TWMS J. App. and Eng. Math. V.16, N.2, 2026, pp. 204-213

ON HYPER COMPLEX NUMBERS WITH HIGHER ORDER BALANCING NUMBERS COMPONENTS

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ABSTRACT. In this article, we define higher-order balancing numbers. Next, we employ higher-order balancing numbers to present a novel family of hyper complex numbers. These families are referred to as the higher-order balancing 2^r-ions. We give various algebraic properties of this higher-order balancing 2^r-ions, such as the recurrence relation, the generating function, Binet's formula, Catalan's identity, Cassini's identity, d'Ocagne's identity and Vajda's identity and so on. Furthermore, we derive the matrix representation of the higher-order balancing 2^r-ions, therefore establishing Cassini's identity as a new type.

Keywords: Hyper complex numbers, Higher-order balancing numbers, Recurrence relation.

AMS Subject Classification: 11B37, 11B83, 11R52.

1. INTRODUCTION

One of the simplest and most celebrated integer sequence is the Fibonacci sequence. Many mathematicians have studied the generalizations of Fibonacci sequences. Another renowned and well-known sequence is the balancing sequence. Balancing number sequence was introduced by Panda and Ray [15]. The recurrence relation for balancing number is $B_{n+1} = 6B_n - B_{n-1}$ with initials $B_0 = 0$ and $B_1 = 1$. The characteristics equation is $x^2 - 6x + 1 = 0$ with roots $\gamma_1 = 3 + \sqrt{8}$ and $\gamma_2 = 3 - \sqrt{8}$. The Binet's formula for the balancing number are given by

$$B_n = \frac{\gamma_1^n - \gamma_2^n}{2\sqrt{8}}. \tag{1}$$

The recurrence relation for Lucas-balancing number is $D_{n+1} = 6D_n - D_{n-1}$ with initials $D_0 = 1$ and $D_1 = 3$, the characteristic equation is same as the balancing number and it's

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§ Manuscript received: January 08, 2025; accepted: May 27, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.2;

Binet's formula is

$$D_n = \frac{\gamma_1^n + \gamma_2^n}{2}. \quad (2)$$

The study of normed division algebra with a number sequence begins with the earlier work of A.F. Horadam [3] on quaternions with Fibonacci and Lucas numbers. In 1843, William Rowan Hamilton discovered the quaternions by extending the concept of the set of complex numbers $\mathbb{C}$ to the set of quaternions; where quaternions $\mathbb{Q}$ is $4 = 2^2$-dimensional algebra over $\mathbb{R}$. This algebra is non-commutative and associative. Inspired by W.R. Hamilton's work, in 1843, J.T. Graves discovered the octonions($\mathbb{O}$), which is an $8 = 2^3$-dimensional algebra over $\mathbb{R}$. The octonions($\mathbb{O}$) are a non-commutative and non-associative algebra. Another generalization is sedenion($\mathbb{S}$) algebra, which is a non-commutative, non-associative, non-alternative, but power-associative $16 = 2^4$-dimensional algebra with a quadratic norm and whose elements are constructed from real numbers $\mathbb{R}$. In 1845, A. Cayley rediscovered these algebras and in their respect, they are sometimes also referred to as the Cayley numbers. The subsequent doubling process applied to sedenion($\mathbb{S}$) generates the trigintaduonions($\mathbb{T}$) is $32 = 2^5$-dimensional algebra over $\mathbb{R}$. This doubling process can be extended beyond the trigintaduonions to construct the 2^r-ions (or hyper complex numbers). The real 2^r-ions algebra is a 2^r-dimensional $\mathbb{R}$-linear space with basis $\{e_0, e_1, e_2, \ldots, e_{2^r-1}\}$, where e_0 is referred as the unit element and $\{e_1, e_2, \ldots, e_{2^r-1}\}$ are imaginaries.

Numerous mathematicians have investigated quaternions, octonions with balancing and Lucas-balancing number components. The nth balancing quaternion and the nth Lucas-balancing quaternion were defined by Patel and Ray [10]. Gaussian balancing and Gaussian Lucas-balancing quaternions were introduced by Asci and Aydinyuz [1]. They show matrix representations for these quaternions. Tasci [16] studied the bicomplex balancing and bicomplex Lucas-balancing quaternions. Prasad et al. [11] presented the k-balancing and k-Lucas-balancing octonions and hyperbolic octonions. Subsequently, Göcen and Soykan [2] introduced the Horadam 2^s-ions, which are the generalizations of quaternions, octonions, etc.

