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EXPLORING FRACTIONAL CALCULUS OPERATORS IN CONTEXT WITH EXTENDED HYPERGEOMETRIC AND CONFLUENT HYPERGEOMETRIC FUNCTION: IMAGE FORMULAS AND APPLICATIONS

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Abstract. Fractional calculus in mathematics has various applications in engineering and science, inequality theory and is also used in solving various integral equations. In the past few years, fractional calculus operators that contains different special functions have been discussed by many researchers. In our paper, our objective is to discuss image formulas for different fractional integral and differential operators using the extended hypergeometric and extended confluent hypergeometric function involving Appell series and Lauricella function. Fractional calculus operators that has Appell function in the kernel and Saigo fractional operator are used in this paper. The results investigated in this manuscript are general, novel and are used to discuss various special cases and more fascinating results involving other special functions and fractional calculus operators. We have also discussed the application and future scope along with a brief comparison with the existing literature.

Keywords: Image formula, Extended hypergeometric function, Extended confluent hypergeometric function, Fractional calculus, Saigo fractional operator.

AMS Subject Classification: 26A33, 33B15, 33C05, 33C15, 33C99

1. Introduction

Fractional Calculus in mathematics involves the investigation of derivatives and integrals of any arbitrary order, real or complex. The subject has gained a lot of admiration and acceptance in last few decades because of its different applications in the areas of science and engineering like visco-elasticity, optics, oscillation, diffusion, electrochemistry, wave propagation and various others. (see, e.g., [1, 2, 6, 7, 4, 3, 5]). It was around eighteenth century when several mathematicians, namely Fourier, Abel, Liouville, and Riemann were

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involved in the development of fractional calculus.

Several authors have introduced the generalization of various special functions, fractional operators associated with certain special functions (see, e.g., [8, 10, 12, 9, 21, 22, 24]) and their applications [20, 23]. A new extension of beta function was introduced that involved Appell series and Lauricella function [8]. For Appell series and Lauricella function refer [13, Eq.(1.4.1-1.4.4), p.23; Eq.(2.1.1-2.1.4), pg.41] respectively.

In the same paper [8], extended Gauss hypergeometric and confluent hypergeometric functions were introduced involving the extended beta function.

Definition 1.1. *The new extension of Gauss hypergeometric function containing Appell series $F_1(.)$,where $\Re(\varrho_3) > \Re(\varrho_2) > 0$, with $\Re(p), \Re(q) \geq 0$ and $|\alpha| < 1$ is given by* [8, Eq. (44), p.10]

$$F_{p,q}^{F_1}(\varrho_1, \varrho_2; \varrho_3; \alpha) = \sum_{n=0}^{\infty} (\varrho_1)_n \frac{B_{p,q}^{F_1}(\varrho_2 + n, \varrho_3 - \varrho_2)}{B(\varrho_2, \varrho_3 - \varrho_2)} \frac{(\alpha)^n}{n!}. \tag{1}$$

Definition 1.2. *Let $\Re(\varrho_3) > \Re(\varrho_2) > 0$ with $\Re(p), \Re(q) \geq 0$ and $|\alpha| < 1$, then the extension of confluent hypergeometric function containing Appell series $F_1(.)$ is given by* [8, Eq. (45), p.10]

$$\Phi_{p,q}^{F_1}(\varrho_2; \varrho_3; \alpha) = \sum_{n=0}^{\infty} \frac{B_{p,q}^{F_1}(\varrho_2 + n, \varrho_3 - \varrho_2)}{B(\varrho_2, \varrho_3 - \varrho_2)} \frac{(\alpha)^n}{n!}, \tag{2}$$

where

$$B_{p,q}^{F_1}(\varrho_1, \varrho_2) = \int_0^1 t^{\varrho_1 - 1}(1-t)^{\varrho_2 - 1} F_1\left(a_1, b_1', c_1''; d; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt, \tag{3}$$

is the generalized beta function involving Appell series $F_1(.)$ [8, Eq.(20), p.4]*, with $\Re(a_1), \Re(b_1'), \Re(c_1''), \Re(d) > 0$, $\Re(p), \Re(q) \geq 0, \Re(\varrho_1) > 0$ and $\Re(\varrho_2) > 0$.*

The extension of Gauss hypergeometric and confluent hypergeometric function involving Appell series $F_2(.)$, $F_3(.)$, $F_4(.)$ and Lauricella function $F_A^{(m)}$, $F_B^{(m)}$, $F_C^{(m)}$ and $F_D^{(m)}$ are defined in the similar way [8].

The main objective of this research work is to investigate and examine the relation and application of fractional calculus operators such as operators with Appell function in the kernel ands Saigo operators to extended hypergeometric and confluent hypergeometric functions.

The study intends to create a thorough framework for integrating fractional calculus operators with extended hypergeometric and confluent hypergeometric functions. Explore the theoretical foundations and derivations of these operators in relation to these functions.

In section 2, we have discussed the relation of the generalized hypergeometric function and generalized confluent hypergeometric function with fractional integral operator involving Appell function in the kernel. The results for Gauss hypergeometric and confluent hypergeometric function are discussed as special cases. In section 3, we have discussed results for the generalized hypergeometric function and generalized confluent hypergeometric function with fractional differential operator involving Appell function in the kernel, along with the special cases. In section 4, results involving Saigo fractional operator is investigated. Section 5, deals with the conclusion, comparative study and novelty of our research work. Section 6 mentions certain applications and future scope related to our research work.

2. Image formulas involving the Fractional Integral Operator with Appell function in the kernel

Definition 2.1. *The fractional integral operator with the Appell function $F_3(.)$ in the kernel* [14] *is*

$$\left(\Im_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta} f\right)(y) = \frac{y^{-\gamma}}{\Gamma(\eta)} \int_0^y (y-t)^{\eta-1} t^{-\gamma'} F_3\left(\gamma,\gamma',\epsilon,\epsilon';\eta;1-\frac{t}{y},1-\frac{y}{t}\right) f(t)dt, \tag{4}$$

$$\left(\Im_{0^-}^{\gamma,\gamma',\epsilon,\epsilon',\eta} f\right)(y) = \frac{y^{-\gamma'}}{\Gamma(\eta)} \int_y^\infty (t-y)^{\eta-1} t^{-\gamma} F_3\left(\gamma,\gamma',\epsilon,\epsilon';\eta;1-\frac{t}{y},1-\frac{y}{t}\right) f(t)dt, \tag{5}$$

where $\gamma,\gamma',\epsilon,\epsilon',\eta \in \mathbb{C}, \Re(\eta) > 0$ and $y \in \mathbb{R}^+$.

We will use the following Lemmas [15] in our main result.

