{\infty}\big[(1+\rho)^2k^2\rho^k+2(1+\rho)(3+2\rho)k\rho^k+(3\rho^2+9\rho+8)\rho^k\big]\Big\} \\
=&\frac{\pi}{16}\Big\{\rho(1+\rho)^2(1+4\rho+\rho^2)\frac{1}{(1-\rho)}+2\rho(1+\rho)^2(3+2\rho) \\
&+\rho(8+9\rho+3\rho^2)(1-\rho)+\frac{\rho}{2}(1+\rho)^3 \\
&+\rho(1+\rho)(3+2\rho)(1-\rho)+\frac{1}{2}(8+9\rho+3\rho^2)(1-\rho)^2\Big\}. \qquad (13)
\end{aligned}$$

Given the estimate (13) and the fact that the function $\frac{w(t)}{w_1(t)}$is non-decreasing with respect to t, it follows that

$$\begin{aligned}
R_{11}(\rho) =& \int_0^{1-\rho}\frac{w(t)}{w_1(t)}|K_{3,\rho}(t)|dt \\
\leq& \frac{w(1-\rho)}{w_1(1-\rho)}\int_0^{1-\rho}|K_{3,\rho}(t)|dt \\
\leq& \frac{\pi}{16}\Big\{\rho(1+\rho)^2(1+4\rho+\rho^2)+(1/2)\rho(13+9\rho)(1+\rho)^2(1-\rho) \\
&+\rho(5\rho^2+14\rho+11)(1-\rho)^2+(1/2)(3\rho^2+9\rho+8)(1-\rho)^3\Big\}\frac{w(1-\rho)}{w_1(1-\rho)}. \qquad (14)
\end{aligned}$$

Now, let us find a different upper bound for the absolute value of the kernel $K_{3,\rho}(t)$, as stated in the part (ii) of Lemma 4.3. Using the inequalities $\pi\sin\varpi \geq 2\varpi$ for $\varpi \in [0,\pi/2]$ and $|\sin\varpi| \leq 1$ for any ϖ, we get

$$\begin{aligned}
|K_{3,\rho}(t)| \leq& \frac{\pi^2(1-\rho)^3}{8t^2}\sum_{k=0}^{\infty}\big[(1-\rho)(1+\rho)^2k^2\rho^k+2(1+\rho)(3\rho^2+2\rho+3)k\rho^k \\
&+2(4\rho^3+9\rho^2+3\rho+4)\rho^k\big]. \qquad (15)
\end{aligned}$$

In (15), the equalities from (9)–(11) are used to find

$$\begin{aligned}
|K_{3,\rho}(t)| \leq& \frac{\pi^2(1-\rho)^3}{8t^2}\Big[(1-\rho)(1+\rho)^2\frac{\rho(1+\rho)}{(1-\rho)^3} \\
&+2(1+\rho)(3\rho^2+2\rho+3)\frac{\rho}{(1-\rho)^2}+2(4\rho^3+9\rho^2+3\rho+4)\frac{1}{1-\rho}\Big] \\
=&\frac{\pi^2}{8}\big[\rho(7+6\rho+5\rho^2)(1-\rho^2)+2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)^2\big]t^{-2}. \qquad (16)
\end{aligned}$$

We now estimate $R_{12}(\rho)$ from above. Using the inequality (16) and the fact that $\frac{w(t)}{tw_1(t)}$ decreases as t increases, we get

$$\begin{aligned}
R_{12}(\rho) &= \int_{1-\rho}^{\pi} \frac{w(t)}{w_1(t)} |K_{3,\rho}(t)|\, dt \\
&\leq \frac{\pi^2}{8} \left[\rho(7+6\rho+5\rho^2)(1-\rho^2) + 2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)^2\right] \int_{1-\rho}^{\pi} \frac{w(t)}{w_1(t)} \frac{dt}{t^2} \\
&\leq \frac{\pi^2}{8} \left[\rho(7+6\rho+5\rho^2)(1+\rho) + 2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)\right] \\
&\quad \times \frac{w(1-\rho)}{w_1(1-\rho)} \ln\left(\frac{\pi}{1-\rho}\right). \qquad (17)
\end{aligned}$$