A hyper complex number is given by Kantor and Solodovnikov (1989) as an element of a unital, but not necessarily associative or commutative, finite-dimensional algebra over the real numbers. The elements are generated with real number coefficients $(a_0, a_1, \ldots, a_n)$ for a basis $\{1, i_1, i_2, \ldots, i_n\}$. Where possible, it is conventional to choose the basis so that $ik^2 \in \{-1, 0, 1\}$.

Many researchers have recently focused on studying higher-order numbers. Randid [14], for instance, established the higher-order Fibonacci numbers. Also, higher-order Fibonacci quaternions were defined by Kizilateş and Kone [4]. Furthermore, higher-order Fibonacci hyper complex numbers were introduced by Kızılateş and Kone [5]. Subsequently, Özkan and Uysal [9] presented higher-order Jacobsthal and Jacobsthal-Lucas numbers, as well as higher-order Jacobsthal and Jacobsthal-Lucas quaternions. Then Özimamoğlu [7, 8] introduced the hyper complex numbers with higher-order Pell number components and Jacobsthal number components respectively. In [12, 13] Prasad et al. studied the higher-order Mersenne numbers and higher-order Balancing numbers respectively.

Motivated by some of the previously listed articles, we present higher-order balancing numbers $B_n^{(s)}$. We define higher-order balancing 2^r-ions (higher-order balancing hyper complex numbers) $\mathcal{HCB}_n^{(s)}$ whose components are balancing numbers. We find recurrence relation, Binet's formula, generating function, exponential generating function, various identities for $\mathcal{HC}B_n^{(s)}$. After that, we construct a matrix with higher-order balancing 2^r-ions entries and obtain Cassini's identity by using the matrices.

2. Higher order balancing 2^r-ions

In this section, we introduce the higher-order balancing 2^r-ions. Also, we derive new properties and identities of them. Throughout this article, let

$$\hat{\gamma_1} = \sum_{i=0}^{2^r-1} \gamma_1^{is} e_i = e_0 + \gamma_1^s e_1 + \gamma_1^{2s} e_2 + \ldots + \gamma_1^{(2^r-1)s} e_{2^r-1},$$
$$\hat{\gamma_2} = \sum_{i=0}^{2^r-1} \gamma_2^{is} e_i = e_0 + \gamma_2^s e_1 + \gamma_2^{2s} e_2 + \ldots + \gamma_2^{(2^r-1)s} e_{2^r-1}.$$

Now, with the help of the Equation 1, we describe a generalization of balancing numbers as follows:

Definition 2.1. *The higher-order balancing numbers for $s \geq 1$ integer are defined by*

$$B_n^{(s)} = \frac{B_{ns}}{B_s} = \frac{\gamma_1^{ns} - \gamma_2^{ns}}{\gamma_1^s - \gamma_2^s}. \tag{3}$$

As B_{ns} is divisible by B_s, the ratio $\frac{B_{ns}}{B_s}$ is an integer. Therefore, all higher-order balancing numbers $B_n^{(s)}$ are integers. Let $s = 1$, then the higher-order balancing numbers $B_n^{(1)}$ become the well-known balancing numbers B_n. We present the higher-order balancing numbers $B_n^{(s)}$ for some n and s in Table 1.

$B_n^{(s)}$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
$B_0^{(s)}$	0	0	0	0	0
$B_1^{(s)}$	1	1	1	1	1
$B_2^{(s)}$	6	36	198	1154	6726
$B_3^{(s)}$	35	1155	39203	1331715	45239075

Table 1. The higher-order balancing numbers $B_n^{(s)}$ for some n and s.

Definition 2.2. *The higher-order balancing hyper complex numbers $\mathcal{HC}B_n^{(s)}$ (or higher-order balancing 2^r-ions) are defined by*

$$\mathcal{HC}B_n^{(s)} = \sum_{i=0}^{2^r-1} B_{n+i}^{(s)} e_i = B_n^{(s)} e_0 + B_{n+1}^{(s)} e_1 + B_{n+2}^{(s)} e_2 + \ldots + B_{n+2^r-1}^{(s)} e_{2^r-1},$$

where $B_n^{(s)}$ is the n-th higher-order balancing number.