Lemma 2.1. *Let $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\gamma) > \max\{0, \Re(\gamma-\gamma'-\epsilon-\eta), \Re(\gamma'-\epsilon')\}$, then*

$$\left(\Im_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta} t^{\xi-1}\right)(y) = \frac{\Gamma(\xi)\Gamma(\xi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\xi+\epsilon'-\gamma')}{\Gamma(\xi+\epsilon')\Gamma(\xi+\eta-\gamma-\gamma')\Gamma(\xi+\eta-\gamma'-\epsilon)} y^{\xi-\gamma-\gamma'+\eta-1}. \tag{6}$$

Lemma 2.2. *Let $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\xi) > \max\{\Re(\epsilon), \Re(-\gamma-\gamma'+\eta), \Re(-\gamma-\epsilon'+\eta)\}$, then*

$$\left(\Im_{0^-}^{\gamma,\gamma',\epsilon,\epsilon',\eta} t^{-\xi}\right)(y) = \frac{\Gamma(-\epsilon+\xi)\Gamma(\gamma+\gamma'-\eta+\xi)\Gamma(\gamma+\epsilon'-\eta+\xi)}{\Gamma(\xi)\Gamma(\gamma-\epsilon+\xi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\xi)} y^{-\gamma-\gamma'+\eta-\xi}. \tag{7}$$

2.1. For the Generalized Hypergeometric Function.

Theorem 2.1. *Let $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\gamma) > \max\{0, \Re(\gamma-\gamma'-\epsilon-\eta), \Re(\gamma'-\epsilon')\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$. Then*

$$\begin{aligned}
&\left(\Im_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1} F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x) \\
&= x^{\phi-\gamma-\gamma'+\eta-1} \frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)} F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha x) \\
&* {}_4F_3\left[\begin{matrix} \phi & \phi+\eta-\gamma-\gamma'-\epsilon & \phi+\epsilon'-\gamma' & 1 \\ \phi+\epsilon' & \phi+\eta-\gamma-\gamma' & \phi+\eta-\gamma'-\epsilon & \end{matrix};\alpha x\right].
\end{aligned} \tag{8}$$

Proof. Consider

$$\begin{aligned}
LHS &= \left(\Im_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1} F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x) \\
&= \left(\Im_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha t)^n}{n!}\right]\right)(x)
\end{aligned}$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\Im_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta} t^{\phi+n-1}\right](x). \tag{9}$$

Using Lemma 2.1 in the above equation (9), we get

$$LHS = \sum_{n=0}^{\infty}(\varrho_1)_n \frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\phi+n)\Gamma(\phi+n+\eta-\gamma-\gamma'-\epsilon)}{\Gamma(\phi+n+\epsilon')\Gamma(\phi+n+\eta-\gamma-\gamma')}$$
$$\times \frac{\Gamma(\phi+n+\epsilon'-\gamma')}{\Gamma(\phi+n+\eta-\gamma'-\epsilon)}x^{\phi+n-\gamma-\gamma'+\eta-1}$$
$$= \frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}\sum_{n=0}^{\infty}(\varrho_1)_n \frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}$$
$$\times x^{\phi+n-\gamma-\gamma'+\eta-1}\frac{(\phi)_n(\phi+\eta-\gamma-\gamma'-\epsilon)_n(\phi+\epsilon'-\gamma')_n}{(\phi+\epsilon')_n(\phi+\eta-\gamma-\gamma')_n(\phi+\eta-\gamma'-\epsilon)_n}. \tag{10}$$

Using Hadamard Product in (10), we get

$$LHS = \frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}x^{\phi-\gamma-\gamma'+\eta-1}$$
$$\times \sum_{n=0}^{\infty}(\varrho_1)_n \frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}(\alpha x)^n \frac{(\phi)_n(\phi+\eta-\gamma-\gamma'-\epsilon)_n(\phi+\epsilon'-\gamma')_n}{(\phi+\epsilon')_n(\phi+\eta-\gamma-\gamma')_n(\phi+\eta-\gamma'-\epsilon)_n}$$
$$\times \frac{(1)_n}{n!}.$$
$$= x^{\phi-\gamma-\gamma'+\eta-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}F^{F_1}_{p_1,p_2}(\varrho_1,\varrho_2;\varrho_3;\alpha x)$$
$$* {}_4F_3\left[\begin{matrix}\phi & \phi+\eta-\gamma-\gamma'-\epsilon & \phi+\epsilon'-\gamma' & 1 \\ & & & ;\alpha x \\ \phi+\epsilon' & \phi+\eta-\gamma-\gamma' & \phi+\eta-\gamma'-\epsilon & \end{matrix}\right].$$

Hence, the desired result (8). □

Theorem 2.2. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi\in\mathbb{C}$, $\Re(\eta)>0$, $\Re(\xi)>\max\{\Re(\epsilon),\Re(-\gamma-\gamma'+\eta),\Re(-\gamma-\epsilon'+\eta)\}$, $\Re(\varrho_3)>\Re(\varrho_2)>0,\Re(p_1),\Re(p_2)>0$, $|\frac{\alpha}{t}|<1$,

$$\left(\Im^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}F^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)$$
$$= x^{-\gamma-\gamma'+\eta-\phi}\frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}F^{F_1}_{p_1,p_2}(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x})$$
$$* {}_4F_3\left[\begin{matrix}-\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & 1 \\ & & & ;\frac{\alpha}{x} \\ \phi & \phi+\gamma-\epsilon+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & \end{matrix}\right]. \tag{11}$$

Proof. Consider

$$LHS = \left(\Im^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}F^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)$$
$$= \left(\Im^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}\left(\frac{\alpha}{t}\right)^n\right]\right)(x).$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}(\varrho_1)_n\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\Im^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}t^{-\phi-n}\right](x). \tag{12}$$

Using Lemma 2.2 in the above equation (12), we get

$$
\begin{aligned}
LHS &= \sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(-\epsilon+\phi+n)\Gamma(\gamma+\gamma'-\eta+\phi+n)}{\Gamma(\phi+n)\Gamma(\gamma-\epsilon+\phi+n)} \\
&\times \frac{\Gamma(\gamma+\epsilon'-\eta+\phi+n)}{\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi+n)} x^{-\gamma-\gamma'+\eta-\phi-n} \\
&= \frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}\sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!} \\
&\times x^{-\gamma-\gamma'+\eta-\phi-n}\frac{(-\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n}{(\varrho+\epsilon')_n(\varrho+\eta-\gamma-\gamma')_n(\varrho+\eta-\gamma'-\epsilon)_n}. \qquad (13)
\end{aligned}
$$

Using Hadamard Product in (13), we get

$$
\begin{aligned}
LHS =& \frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)} x^{-\gamma-\gamma'+\eta-\phi} \\
&\times \sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}(\frac{\alpha}{x})^n \frac{(-\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n}{(\varrho+\epsilon')_n(\varrho+\eta-\gamma-\gamma')_n(\varrho+\eta-\gamma'-\epsilon)_n} \\
&\times \frac{(1)_n}{n!} \\
&= x^{-\gamma-\gamma'+\eta-\phi}\frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)} F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x}) \\
&* {}_4F_3\left[\begin{matrix} -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & 1 \\ & & & ;\frac{\alpha}{x} \\ \phi & \phi+\gamma-\epsilon+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & \end{matrix}\right].
\end{aligned}
$$

Hence, the desired result (11). $\square$

Special Cases

When $p_2=0$ and, if $p_1=1, r=0, a_1, b_1', d=1$, theorems 2.1 and 2.2 give the results for the Gauss hypergeometric function.