Combining the results of (7), (14), and (17), we have

$$\begin{aligned}
&\frac{\|P_{3,\rho}(f)(\cdot+y) - 2P_{3,\rho}(f)(\cdot) + P_{3,\rho}(f)(\cdot-y)\|_r}{w_1(|y|)} \\
&= \mathcal{O}(1) \Big\{ \big[\rho(1+\rho)^2(1+4\rho+\rho^2) + (1/2)\rho(13+9\rho)(1+\rho)^2(1-\rho) \\
&\quad + \rho(5\rho^2+14\rho+11)(1-\rho)^2 + (1/2)(3\rho^2+9\rho+8)(1-\rho)^3\big] \\
&\quad + \big[\rho(7+6\rho+5\rho^2)(1+\rho) + 2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)\big] \ln\left(\frac{\pi}{1-\rho}\right) \Big\} \frac{w(1-\rho)}{w_1(1-\rho)}. \qquad (18)
\end{aligned}$$

To finish the proof, we need to find the upper limit of $\|P_{3,\rho}(f)\|_r$. We can do this using a method similar to the one explained before. In particular, by using Lemma 4.1 on equation (5), we get

$$\|P_{3,\rho}(f)\|_r \leq \frac{1}{\pi} \int_0^{\pi} \|\varphi_{\cdot}(t)\|_r |K_{3,\rho}(t)|\, dt. \qquad (19)$$

Then, by applying Lemma 4.2 (i) to equation (19), we find

$$\begin{aligned}
\|P_{3,\rho}(f)\|_r &= \mathcal{O}(1) \int_0^{\pi} w(t) |K_{3,\rho}(t)|\, dt \\
&= \mathcal{O}(1) \Bigg(\underbrace{\int_0^{1-\rho} w(t)|K_{3,\rho}(t)|\, dt}_{=:R_{21}(\rho)} + \underbrace{\int_{1-\rho}^{\pi} w(t)|K_{3,\rho}(t)|\, dt}_{=:R_{22}(\rho)} \Bigg). \qquad (20)
\end{aligned}$$

At this point, we analyze the individual contributions of $R_{21}(\rho)$ and $R_{22}(\rho)$. Using the estimate (13) for $|K_{3,\rho}(t)|$ and the fact that $\eta_1(t)$ is monotonic, we obtain

$$\begin{aligned}
R_{21}(\rho) &\leq w(1-\rho) \int_0^{1-\rho} |K_{3,\rho}(t)|\, dt \\
&\leq \frac{\pi}{16} \Big\{ \rho(1+\rho)^2(1+4\rho+\rho^2) + (1/2)\rho(13+9\rho)(1+\rho)^2(1-\rho) \\
&\quad + \rho(11+14\rho+5\rho^2)(1-\rho)^2 + (1/2)(3\rho^2+9\rho+8)(1-\rho)^3 \Big\} w_1(1-\rho) \frac{w(1-\rho)}{w_1(1-\rho)} \\
&\leq \frac{\pi w_1(\pi)}{16} \Big\{ \rho(1+\rho)^2(1+4\rho+\rho^2) + (1/2)\rho(13+9\rho)(1+\rho)^2(1-\rho) \\
&\quad + \rho(11+14\rho+5\rho^2)(1-\rho)^2 + (1/2)(3\rho^2+9\rho+8)(1-\rho)^3 \Big\} \frac{w(1-\rho)}{w_1(1-\rho)}. \qquad (21)
\end{aligned}$$

For $R_{22}(\rho)$, we use the estimate from (16) and then continue as follows

$$\begin{aligned}
R_{22}(\rho) &\le \frac{\pi^2}{8}\left[\rho(7+6\rho+5\rho^2)(1-\rho^2)+2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)^2\right]\\
&\quad\times\int_{1-\rho}^{\pi} w_1(t)\frac{w(t)}{w_1(t)}\frac{dt}{t^2}\\
&\le \frac{\pi^2 w_1(\pi)}{8}\left[\rho(7+6\rho+5\rho^2)(1+\rho)+2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)\right]\\
&\quad\times\frac{w(1-\rho)}{w_1(1-\rho)}\ln\left(\frac{\pi}{1-\rho}\right). \qquad (22)
\end{aligned}$$