Some special situations for $\mathcal{HC}B_n^{(s)}$ in Definition 2.2 are as follows:

1. If we take $r = 0$, we get the higher-order balancing numbers $B_n^{(s)}$[equation 3]
2. If we take $r = 1$, we get the higher-order balancing complex numbers $\mathbb{C}B_n^{(s)}$
3. If we take $r = 2$, we get the higher-order balancing quaternions $\mathbb{Q}B_n^{(s)}$
4. If we take $r = 3$, we get the higher-order balancing octonions $\mathbb{O}B_n^{(s)}$
5. If we take $r = 4$, we get the higher-order balancing sedenions $\mathbb{S}B_n^{(s)}$
6. If we take $r = 0$ and $s = 1$, we get the well-known balancing numbers B_n

7. If we take $r = 1$ and $s = 1$, we get the balancing complex numbers $\mathbb{C}B_n$
8. If we take $r = 2$ and $s = 1$, we get the balancing quaternions $\mathbb{Q}B_n$
9. If we take $r = 3$ and $s = 1$, we get the balancing octonions $\mathbb{O}B_n$
10. If we take $r = 4$ and $s = 1$, we get the balancing sedenions $\mathbb{S}B_n$.

The conjugate of the higher-order balancing 2^r-ions $\mathcal{HC}B_n^{(s)}$ is

$$\begin{aligned}\overline{\mathcal{HC}B_n^{(s)}} &= B_n^{(s)}e_0 - \sum_{i=1}^{2^r-1} B_{n+i}^{(s)}e_i \\ &= B_n^{(s)}e_0 - B_{n+1}^{(s)}e_1 - \ldots - B_{n+2^r-1}^{(s)}e_{2^r-1}. \end{aligned} \tag{4}$$

Proposition 2.1. *For higher-order balancing 2^r-ions $\mathcal{HC}B_n^{(s)}$, we get*

$$\mathcal{HC}B_n^{(s)} + \overline{\mathcal{HC}B_n^{(s)}} = 2B_n^{(s)}.$$

Proof. Using Definition 2.2 and Equation 4, we obtain the required result. □

Theorem 2.1 (Binet's formula)**.** *The Binet's formula of the higher-order balancing 2^r-ions $\mathcal{HC}B_n^{(s)}$ is*

$$\mathcal{HC}B_n^{(s)} = \frac{\gamma_1^{ns}\hat{\gamma}_1 - \gamma_2^{ns}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}.$$

Proof. From Definition 2.2 and 2.1, we obtain

$$\begin{aligned}\mathcal{HC}B_n^{(s)} &= B_n^{(s)}e_0 + B_{n+1}^{(s)}e_1 + B_{n+2}^{(s)}e_2 + \ldots + B_{n+2^r-1}^{(s)}e_{2^r-1} \\ &= [\frac{\gamma_1^{ns} - \gamma_2^{ns}}{\gamma_1^s - \gamma_2^s}]e_0 + [\frac{\gamma_1^{(n+1)s} - \gamma_2^{(n+1)s}}{\gamma_1^s - \gamma_2^s}]e_1 + [\frac{\gamma_1^{(n+2)s} - \gamma_2^{(n+2)s}}{\gamma_1^s - \gamma_2^s}]e_2 \\ &+ \ldots + [\frac{\gamma_1^{(n+2^r-1)s} - \gamma_2^{(n+2^r-1)s}}{\gamma_1^s - \gamma_2^s}]e_{2^r-1}.\end{aligned}$$

After some mathematical calculations, we get

$$\mathcal{HC}B_n^{(s)} = \frac{\gamma_1^{ns}\hat{\gamma}_1 - \gamma_2^{ns}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}.$$

This completes the proof. □

Corollary 2.1. *For some special values of s, by Theorem 2.1, the Binet's formulas of $\mathcal{HC}B_n^{(1)}$ are given as follows:*

(i) *For $r = 1$, we derive the Binet's formula of the balancing complex numbers as*

$$\mathbb{C}B_n = \frac{\gamma_1^n\hat{\gamma}_1 - \gamma_2^n\hat{\gamma}_2}{2\sqrt{8}} \ (balancing\ 2^1 - ions)$$

(ii) *For $r = 2$, we derive the Binet's formula of the balancing quaternions as*

$$\mathbb{Q}B_n = \frac{\gamma_1^n\hat{\gamma}_1 - \gamma_2^n\hat{\gamma}_2}{2\sqrt{8}} \ (balancing\ 2^2 - ions)$$