Corollary 2.1. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi\in\mathbb{C}$, $\Re(\eta)>0$, $\Re(\gamma)>\max\{0,\Re(\gamma-\gamma'-\epsilon-\eta),\Re(\gamma'-\epsilon')\}$, $\Re(\varrho_3)>\Re(\varrho_2)>0,\Re(p_1),\Re(p_2)>0$, $|\alpha t|<1$. *Then*

$$
\begin{aligned}
&\left(\Im_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}{}_2F_1(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x) \\
&= x^{\phi-\gamma-\gamma'+\eta-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}{}_2F_1(\varrho_1,\varrho_2;\varrho_3;\alpha x) \\
&* {}_4F_3\left[\begin{matrix} \phi & \phi+\eta-\gamma-\gamma'-\epsilon & \phi+\epsilon'-\gamma' & 1 \\ & & & ;\alpha x \\ \phi+\epsilon' & \phi+\eta-\gamma-\gamma' & \phi+\eta-\gamma'-\epsilon & \end{matrix}\right]. \qquad (14)
\end{aligned}
$$

Corollary 2.2. *Let* $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\xi) > \max\{\Re(\epsilon), \Re(-\gamma-\gamma'+\eta), \Re(-\gamma-\epsilon'+\eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$,

$$\begin{aligned}
&\left(\mathfrak{I}_{0^-}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{-\phi}{}_2F_1\left(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{t}\right)\right]\right)(x) \\
&= x^{-\gamma-\gamma'+\eta-\phi}\frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{}_2F_1(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{x}) \\
&* {}_4F_3\left[\begin{matrix}-\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & 1 \\ & & & ; \frac{\alpha}{x} \\ \phi & \phi+\gamma-\epsilon+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & \end{matrix}\right].
\end{aligned} \tag{15}$$

2.2. **For the Generalized Confluent Hypergeometric Function.**

Theorem 2.3. *Suppose* $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\gamma) > \max\{0, \Re(\gamma-\gamma'-\epsilon-\eta), \Re(\gamma'-\epsilon')\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$. *Then*

$$\begin{aligned}
&\left(\mathfrak{I}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\Psi_{p_1,p_2}^{F_1}(\varrho_2; \varrho_3; \alpha t)\right]\right)(x) \\
&= x^{\phi-\gamma-\gamma'+\eta-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}\Psi_{p_1,p_2}^{F_1}(\varrho_2; \varrho_3; \alpha x) \\
&* {}_4F_3\left[\begin{matrix}\phi & \phi+\eta-\gamma-\gamma'-\epsilon & \phi+\epsilon'-\gamma' & 1 \\ & & & ; \alpha x \\ \phi+\epsilon' & \phi+\eta-\gamma-\gamma' & \phi+\eta-\gamma'-\epsilon & \end{matrix}\right].
\end{aligned} \tag{16}$$

Proof. Consider

$$\begin{aligned}
LHS &= \left(\mathfrak{I}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\Psi_{p_1,p_2}^{F_1}(\varrho_2; \varrho_3; \alpha t)\right]\right)(x) \\
&= \left(\mathfrak{I}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha t)^n}{n!}\right]\right)(x)
\end{aligned}$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{I}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}t^{\phi+n-1}\right](x). \tag{17}$$

Using Lemma 2.1 in the above equation (17), we get

$$\begin{aligned}
LHS &= \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\phi+n)\Gamma(\phi+n+\eta-\gamma-\gamma'-\epsilon)}{\Gamma(\phi+n+\epsilon')\Gamma(\phi+n+\eta-\gamma-\gamma')} \\
&\times \frac{\Gamma(\phi+n+\epsilon'-\gamma')}{\Gamma(\phi+n+\eta-\gamma'-\epsilon)}x^{\phi+n-\gamma-\gamma'+\eta-1} \\
&= \frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!} \\
&\times x^{\phi+n-\gamma-\gamma'+\eta-1}\frac{(\phi)_n(\phi+\eta-\gamma-\gamma'-\epsilon)_n(\phi+\epsilon'-\gamma')_n}{(\phi+\epsilon')_n(\phi+\eta-\gamma-\gamma')_n(\phi+\eta-\gamma'-\epsilon)_n}.
\end{aligned} \tag{18}$$

Using Hadamard Product in (18), we get

$$\begin{aligned}LHS =&\frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}x^{\phi-\gamma-\gamma'+\eta-1}\\ &\times\sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}(\alpha x)^n\frac{(\phi)_n(\phi+\eta-\gamma-\gamma'-\epsilon)_n(\phi+\epsilon'-\gamma')_n}{(\phi+\epsilon')_n(\phi+\eta-\gamma-\gamma')_n(\phi+\eta-\gamma'-\epsilon)_n}\frac{(1)_n}{n!}.\\ &=x^{\phi-\gamma-\gamma'+\eta-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}\Psi^{F_1}_{p_1,p_2}(\varrho_2;\varrho_3;\alpha x)\\ &*{}_4F_3\left[\begin{matrix}\phi & \phi+\eta-\gamma-\gamma'-\epsilon & \phi+\epsilon'-\gamma' & 1\\ & & & ;\alpha x\\ \phi+\epsilon' & \phi+\eta-\gamma-\gamma' & \phi+\eta-\gamma'-\epsilon & \end{matrix}\right].\end{aligned}$$

Hence, we get the desired result (16). □

Theorem 2.4. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi\in\mathbb{C}$, $\Re(\eta)>0$, $\Re(\xi)>\max\{\Re(\epsilon),\Re(-\gamma-\gamma'+\eta),\Re(-\gamma-\epsilon'+\eta)\}$, $\Re(\varrho_3)>\Re(\varrho_2)>0,\Re(p_1),\Re(p_2)>0$, $|\frac{\alpha}{t}|<1$,

$$\begin{aligned}&\left(\mathfrak{I}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}\Psi^{F_1}_{p_1,p_2}\left(\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)\\ &=x^{-\gamma-\gamma'+\eta-\phi}\frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}\Psi^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x}\right)\\ &*{}_4F_3\left[\begin{matrix}-\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & 1\\ & & & ;\frac{\alpha}{x}\\ \phi & \phi+\gamma-\epsilon+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & \end{matrix}\right].\end{aligned}\tag{19}$$

Proof. Consider

$$\begin{aligned}LHS&=\left(\mathfrak{I}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}\Psi^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)\\ &=\left(\mathfrak{I}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}\sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}\left(\frac{\alpha}{t}\right)^n\right]\right)(x).\end{aligned}$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS=\sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{I}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}t^{-\phi-n}\right](x).\tag{20}$$