By putting together estimates from (20), (21), and (22), we find that

$$\begin{aligned}
&\|P_{3,\rho}(f)\|_r\\
&=\mathcal{O}(1)\Big\{\rho(1+\rho)^2(1+4\rho+\rho^2)+(1/2)\rho(13+9\rho)(1+\rho)^2(1-\rho)\\
&\quad+\rho(11+14\rho+5\rho^2)(1-\rho)^2+(1/2)(3\rho^2+9\rho+8)(1-\rho)^3\\
&\quad+\left[\rho(7-2\rho)(1+\rho)-2(4\rho^3+9\rho^2+3\rho-4)(1-\rho)\right]\ln\left(\frac{\pi}{1-\rho}\right)\Big\}\frac{w(1-\rho)}{w_1(1-\rho)}. \qquad (23)
\end{aligned}$$

Finally, by putting together the results from relations (6), (18), and (23), we conclude

$$\begin{aligned}
&\|P_{3,\rho}(f)\|_r^{(w_1)}\\
&=\mathcal{O}(1)\Big\{\Big[2\rho(1+\rho)^2(1+4\rho+\rho^2)+\rho(13+9\rho)(1+\rho)^2(1-\rho)\\
&\quad+2\rho(11+14\rho+5\rho^2)(1-\rho)^2+(3\rho^2+9\rho+8)(1-\rho)^3\Big]\\
&\quad+2\left[\rho(7+6\rho+5\rho^2)(1+\rho)+2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)\right]\ln\left(\frac{\pi}{1-\rho}\right)\Big\}\frac{w(1-\rho)}{w_1(1-\rho)},
\end{aligned}$$

which is (4).

This completes the proof. □

We select the functions $w(t)=t^{\nu_1}$ and $w_1(t)=t^{\nu_2}$ in the context of Theorem 5.1, where the parameters ν_1 and ν_2 satisfy $0\le\nu_2<\nu_1\le 1$. With these conditions, the ratio $w(t)/w_1(t)=t^{\nu_1-\nu_2}$ is increasing, while $w(t)/(tw_1(t))=1/(t^{1-\nu_1+\nu_2})$ is decreasing. As a result, we obtain the following corollary.

Corollary 5.1. *Let $f\in Z_{\nu_1,r}$, $r\ge 1$, and $0\le\nu_2<\nu_1\le 1$. Then, for $0<\rho<1$ and $r\ge 1$, the deviation of the three-harmonic Poisson integral $P_{3,\rho}(f;x)$ from f measured in the norm $\|\cdot\|_{\nu_2,r}$, is given by*

$$\begin{aligned}
&\|P_{3,\rho}(f)(\cdot)-f(\cdot)\|_{\nu_2,r}\\
&=\mathcal{O}(1)\Big\{\Big[2\rho(1+\rho)^2(1+4\rho+\rho^2)+\rho(13+9\rho)(1+\rho)^2(1-\rho)\\
&\quad+2\rho(11+14\rho+5\rho^2)(1-\rho)^2+(3\rho^2+9\rho+8)(1-\rho)^3\Big]\\
&\quad+2\left[\rho(7+6\rho+5\rho^2)(1+\rho)+2(4\rho^3+9\rho^2+3\rho+4)(1-\rho)\right]\ln\left(\frac{\pi}{1-\rho}\right)\Big\}(1-\rho)^{\nu_1-\nu_2}.
\end{aligned}$$

In terms of the approximation achieved in our result, we make the following remark.

Remark 5.1. *The degree of approximation described in Corollary 5.1 can be bounded by*

$$\Big\{3\big[2+7(1-\rho)\big]+\big[9+10(1-\rho)\big]\ln\left(\frac{\pi}{1-\rho}\right)\Big\}(1-\rho)^{\nu_1-\nu_2},$$

up to a constant, and it approaches zero as $\rho \to 1^-$*. This shows the high effectiveness of the approximation. It also highlights the usefulness of three-harmonic Poisson integrals for approximating periodic functions in the generalized Zygmund class.*

In the special case when $\nu_1 = 1$ *and* $\nu_2 = 0$*, the above bound simplifies further to*

$$(1-\rho) + (1-\rho)^2 \ln\left(\frac{\pi}{1-\rho}\right).$$

6. Conclusion

In summary, this paper explores the error in approximating periodic functions within generalized Zygmund classes using three-harmonic Poisson integrals. By employing two moduli of continuity of order two, the analysis provides a detailed understanding of the approximation process and demonstrates the effectiveness of the method. The findings highlight the potential of three-harmonic Poisson integrals as a reliable tool for approximation, with a clear characterization of the error and its behavior. Future research will focus on extending these methods to other function classes and uncovering broader applications of this approach.

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