(iii) *For $r = 3$, we derive the Binet's formula of the balancing octonions as*

$$\mathbb{O}B_n = \frac{\gamma_1^n\hat{\gamma}_1 - \gamma_2^n\hat{\gamma}_2}{2\sqrt{8}} \ (balancing\ 2^3 - ions)$$

(iv) *For $r = 4$, we derive the Binet's formula of the balancing sedenions as*

$$\mathbb{S}B_n = \frac{\gamma_1^n \hat{\gamma}_1 - \gamma_2^n \hat{\gamma}_2}{2\sqrt{8}} \text{ (balancing } 2^4 - ions)$$

(v) *For $r \in \mathbb{Z}^+$, we derive the Binet's formula of the balancing 2^r-ions as*

$$\mathcal{HCB}_n^{(1)} = \frac{\gamma_1^n \hat{\gamma}_1 - \gamma_2^n \hat{\gamma}_2}{2\sqrt{8}} \text{ (balancing } 2^r - ions).$$

Theorem 2.2. *For $n \in \mathbb{Z}^+$, we get the following recurrence relation:*

$$\mathcal{HCB}_{n+1}^{(s)} = 2D_s \mathcal{HCB}_n^{(s)} - \mathcal{HCB}_{n-1}^{(s)}, \text{ where } D_s \text{ is the Lucas-balancing number.}$$

Proof. By using Binet's formula in Theorem 2.1 and Equation 2, we obtain

$$\begin{aligned}
\mathcal{HCB}_{n+1}^{(s)} &= \frac{\gamma_1^{(n+1)s}\hat{\gamma}_1 - \gamma_2^{(n+1)s}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s} \\
&= \frac{1}{\gamma_1^s - \gamma_2^s}(\gamma_1^{(n+1)s}\hat{\gamma}_1 - \gamma_2^{(n+1)s}\hat{\gamma}_2) \\
&= \frac{1}{\gamma_1^s - \gamma_2^s}(\gamma_1^{(n+1)s}\hat{\gamma}_1 - \gamma_1^s\gamma_2^{ns}\hat{\gamma}_2 + \gamma_1^s\gamma_2^{ns}\hat{\gamma}_2 - \gamma_2^{(n+1)s}\hat{\gamma}_2) \\
&= \frac{1}{\gamma_1^s - \gamma_2^s}\left[\gamma_1^s(\gamma_1^{ns}\hat{\gamma}_1 - \gamma_2^{ns}\hat{\gamma}_2) + (\gamma_1^s\gamma_2^{ns}\hat{\gamma}_2 - \gamma_2^{(n+1)s}\hat{\gamma}_2)\right] \\
&= \gamma_1^s \mathcal{HCB}_n^{(s)} + \frac{1}{\gamma_1^s - \gamma_2^s}(\gamma_1^s\gamma_2^{ns}\hat{\gamma}_2 - \gamma_2^{(n+1)s}\hat{\gamma}_2) \\
&= (\gamma_1^s + \gamma_2^s)\mathcal{HCB}_n^{(s)} - \gamma_2^s \mathcal{HCB}_n^{(s)} + \frac{1}{\gamma_1^s - \gamma_2^s}(\gamma_1^s\gamma_2^{ns}\hat{\gamma}_2 - \gamma_2^{(n+1)s}\hat{\gamma}_2) \\
&= 2D_s \mathcal{HCB}_n^{(s)} + \frac{1}{\gamma_1^s - \gamma_2^s}\left[-\gamma_2^s\gamma_1^{ns}\hat{\gamma}_1 + \gamma_2^s\gamma_2^{ns}\hat{\gamma}_2 + \gamma_1^s\gamma_2^{ns}\hat{\gamma}_2 - \gamma_2^{(n+1)s}\hat{\gamma}_2\right] \\
&= 2D_s \mathcal{HCB}_n^{(s)} + \frac{1}{\gamma_1^s - \gamma_2^s}(\gamma_1\gamma_2)^s(-\gamma_1^{(n-1)s}\hat{\gamma}_1 + \gamma_2^{(n-1)s}\hat{\gamma}_2) \\
&= 2D_s \mathcal{HCB}_n^{(s)} - \frac{\gamma_1^{(n-1)s}\hat{\gamma}_1 - \gamma_2^{(n-1)s}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s} \\
&= 2D_s \mathcal{HCB}_n^{(s)} - \mathcal{HCB}_{n-1}^{(s)}.
\end{aligned}$$