Using Lemma 2.2 in the above equation (20), we get

$$\begin{aligned}LHS=&\sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(-\epsilon+\phi+n)\Gamma(\gamma+\gamma'-\eta+\phi+n)}{\Gamma(\phi+n)\Gamma(\gamma-\epsilon+\phi+n)}\\ &\times\frac{\Gamma(\gamma+\epsilon'-\eta+\phi+n)}{\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi+n)}x^{-\gamma-\gamma'+\eta-\phi-n}\\ &=\frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}\sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\\ &\times x^{-\gamma-\gamma'+\eta-\phi-n}\frac{(-\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n}{(\varrho+\epsilon')_n(\varrho+\eta-\gamma-\gamma')_n(\varrho+\eta-\gamma'-\epsilon)_n}.\end{aligned}\tag{21}$$

Using Hadamard Product in (21), we get

$$LHS = \frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)} x^{-\gamma-\gamma'+\eta-\phi}$$
$$\times \sum_{n=0}^{\infty} \frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!} \left(\frac{\alpha}{x}\right)^n \frac{(-\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n}{(\varrho+\epsilon')_n(\varrho+\eta-\gamma-\gamma')_n(\varrho+\eta-\gamma'-\epsilon)_n}$$
$$\times \frac{(1)_n}{n!}$$
$$= x^{-\gamma-\gamma'+\eta-\phi} \frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)} \Psi^{F_1}_{p_1,p_2}\left(\varrho_2;\varrho_3;\frac{\alpha}{x}\right)$$
$$* {}_4F_3\left[\begin{matrix} -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & 1 \\ \phi & \phi+\gamma-\epsilon+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & \end{matrix};\frac{\alpha}{x}\right].$$

Hence, the desired result (19). □

Special Cases

When $p_2 = 0$ and, if $p_1 = 1, r = 0, a_1, b_1', d = 1$, theorems 2.3 and 2.4 give the results for the confluent hypergeometric function.

Corollary 2.3. *If* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\gamma) > \max\{0, \Re(\gamma-\gamma'-\epsilon-\eta), \Re(\gamma'-\epsilon')\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$. *Then*

$$\left(\mathfrak{I}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^+}\left[t^{\phi-1}{}_1\Psi_1(\varrho_2;\varrho_3;\alpha t)\right]\right)(x)$$
$$= x^{\phi-\gamma-\gamma'+\eta-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\gamma-\gamma'-\epsilon)\Gamma(\phi+\epsilon'-\gamma')}{\Gamma(\phi+\epsilon')\Gamma(\phi+\eta-\gamma-\gamma')\Gamma(\phi+\eta-\gamma'-\epsilon)}{}_1\Psi_1(\varrho_2;\varrho_3;\alpha x)$$
$$* {}_4F_3\left[\begin{matrix} \phi & \phi+\eta-\gamma-\gamma'-\epsilon & \phi+\epsilon'-\gamma' & 1 \\ \phi+\epsilon' & \phi+\eta-\gamma-\gamma' & \phi+\eta-\gamma'-\epsilon & \end{matrix};\alpha x\right]. \tag{22}$$

Corollary 2.4. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\xi) > \max\{\Re(\epsilon), \Re(-\gamma-\gamma'+\eta), \Re(-\gamma-\epsilon'+\eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$,

$$\left(\mathfrak{I}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}{}_1\Psi_1\left(\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)$$
$$= x^{-\gamma-\gamma'+\eta-\phi}\frac{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{\Gamma(\phi)\Gamma(\gamma-\epsilon+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{}_1\Psi_1(\varrho_2;\varrho_3;\frac{\alpha}{x})$$
$$* {}_4F_3\left[\begin{matrix} -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & 1 \\ \phi & \phi+\gamma-\epsilon+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & \end{matrix};\frac{\alpha}{x}\right]. \tag{23}$$

3. Image formula using fractional differential operator associated with the Appell function

Definition 3.1. *The fractional differential operator with the Appell function* $F_3(.)$ *in the kernel* [14] *are*

$$\left(\mathfrak{D}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}f\right)(y) = \left(-\frac{d}{dy}\right)^{[\Re(\eta)]+1}\left(\mathfrak{I}^{-\gamma',-\gamma,-\epsilon',-\epsilon+[\Re(\eta)]+1,-\eta+[\Re(\eta)]+1}_{0^-}f\right)(y) \tag{24}$$

and

$$\left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta} f\right)(y) = \left(\frac{d}{dy}\right)^{[\Re(\eta)]+1} \left(\mathfrak{I}_{0^+}^{-\gamma',-\gamma,-\epsilon'+[\Re(\eta)]+1,-\epsilon,-\eta+[\Re(\eta)]+1} f\right)(y), \tag{25}$$

where $\gamma,\gamma',\epsilon,\epsilon',\eta \in \mathbb{C}, \Re(\eta) > 0$ *and* $y \in \mathbb{R}^+$.

We will use the following Lemmas [15] in our main result.

Lemma 3.1. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\xi) > \max\{0, \Re(-\gamma+\epsilon'), \Re(-\gamma-\gamma'-\epsilon+\eta)\}$, *then*

$$\left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta} t^{\xi-1}\right)(y) = \frac{\Gamma(\xi)\Gamma(-\epsilon+\gamma+\xi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\xi)}{\Gamma(-\epsilon+\xi)\Gamma(\gamma+\gamma'-\eta+\xi)\Gamma(\gamma+\epsilon'-\eta+\xi)} y^{\gamma+\gamma'-\eta+\xi-1}. \tag{26}$$

Lemma 3.2. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\xi) > \max\{\Re(-\epsilon'), \Re(\gamma'+\epsilon-\eta), \Re(\gamma+\gamma'-\eta) + [\Re(\eta)] + 1\}$, *then*

$$\left(\mathfrak{D}_{0^-}^{\gamma,\gamma',\epsilon,\epsilon',\eta} t^{-\xi}\right)(y) = \frac{\Gamma(\epsilon'+\xi)\Gamma(-\gamma-\gamma'+\eta+\xi)\Gamma(-\gamma'-\epsilon+\eta+\xi)}{\Gamma(\xi)\Gamma(-\gamma'+\epsilon'+\xi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\xi)} y^{\gamma+\gamma'-\eta-\xi}. \tag{27}$$

3.1. **For the Generalized Hypergeometric Function.**

Theorem 3.1. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$, $\Re(\xi) > \max\{0, \Re(-\gamma+\epsilon'), \Re(-\gamma-\gamma'-\epsilon+\eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$, *then*

$$\begin{aligned}
&\left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1} F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x)\\
&= x^{\gamma+\gamma'-\eta+\phi-1} \frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)} F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha x)\\
&* {}_4F_3\left[\begin{matrix} \phi & -\epsilon+\gamma+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & 1 \\ & & & ;\alpha x \\ -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & \end{matrix}\right].
\end{aligned} \tag{28}$$

Proof. Consider

$$\begin{aligned}
LHS &= \left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1} F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x)\\
&= \left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha t)^n}{n!}\right]\right)(x)
\end{aligned}$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta} t^{\phi+n-1}\right](x). \tag{29}$$

Using Lemma 3.1 in the above equation (29), we get

$$\begin{aligned}
LHS &= \sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\phi+n)\Gamma(-\epsilon+\gamma+\phi+n)}{\Gamma(-\epsilon+\phi+n)\Gamma(\gamma+\gamma'-\eta+\phi+n)}\\
&\times \frac{\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi+n)}{\Gamma(\gamma+\epsilon'-\eta+\phi+n)} x^{\gamma+\gamma'-\eta+n-1}\\
&= \frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}\sum_{n=0}^{\infty}(\varrho_1)_n \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\\
&\times x^{\gamma+\gamma'-\eta+\phi+n-1}\frac{(\phi)_n(-\epsilon+\gamma+\phi)_n(\gamma+\gamma'+\epsilon'-\eta+\phi)_n}{(\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n}.
\end{aligned} \tag{30}$$

Using Hadamard Product in (30), we get

$$
\begin{aligned}
LHS =& \frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)} x^{\gamma+\gamma'-\eta+\phi-1} \\
&\times \sum_{n=0}^{\infty} (\varrho_1)_n \frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!} (\alpha x)^n \frac{(\phi)_n(-\epsilon+\gamma+\phi)_n(\gamma+\gamma'+\epsilon'-\eta+\phi)_n}{(\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n} \\
&\times \frac{(1)_n}{n!}. \\
&= x^{\gamma+\gamma'-\eta+\phi-1} \frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)} F^{F_1}_{p_1,p_2}(\omega_1,\omega_2;\omega_3;\alpha x) \\
&* {}_4F_3 \left[\begin{matrix} \phi & -\epsilon+\gamma+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & 1 \\ & & & ;\alpha x \\ -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & \end{matrix} \right].
\end{aligned}
$$

Hence, the desired result (28). ☐

Theorem 3.2. *Let* $\gamma,\gamma',\epsilon,\epsilon',\eta,\xi \in \mathbb{C}$ *such that* $\Re(\xi) > \max\{\Re(-\epsilon'), \Re(\gamma'+\epsilon-\eta), \Re(\gamma+\gamma'-\eta)+[\Re(\eta)]+1\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. *Then*

$$
\begin{aligned}
&\left(\mathfrak{D}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}F^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x) \\
&= x^{\gamma+\gamma'-\eta-\phi}\frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)} F^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x}\right) \\
&* {}_4F_3 \left[\begin{matrix} \epsilon'+\phi & -\gamma-\gamma'+\eta+\phi & -\gamma'-\epsilon+\eta+\phi & 1 \\ & & & ;\frac{\alpha}{x} \\ \phi & -\gamma'-\epsilon'+\phi & -\gamma-\gamma'-\epsilon+\eta+\phi & \end{matrix} \right].
\end{aligned}
\tag{31}
$$

Proof. Consider

$$
\begin{aligned}
LHS &= \left(\mathfrak{D}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^-}\left[t^{-\phi}F^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x) \\
&= \left(\mathfrak{D}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^+}\left[t^{-\phi}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{\alpha^n}{t^n n!}\right]\right)(x).
\end{aligned}
$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$
LHS = \sum_{n=0}^{\infty}(\varrho_1)_n\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{D}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0^+}t^{-\phi-n}\right](x). \tag{32}
$$

Using Lemma 3.2 in the above equation (32), we get

$$
\begin{aligned}
LHS =& \sum_{n=0}^{\infty}(\varrho_1)_n\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\epsilon'+\phi+n)\Gamma(-\gamma-\gamma'+\eta+\phi+n)}{\Gamma(\phi+n)\Gamma(-\gamma'-\epsilon'+\phi+n)} \\
&\times \frac{\Gamma(-\gamma'-\epsilon+\eta+\phi+n)}{\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi+n)} x^{\gamma+\gamma'-\eta-n-\phi} \\
=& \frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)} \\
&\times \frac{(\alpha)^n}{n!}x^{\gamma+\gamma'-\eta+\phi-n}\frac{(\epsilon'+\phi)_n(-\gamma-\gamma'+\eta+\phi)_n(-\gamma'-\epsilon+\eta+\phi)_n}{(\phi)_n(-\gamma'-\epsilon'+\phi)_n(-\gamma-\gamma'-\epsilon+\eta+\phi)_n}.
\end{aligned}
\tag{33}
$$

Using Hadamard Product in (33), we get

$$
\begin{aligned}
LHS =& \frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)} x^{\gamma+\gamma'-\eta+\phi} \\
&\times \sum_{n=0}^{\infty} (\varrho_1)_n \frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!} \left(\frac{\alpha}{x}\right)^n \\
&\times \frac{(\epsilon'+\phi)_n(-\gamma-\gamma'+\eta+\phi)_n(-\gamma'-\epsilon+\eta+\phi)_n}{(\phi)_n(-\gamma'-\epsilon'+\phi)_n(-\gamma-\gamma'-\epsilon+\eta+\phi)_n} \frac{(1)_n}{n!}. \\
&= x^{\gamma+\gamma'-\eta-\phi} \frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)} F^{F_1}_{p_1,p_2}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x}\right) \\
&* {}_4F_3 \left[\begin{matrix} \epsilon'+\phi & -\gamma-\gamma'+\eta+\phi & -\gamma'-\epsilon+\eta+\phi & 1 \\ \phi & -\gamma'-\epsilon'+\phi & -\gamma-\gamma'-\epsilon+\eta+\phi & \end{matrix} ; \frac{\alpha}{x}\right].
\end{aligned}
$$

Hence, the desired result (31). $\square$

Special Cases

When $p_2 = 0$ and, if $p_1 = 1, r = 0, a_1, b'_1, d = 1$, theorems 3.1 and 3.2 give the results for the Gauss hypergeometric function.