Which completes the proof. □

Theorem 2.3. *The generating function of the higher-order balancing 2^r-ions $\mathcal{HCB}_n^{(s)}$ is*

$$\mathcal{HCB}_n^{(s)}(t) = \frac{(\hat{\gamma}_1 - \hat{\gamma}_2) - (\gamma_2^s\hat{\gamma}_1 - \gamma_1^s\hat{\gamma}_2)t}{(\gamma_1^s - \gamma_2^s)(1 - 2D_s t + t^2)}, \text{ where } D_s \text{ is the Lucas-balancing number.}$$

Proof. The generating function of $\mathcal{HC}B_n^{(s)}$ is given by

$$\begin{aligned}\mathcal{HC}B_n^{(s)}(t) &= \sum_{n=0}^{\infty} \mathcal{HC}B_n^{(s)} t^n \\ &= \sum_{n=0}^{\infty} \Big[\frac{\gamma_1^{ns}\hat{\gamma}_1 - \gamma_2^{ns}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}\Big] t^n \\ &= \frac{1}{\gamma_1^s - \gamma_2^s}\Big[\hat{\gamma}_1 \sum_{n=0}^{\infty}(\gamma_1^s t)^n - \hat{\gamma}_2 \sum_{n=0}^{\infty}(\gamma_2^s t)^n\Big] \\ &= \frac{1}{\gamma_1^s - \gamma_2^s}\Big[\hat{\gamma}_1 \frac{1}{1-\gamma_1^s t} - \hat{\gamma}_2 \frac{1}{1-\gamma_2^s t}\Big],\end{aligned}$$

after some mathematical calculations, we get

$$\mathcal{HC}B_n^{(s)}(t) = \frac{(\hat{\gamma}_1 - \hat{\gamma}_2) - (\gamma_2^s \hat{\gamma}_1 - \gamma_1^s \hat{\gamma}_2)t}{(\gamma_1^s - \gamma_2^s)(1 - 2D_s t + t^2)}.$$

Thus, the result is obtained. □

Corollary 2.2. *For some special values of r, by Theorem 2.3 the generating functions of $\mathcal{HC}B_n^{(1)}(t)$ are given as follows:*

(i) *For $r = 1$, we derive the generating function of the balancing complex numbers as*

$$\mathbb{C}B_n(t) = \frac{te_0 + e_1}{1 - 6t + t^2} \ (\textit{balancing } 2^1 - \textit{ions})$$

(ii) *For $r = 2$, we derive the generating function of the balancing quaternions as*

$$\mathbb{Q}B_n(t) = \frac{te_0 + e_1 + (6-t)e_2 + (35-6t)e_3}{1 - 6t + t^2} \ (\textit{balancing } 2^2 - \textit{ions})$$

(iii) *For $r = 3$, we derive the generating function of the balancing octonions as*

$$\mathbb{O}B_n(t) = \frac{te_0 + \sum_{i=1}^{7}(B_i - B_{i-1}t)e_i}{1 - 6t + t^2} \ (\textit{balancing } 2^3 - \textit{ions})$$

(iv) *For $r = 4$, we derive the generating function of the balancing sedenions as*

$$\mathbb{S}B_n(t) = \frac{te_0 + \sum_{i=1}^{15}(B_i - B_{i-1}t)e_i}{1 - 6t + t^2} \ (\textit{balancing } 2^4 - \textit{ions})$$

(v) *For $r \in \mathbb{Z}^+$, we derive the generating function of the balancing 2^r-ions as*

$$\mathcal{HC}B_n^{(1)}(t) = \frac{te_0 + \sum_{i=1}^{2^r-1}(B_i - B_{i-1}t)e_i}{1 - 6t + t^2} \ (\textit{balancing } 2^r - \textit{ions}).$$

Theorem 2.4. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$, the generating function of balancing 2^r-ions $\mathcal{HC}B_{n+m}^{(s)}$ is*

$$\mathcal{HC}B_{n+m}^{(s)}(t) = \frac{\mathcal{HC}B_m^{(s)} - \mathcal{HC}B_{m-1}^{(s)} t}{1 - 2D_s t + t^2},$$

where D_s is the Lucas-balancing number.

Proof. We have

$$\mathcal{HC}B_{n+m}^{(s)}(t) = \sum_{n=0}^{\infty} \mathcal{HC}B_{n+m}^{(s)} t^n.$$

Using Binet's formula in Theorem 2.1 and some mathematical calculation, we obtain

$$\mathcal{HCB}_{n+m}^{(s)}(t) = \frac{\mathcal{HCB}_{m}^{(s)} - \mathcal{HCB}_{m-1}^{(s)}t}{1 - 2D_s t + t^2}.$$

□

Theorem 2.5. *The exponential generating function of the higher-order balancing 2^r-ions $\mathcal{HCB}_{n+m}^{(s)}$ is*

$$E_{\mathcal{HCB}_{n+m}^{(s)}}(t) = \frac{e^{\gamma_1^s t}\hat{\gamma}_1 - e^{\gamma_2^s t}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}.$$

Proof. We have

$$\begin{aligned} E_{\mathcal{HCB}_{n+m}^{(s)}}(t) &= \sum_{n=0}^{\infty} \mathcal{HCB}_{n+m}^{(s)} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big[\frac{\gamma_1^{ns}\hat{\gamma}_1 - \gamma_2^{ns}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}\Big] \frac{t^n}{n!} \\ &= \frac{1}{\gamma_1^s - \gamma_2^s}\Big[\hat{\gamma}_1 \sum_{n=0}^{\infty} \frac{(\gamma_1^s t)^n}{n!} - \hat{\gamma}_2 \sum_{n=0}^{\infty} \frac{(\gamma_2^s t)^n}{n!}\Big] \\ &= \frac{e^{\gamma_1^s t}\hat{\gamma}_1 - e^{\gamma_2^s t}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}. \end{aligned}$$

Thus, the proof is completed. □

Theorem 2.6 (Vajda's identity)**.** *For any integers n, m and k, we get*

$$\mathcal{HCB}_{n+m}^{(s)}\mathcal{HCB}_{n+k}^{(s)} - \mathcal{HCB}_{n}^{(s)}\mathcal{HCB}_{n+m+k}^{(s)} = \frac{B_m^s(\gamma_1^{ks}\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^{ks}\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}.$$

Proof. From Binet's formula of the higher-order for balancing 2^r-ions in Theorem 2.1 and Equation 3, we obtain

$$\begin{aligned} \mathcal{HCB}_{n+m}^{(s)}\mathcal{HCB}_{n+k}^{(s)} - \mathcal{HCB}_{n}^{(s)}\mathcal{HCB}_{n+m+k}^{(s)} &= \Big[\frac{\gamma_1^{(n+m)s}\hat{\gamma}_1 - \gamma_2^{(n+m)s}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}\Big]\Big[\frac{\gamma_1^{(n+k)s}\hat{\gamma}_1 - \gamma_2^{(n+k)s}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}\Big] \\ &\quad - \Big[\frac{\gamma_1^{ns}\hat{\gamma}_1 - \gamma_2^{ns}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}\Big]\Big[\frac{\gamma_1^{(n+m+k)s}\hat{\gamma}_1 - \gamma_2^{(n+m+k)s}\hat{\gamma}_2}{\gamma_1^s - \gamma_2^s}\Big] \\ &= \frac{1}{(\gamma_1^s - \gamma_2^s)^2}\Big[- \gamma_1^{(n+m)s}\gamma_2^{(n+k)s}\hat{\gamma}_1\hat{\gamma}_2 \\ &\quad + \gamma_1^{ns}\gamma_2^{(n+m+k)s}\hat{\gamma}_1\hat{\gamma}_2 - \gamma_2^{(n+m)s}\gamma_1^{(n+k)s}\hat{\gamma}_2\hat{\gamma}_1 \\ &\quad + \gamma_2^{ns}\gamma_1^{(n+m+k)s}\hat{\gamma}_2\hat{\gamma}_1\Big] \\ &= \frac{1}{(\gamma_1^s - \gamma_2^s)^2}\gamma_1^{ns}\gamma_2^{ns}\Big[- \gamma_1^{ms}\gamma_2^{ks}\hat{\gamma}_1\hat{\gamma}_2 + \gamma_2^{(m+k)s}\hat{\gamma}_1\hat{\gamma}_2 \\ &\quad - \gamma_2^{(n+m)s}\gamma_1^{(n+k)s}\hat{\gamma}_2\hat{\gamma}_1 + \gamma_2^{ns}\gamma_1^{(n+m+k)s}\hat{\gamma}_2\hat{\gamma}_1\Big] \\ &= \frac{1}{(\gamma_1^s - \gamma_2^s)^2}\Big[- \gamma_2^{ks}\hat{\gamma}_1\hat{\gamma}_2(\gamma_1^{ms} - \gamma_2^{ms}) \\ &\quad + \gamma_1^{ks}\hat{\gamma}_2\hat{\gamma}_1(\gamma_1^{ms} - \gamma_2^{ms})\Big] \end{aligned}$$