Corollary 3.1. *Let* $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}$, $\Re(\xi) > \max\{0, \Re(-\gamma+\epsilon'), \Re(-\gamma-\gamma'-\epsilon+\eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$, *then*

$$
\begin{aligned}
&\left(\mathfrak{D}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0+}\left[t^{\phi-1}{}_2F_1(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x) \\
&= x^{\gamma+\gamma'-\eta+\phi-1} \frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)} {}_2F_1(\varrho_1,\varrho_2;\varrho_3;\alpha x) \\
&* {}_4F_3 \left[\begin{matrix} \phi & -\epsilon+\gamma+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & 1 \\ -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & \end{matrix} ; \alpha x\right].
\end{aligned} \tag{34}
$$

Corollary 3.2. *Let* $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}$, $\Re(\xi) > \max\{\Re(-\epsilon'), \Re(\gamma'+\epsilon-\eta), \Re(\gamma+\gamma'-\eta) + [\Re(\eta)] + 1\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. *Then*

$$
\begin{aligned}
&\left(\mathfrak{D}^{\gamma,\gamma',\epsilon,\epsilon',\eta}_{0-}\left[t^{-\phi}{}_2F_1\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x) \\
&= x^{\gamma+\gamma'-\eta-\phi} \frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)} {}_2F_1\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x}\right) \\
&* {}_4F_3 \left[\begin{matrix} \epsilon'+\phi & -\gamma-\gamma'+\eta+\phi & -\gamma'-\epsilon+\eta+\phi & 1 \\ \phi & -\gamma'-\epsilon'+\phi & -\gamma-\gamma'-\epsilon+\eta+\phi & \end{matrix} ; \frac{\alpha}{x}\right].
\end{aligned} \tag{35}
$$

3.2. **For the Generalized Confluent Hypergeometric Function.**

Theorem 3.3. *Let* $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}, \Re(\xi) > \max\{0, \Re(-\gamma+\epsilon'), \Re(-\gamma-\gamma'-\epsilon+\eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$, *then*

$$\begin{aligned}&\left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\Psi_{p_1,p_2}^{F_1}(\varrho_2;\varrho_3;\alpha t)\right]\right)(x)\\ &= x^{\gamma+\gamma'-\eta+\phi-1}\frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}\Psi_{p_1,p_2}^{F_1}(\varrho_2;\varrho_3;\alpha x)\\ &* {}_4F_3\left[\begin{matrix}\phi & -\epsilon+\gamma+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & 1\\ & & & ;\alpha x\\ -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & \end{matrix}\right].\end{aligned} \tag{36}$$

Proof. Consider

$$\begin{aligned}LHS &= \left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\Psi_{p_1,p_2}^{F_1}(\varrho_2;\varrho_3;\alpha t)\right]\right)(x)\\ &= \left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha t)^n}{n!}\right]\right)(x)\end{aligned}$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}t^{\phi+n-1}\right](x). \tag{37}$$

Using Lemma 3.1 in the above equation (37), we get

$$\begin{aligned}LHS &= \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\phi+n)\Gamma(-\epsilon+\gamma+\phi+n)}{\Gamma(-\epsilon+\phi+n)\Gamma(\gamma+\gamma'-\eta+\phi+n)}\\ &\times\frac{\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi+n)}{\Gamma(\gamma+\epsilon'-\eta+\phi+n)}x^{\gamma+\gamma'-\eta+n-1}\\ &= \frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\\ &\times x^{\gamma+\gamma'-\eta+\phi+n-1}\frac{(\phi)_n(-\epsilon+\gamma+\phi)_n(\gamma+\gamma'+\epsilon'-\eta+\phi)_n}{(\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n}.\end{aligned} \tag{38}$$

Using Hadamard Product in (38), we get

$$\begin{aligned}LHS =& \frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}x^{\gamma+\gamma'-\eta+\phi-1}\\ &\times\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}(\alpha x)^n\frac{(\phi)_n(-\epsilon+\gamma+\phi)_n(\gamma+\gamma'+\epsilon'-\eta+\phi)_n}{(\epsilon+\phi)_n(\gamma+\gamma'-\eta+\phi)_n(\gamma+\epsilon'-\eta+\phi)_n}\frac{(1)_n}{n!}.\\ &= x^{\gamma+\gamma'-\eta+\phi-1}\frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}\Psi_{p_1,p_2}^{F_1}(\omega_1,\omega_1;\omega_3;\alpha x)\\ &* {}_4F_3\left[\begin{matrix}\phi & -\epsilon+\gamma+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & 1\\ & & & ;\alpha x\\ -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & \end{matrix}\right].\end{aligned}$$

Hence, the desired result (28). □

Theorem 3.4. *Let $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}$ such that $\Re(\xi) > \max\{\Re(-\epsilon'), \Re(\gamma' + \epsilon - \eta), \Re(\gamma + \gamma' - \eta) + [\Re(\eta)] + 1\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. Then*

$$\left(\mathfrak{D}_{0^-}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{-\phi}\Psi_{p_1,p_2}^{F_1}\left(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{t}\right)\right]\right)(x)$$
$$= x^{\gamma+\gamma'-\eta-\phi}\frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)}\Psi_{p_1,p_2}^{F_1}\left(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{x}\right)$$
$$* {}_4F_3\left[\begin{matrix}\epsilon'+\phi & -\gamma-\gamma'+\eta+\phi & -\gamma'-\epsilon+\eta+\phi & 1\\ \phi & -\gamma'-\epsilon'+\phi & -\gamma-\gamma'-\epsilon+\eta+\phi & \end{matrix}; \frac{\alpha}{x}\right]. \tag{39}$$

Proof. Consider

$$LHS = \left(\mathfrak{D}_{0^-}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{-\phi}\Psi_{p_1,p_2}^{F_1}\left(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{t}\right)\right]\right)(x)$$
$$= \left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{-\phi}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{\alpha^n}{t^n n!}\right]\right)(x).$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}t^{-\phi-n}\right](x). \tag{40}$$

Using Lemma 3.2 in the above equation (40), we get

$$LHS = \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\epsilon'+\phi+n)\Gamma(-\gamma-\gamma'+\eta+\phi+n)}{\Gamma(\phi+n)\Gamma(-\gamma'-\epsilon'+\phi+n)}$$
$$\times \frac{\Gamma(-\gamma'-\epsilon+\eta+\phi+n)}{\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi+n)}x^{\gamma+\gamma'-\eta-n-\phi}$$
$$= \frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}$$
$$\times x^{\gamma+\gamma'-\eta+\phi-n}\frac{(\epsilon'+\phi)_n(-\gamma-\gamma'+\eta+\phi)_n(-\gamma'-\epsilon+\eta+\phi)_n}{(\phi)_n(-\gamma'-\epsilon'+\phi)_n(-\gamma-\gamma'-\epsilon+\eta+\phi)_n}. \tag{41}$$

Using Hadamard Product in (41), we get

$$LHS = \frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)}x^{\gamma+\gamma'-\eta+\phi}$$
$$\times \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)n!}\left(\frac{\alpha}{x}\right)^n$$
$$\times \frac{(\epsilon'+\phi)_n(-\gamma-\gamma'+\eta+\phi)_n(-\gamma'-\epsilon+\eta+\phi)_n}{(\phi)_n(-\gamma'-\epsilon'+\phi)_n(-\gamma-\gamma'-\epsilon+\eta+\phi)_n}\frac{(1)_n}{n!}.$$
$$= x^{\gamma+\gamma'-\eta-\phi}\frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)}\Psi_{p_1,p_2}^{F_1}\left(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{x}\right)$$
$$* {}_4F_3\left[\begin{matrix}\epsilon'+\phi & -\gamma-\gamma'+\eta+\phi & -\gamma'-\epsilon+\eta+\phi & 1\\ \phi & -\gamma'-\epsilon'+\phi & -\gamma-\gamma'-\epsilon+\eta+\phi & \end{matrix}; \frac{\alpha}{x}\right].$$

Hence, the desired result (31). $\square$

Special Cases
When $p_2 = 0$ and, if $p_1 = 1, r = 0, a_1, b_1', d = 1$, theorems 3.3 and 3.4 give the results for the confluent hypergeometric function.

Corollary 3.3. *Let* $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}$, $\Re(\xi) > \max\{0, \Re(-\gamma+\epsilon'), \Re(-\gamma-\gamma'-\epsilon+\eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$, *then*

$$\begin{aligned}
&\left(\mathfrak{D}_{0^+}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{\phi-1}{}_1\Psi_1(\varrho_2;\varrho_3;\alpha t)\right]\right)(x)\\
&= x^{\gamma+\gamma'-\eta+\phi-1}\frac{\Gamma(\phi)\Gamma(-\epsilon+\gamma+\phi)\Gamma(\gamma+\gamma'+\epsilon'-\eta+\phi)}{\Gamma(-\epsilon+\phi)\Gamma(\gamma+\gamma'-\eta+\phi)\Gamma(\gamma+\epsilon'-\eta+\phi)}{}_1\Psi_1(\varrho_2;\varrho_3;\alpha x)\\
&* {}_4F_3\left[\begin{matrix}\phi & -\epsilon+\gamma+\phi & \gamma+\gamma'+\epsilon'-\eta+\phi & 1 \\ & & & ;\alpha x\\ -\epsilon+\phi & \gamma+\gamma'-\eta+\phi & \gamma+\epsilon'-\eta+\phi & \end{matrix}\right].
\end{aligned}\tag{42}$$

Corollary 3.4. *Let* $\gamma, \gamma', \epsilon, \epsilon', \eta, \xi \in \mathbb{C}$, $\Re(\xi) > \max\{\Re(-\epsilon'), \Re(\gamma'+\epsilon-\eta), \Re(\gamma+\gamma'-\eta) + [\Re(\eta)] + 1\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. *Then*

$$\begin{aligned}
&\left(\mathfrak{D}_{0^-}^{\gamma,\gamma',\epsilon,\epsilon',\eta}\left[t^{-\phi}{}_1\Psi_1\left(\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)\\
&= x^{\gamma+\gamma'-\eta-\phi}\frac{\Gamma(\epsilon'+\phi)\Gamma(-\gamma-\gamma'+\eta+\phi)\Gamma(-\gamma'-\epsilon+\eta+\phi)}{\Gamma(\phi)\Gamma(-\gamma'-\epsilon'+\phi)\Gamma(-\gamma-\gamma'-\epsilon+\eta+\phi)}{}_1\Psi_1\left(\varrho_2;\varrho_3;\frac{\alpha}{x}\right)\\
&* {}_4F_3\left[\begin{matrix}\epsilon'+\phi & -\gamma-\gamma'+\eta+\phi & -\gamma'-\epsilon+\eta+\phi & 1 \\ & & & ;\frac{\alpha}{x}\\ \phi & -\gamma'-\epsilon'+\phi & -\gamma-\gamma'-\epsilon+\eta+\phi & \end{matrix}\right].
\end{aligned}\tag{43}$$

4. Image formula using Saigo fractional integral operator

Definition 4.1. *The Saigo fractional integral operators are defined as* [16, 17]

$$\left(\mathfrak{I}_{0^-}^{\gamma,\epsilon,\eta}f\right)(y) = \frac{1}{\Gamma(\gamma)}\int_y^\infty (t-y)^{\gamma-1}t^{-\gamma-\epsilon}{}_2F_1\left(\gamma+\epsilon,-\eta;\gamma;1-\frac{x}{t}\right)f(t)dt. \tag{44}$$

and

$$\left(\mathfrak{I}_{0^+}^{\gamma,\epsilon,\eta}f\right)(y) = \frac{y^{-\gamma-\epsilon}}{\Gamma(\gamma)}\int_0^y (y-t)^{\gamma-1}{}_2F_1\left(\gamma+\epsilon,-\eta;\gamma;1-\frac{t}{x}\right)f(t)dt. \tag{45}$$

where $y > 0, \gamma, \epsilon, \eta \in \mathbb{C}, \Re(\gamma) > 0$ *and* ${}_2F_1(.)$ *is the Gauss hypergeometric series.*

We will use the following Lemmas [17] in our result:

Lemma 4.1. *Let* $\gamma, \epsilon, \eta \in \mathbb{C}$, $\Re(\gamma) > 0, \Re(\xi) > \max\{0, \Re(\epsilon-\eta)\}$, *then*

$$\left(\mathfrak{I}_{0^+}^{\gamma,\epsilon,\eta}t^{\xi-1}\right)(y) = \frac{\Gamma(\xi)\Gamma(\xi+\eta-\epsilon)}{\Gamma(\xi-\epsilon)\Gamma(\xi+\gamma+\eta)}y^{\xi-\epsilon-1}. \tag{46}$$

Lemma 4.2. *Let* $\gamma, \epsilon, \eta \in \mathbb{C}$, $\Re(\gamma) > 0, \Re(\xi) < 1 + \min[\Re(\epsilon), \Re(\eta)]$, *then*

$$\left(\mathfrak{I}_{0^-}^{\gamma,\epsilon,\eta}t^{\xi-1}\right)(y) = \frac{\Gamma(\epsilon-\xi+1)\Gamma(\eta-\xi+1)}{\Gamma(1-\xi)\Gamma(\gamma+\epsilon+\eta-\xi+1)}y^{\xi-\epsilon-1}. \tag{47}$$

4.1. **For the Generalized Hypergeometric Function.**

Theorem 4.1. *Let* $\gamma, \epsilon, \eta \in \mathbb{C}$ *and* $\Re(\gamma) > 0, \Re(\xi) > \max\{0, \Re(\epsilon - \eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$. *Then*

$$\left(\Im_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x)$$
$$= x^{\phi-\epsilon-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha x) * {}_3F_2\left[\begin{matrix}\phi & \phi+\eta-\epsilon & 1 \\ & & ;\alpha x \\ \phi-\epsilon & \phi+\gamma+\eta & \end{matrix}\right]. \tag{48}$$

Proof. Consider

$$\begin{aligned} LHS &= \left(\Im_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha t)\right]\right)(x) \\ &= \left(\Im_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha t)^n}{n!}\right]\right)(x). \end{aligned}$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\Im_{0^+}^{\gamma,\epsilon,\eta}t^{\phi+n-1}\right](x). \tag{49}$$

Using Lemma 4.1 in the above equation (49), we get

$$\begin{aligned} LHS &= \sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\phi+n)\Gamma(\phi+n+\eta-\epsilon)}{\Gamma(\phi+n-\epsilon)\Gamma(\phi+n+\gamma+\eta)}x^{\phi+n-\epsilon-1} \\ &= \frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!} \\ &\times x^{\phi+n-\epsilon-1}\frac{(\phi)_n(\phi+\eta-\epsilon)_n}{(\phi-\epsilon)_n(\phi+\gamma+\eta)_n}. \end{aligned} \tag{50}$$