$$= \frac{(\gamma_1^{ms} - \gamma_2^{ms})(\gamma_1^{ks}\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^{ks}\hat{\gamma}_1\hat{\gamma}_2)}{(\gamma_1^s - \gamma_2^s)^2}$$
$$= \frac{B_m^{(s)}(\gamma_1^{ks}\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^{ks}\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}.$$

Which completes the proof. □

Corollary 2.3 (Catalan's identity)**.** *Let $n, k \in \mathbb{Z}^+$ be such that $n \geq k$, then we have*

$$\mathcal{HCB}_{n-k}^{(s)}\mathcal{HCB}_{n+k}^{(s)} - (\mathcal{HCB}_n^{(s)})^2 = \frac{-B_k^s(\gamma_1^{ks}\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^{ks}\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}.$$

Proof. For $m = -k$ in Theorem 2.6 and using the result $B_{-n} = -B_n$, we obtain the required result. □

Corollary 2.4 (Cassini's identity)**.** *For $n \in \mathbb{Z}^+$, then we have*

$$\mathcal{HCB}_{n-1}^{(s)}\mathcal{HCB}_{n+1}^{(s)} - (\mathcal{HCB}_n^{(s)})^2 = \frac{\gamma_2^s\hat{\gamma}_1\hat{\gamma}_2 - \gamma_1^s\hat{\gamma}_2\hat{\gamma}_1}{\gamma_1^s - \gamma_2^s}.$$

Proof. For $k = 1$ in Corollary 2.3, by Equation 3, we have

$$\mathcal{HCB}_{n-1}^{(s)}\mathcal{HCB}_{n+1}^{(s)} - (\mathcal{HCB}_n^{(s)})^2 = \frac{-B_1^s(\gamma_1^s\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^s\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}$$
$$= \frac{-(\gamma_1^s\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^s\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}$$
$$= \frac{\gamma_2^s\hat{\gamma}_1\hat{\gamma}_2 - \gamma_1^s\hat{\gamma}_2\hat{\gamma}_1}{\gamma_1^s - \gamma_2^s}.$$

This completes the proof. □

Corollary 2.5 (d'Ocagne's identity)**.** *Let $n \in \mathbb{N}$, $t \in \mathbb{Z}^+$ such that $t > n+1$. Then we get*

$$\mathcal{HCB}_{n+1}^{(s)}\mathcal{HCB}_t^{(s)} - \mathcal{HCB}_n^{(s)}\mathcal{HCB}_{t+1}^{(s)} = \frac{(\gamma_1^{(t-n)s}\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^{(t-n)s}\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}.$$

Proof. For $m = 1$ and $k = t - n$ in Theorem 2.6 and by using Equation 3, we find

$$\mathcal{HCB}_{n+1}^{(s)}\mathcal{HCB}_t^{(s)} - \mathcal{HCB}_n^{(s)}\mathcal{HCB}_{t+1}^{(s)} = \frac{B_1^s(\gamma_1^{(t-n)s}\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^{(t-n)s}\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}$$
$$= \frac{(\gamma_1^{(t-n)s}\hat{\gamma}_2\hat{\gamma}_1 - \gamma_2^{(t-n)s}\hat{\gamma}_1\hat{\gamma}_2)}{\gamma_1^s - \gamma_2^s}.$$

Hence, the desired result is obtained. □

3. A matrix representation for higher order balancing 2^r-ions

In this section, we obtain the matrix representation of the higher-order balancing 2^r-ions. We define two matrices $A^{(s)}$ and $B^{(s)}$ as

$$A^{(s)} = \begin{bmatrix} 2D_s & -1 \\ 1 & 0 \end{bmatrix} \text{ and } B^{(s)} = \begin{bmatrix} \mathcal{HCB}_2^{(s)} & \mathcal{HCB}_1^{(s)} \\ \mathcal{HCB}_1^{(s)} & \mathcal{HCB}_0^{(s)} \end{bmatrix}, \tag{5}$$

where D_s is the Lucas-balancing number. In light of our conclusion, we provide the following theorem.

Theorem 3.1. *For $n \in \mathbf{N}$, then we get*

$$(A^{(s)})^n B^{(s)} = \begin{bmatrix} \mathcal{HCB}_{n+2}^{(s)} & \mathcal{HCB}_{n+1}^{(s)} \\ \mathcal{HCB}_{n+1}^{(s)} & \mathcal{HCB}_{n}^{(s)} \end{bmatrix}.$$

Proof. We use the induction method on n to prove the theorem. For $n = 0$, the equality holds. Assume that the hypothesis is true for $n = i$. Namely,

$$(A^{(s)})^i B^{(s)} = \begin{bmatrix} \mathcal{HCB}_{i+2}^{(s)} & \mathcal{HCB}_{i+1}^{(s)} \\ \mathcal{HCB}_{i+1}^{(s)} & \mathcal{HCB}_{i}^{(s)} \end{bmatrix}. \tag{6}$$

For $n = i + 1$, by Equation 6 and Theorem 2.2, we obtain

$$\begin{aligned}(A^{(s)})^{i+1} B^{(s)} &= A^{(s)} (A^{(s)})^i B^{(s)} \\ &= \begin{bmatrix} 2D_s & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{HCB}_{i+2}^{(s)} & \mathcal{HCB}_{i+1}^{(s)} \\ \mathcal{HCB}_{i+1}^{(s)} & \mathcal{HCB}_{i}^{(s)} \end{bmatrix} \\ &= \begin{bmatrix} 2D_s\mathcal{HCB}_{i+2}^{(s)} - \mathcal{HCB}_{i+1}^{(s)} & 2D_s\mathcal{HCB}_{i+1}^{(s)} - \mathcal{HCB}_{i}^{(s)} \\ \mathcal{HCB}_{i+2}^{(s)} & \mathcal{HCB}_{i}^{(s)} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{HCB}_{i+3}^{(s)} & \mathcal{HCB}_{i+2}^{(s)} \\ \mathcal{HCB}_{i+2}^{(s)} & \mathcal{HCB}_{i+1}^{(s)} \end{bmatrix}.\end{aligned}$$

Therefore, the proof is completed. □

By using the matrices mentioned above, we generate Cassini's identity for higher-order balancing 2^r-ions in the following corollary.

Corollary 3.1. *For $n \in \mathbf{Z}^+$, then we get*

$$\mathcal{HCB}_{n+1}^{(s)}\mathcal{HCB}_{n-1}^{(s)} - (\mathcal{HCB}_{n}^{(s)})^2 = (-1)^{n-1}[\mathcal{HCB}_{2}^{(s)}\mathcal{HCB}_{0}^{(s)} - (\mathcal{HCB}_{1}^{(s)})^2].$$

Proof. From Equation 6 and Theorem 3.1, we have

$$\begin{bmatrix} 2D_s & -1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} \mathcal{HCB}_{2}^{(s)} & \mathcal{HCB}_{1}^{(s)} \\ \mathcal{HCB}_{1}^{(s)} & \mathcal{HCB}_{0}^{(s)} \end{bmatrix} = \begin{bmatrix} \mathcal{HCB}_{n+1}^{(s)} & \mathcal{HCB}_{n}^{(s)} \\ \mathcal{HCB}_{n}^{(s)} & \mathcal{HCB}_{n-1}^{(s)} \end{bmatrix}. \tag{7}$$

If we take the determinant on both sides of Equation 7 , then we find that

$$\mathcal{HCB}_{n+1}^{(s)}\mathcal{HCB}_{n-1}^{(s)} - (\mathcal{HCB}_{n}^{(s)})^2 = (-1)^{n-1}[\mathcal{HCB}_{2}^{(s)}\mathcal{HCB}_{0}^{(s)} - (\mathcal{HCB}_{1}^{(s)})^2].$$

□

4. Conclusions

In summary, we introduce the higher-order balancing 2^r-ions and present Binet's formula, Vajda's identity, Catalan's identity, Cassini's identity, and d'Ocagne's identity, ordinary (and exponential) generating functions, and give matrix representation for this sequence.

In future, it would be interesting to study the higher-order balancing quaternions, octonions, sedenions, trigintaduonions, and higher-order balancing hyper dual numbers. In addition, it's application in matrix algebra, spinor algebra and cryptography may be explored.

Acknowledgement. The authors would like to thank anonymous reviewers whose attentive reading and insightful remarks enabled us to improve the quality of our work in its present form.

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