Using Hadamard Product in (50), we get

$$\begin{aligned} &LHS \\ &= \frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}x^{\phi-\epsilon-1}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}(\alpha x)^n \\ &\times \frac{(\phi)_n(\phi+\eta-\epsilon)_n}{(\phi-\epsilon)_n(\phi+\gamma+\eta)_n}\frac{(1)_n}{n!} \\ &= x^{\phi-\epsilon-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}F_{p_1,p_2}^{F_1}(\varrho_1,\varrho_2;\varrho_3;\alpha x) \\ &* {}_3F_2\left[\begin{matrix}\phi & \phi+\eta-\epsilon & 1 \\ & & ;\alpha x \\ \phi-\epsilon & \phi+\gamma+\eta & \end{matrix}\right]. \end{aligned}$$

Hence, the desired result (48). □

Theorem 4.2. *Let $\gamma, \epsilon, \eta \in \mathbb{C}$ and $\Re(\gamma) > 0, \Re(\xi) < 1+\min[\Re(\epsilon), \Re(\eta)]$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. Then*

$$\left(\mathfrak{I}_{0^-}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}F_{p_1,p_2}^{F_1}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)$$
$$= x^{\phi-\epsilon-1}\frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+1)}$$
$$F_{p_1,p_2}^{F_1}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x}\right) * {}_3F_2\left[\begin{matrix}\epsilon-\phi+1 & \eta-\phi+1 & 1 \\ & & ;\frac{\alpha}{x} \\ 1-\phi & \gamma+\epsilon+\eta-\phi+1 & \end{matrix}\right]. \tag{51}$$

Proof. Consider

$$LHS = \left(\mathfrak{I}_{0^-}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}F_{p_1,p_2}^{F_1}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x)$$
$$= \left(\mathfrak{I}_{0^-}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{\alpha^n}{t^n n!}\right]\right)(x)$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{I}_{0^+}^{\gamma,\epsilon,\eta}t^{\phi-n-1}\right](x). \tag{52}$$

Using Lemma 4.2 in the above equation (52), we get

$$LHS = \sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!} \tag{53}$$
$$\times\frac{\Gamma(\epsilon-\phi+n+1)\Gamma(\eta-\phi+n+1)}{\Gamma(1-\phi+n)\Gamma(\gamma+\epsilon+\eta-\phi+n+1)}x^{\phi-n-\epsilon-1}$$
$$= \frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+n+1)}\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}$$
$$\times x^{\phi-n-\epsilon-1}\frac{(\epsilon-\phi+1)_n(\eta-\phi+1)_n}{(1-\phi)_n(\gamma+\epsilon+\eta-\phi+n+1)_n}. \tag{54}$$

Using Hadamard Product in (53), we get

$$LHS = \frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+n+1)}x^{\phi-\epsilon-1}$$
$$\times\sum_{n=0}^{\infty}(\varrho_1)_n\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}\left(\frac{\alpha}{x}\right)^n\frac{(\epsilon-\phi+1)_n(\eta-\phi+1)_n}{(1-\phi)_n(\gamma+\epsilon+\eta-\phi+n+1)_n}\frac{(1)_n}{n!}$$
$$= x^{\phi-\epsilon-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}F_{p_1,p_2}^{F_1}\left(\varrho_1,\varrho_2;\varrho_3;\frac{\alpha}{x}\right)$$
$$* {}_3F_2\left[\begin{matrix}\epsilon-\phi+1 & \eta-\phi+1 & 1 \\ & & ;\frac{\alpha}{x} \\ 1-\phi & \gamma+\epsilon+\eta-\phi+n+1 & \end{matrix}\right].$$

Hence, we get the desired result (51). □

Special Cases

When $p_2 = 0$ and, if $p_1 = 1, r = 0, a_1, b_1', d = 1$, theorems 4.1 and 4.2 give the results for the Gauss hypergeometric function.

Corollary 4.1. *Let $\gamma, \epsilon, \eta \in \mathbb{C}$ and $\Re(\gamma) > 0, \Re(\xi) > \max\{0, \Re(\epsilon - \eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$. Then*

$$\left(\Im_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}{}_2F_1(\varrho_1, \varrho_2; \varrho_3; \alpha t)\right]\right)(x)$$
$$= x^{\phi-\epsilon-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}{}_2F_1(\varrho_1, \varrho_2; \varrho_3; \alpha x) * {}_3F_2\left[\begin{matrix}\phi & \phi+\eta-\epsilon & 1 & \\ & & & ;\alpha x \\ \phi-\epsilon & \phi+\gamma+\eta & & \end{matrix}\right]. \tag{55}$$

Corollary 4.2. *Let $\gamma, \epsilon, \eta \in \mathbb{C}$ and $\Re(\gamma) > 0, \Re(\xi) < 1+\min[\Re(\epsilon), \Re(\eta)]$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. Then*

$$\left(\Im_{0^-}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}{}_2F_1\left(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{t}\right)\right]\right)(x)$$
$$= x^{\phi-\epsilon-1}\frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+1)}$$
$${}_2F_1\left(\varrho_1, \varrho_2; \varrho_3; \frac{\alpha}{x}\right) * {}_3F_2\left[\begin{matrix}\epsilon-\phi+1 & \eta-\phi+1 & 1 & \\ & & & ;\frac{\alpha}{x} \\ 1-\phi & \gamma+\epsilon+\eta-\phi+1 & & \end{matrix}\right]. \tag{56}$$

4.2. **For the Generalized Confluent Hypergeometric Function.**

Theorem 4.3. *Let $\gamma, \epsilon, \eta \in \mathbb{C}$ and $\Re(\gamma) > 0, \Re(\xi) > \max\{0, \Re(\epsilon - \eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$. Then*

$$\left(\Im_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}\Psi_{p_1,p_2}^{F_1}(\varrho_2; \varrho_3; \alpha t)\right]\right)(x)$$
$$= x^{\phi-\epsilon-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}\Psi_{p_1,p_2}^{F_1}(\varrho_2; \varrho_3; \alpha x) * {}_3F_2\left[\begin{matrix}\phi & \phi+\eta-\epsilon & 1 & \\ & & & ;\alpha x \\ \phi-\epsilon & \phi+\gamma+\eta & & \end{matrix}\right]. \tag{57}$$

Proof. Consider

$$LHS = \left(\Im_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}\Psi_{p_1,p_2}^{F_1}(\varrho_2; \varrho_3; \alpha t)\right]\right)(x)$$
$$= \left(\Im_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha t)^n}{n!}\right]\right)(x).$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$LHS = \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\Im_{0^+}^{\gamma,\epsilon,\eta}t^{\phi+n-1}\right](x). \tag{58}$$

Using Lemma 4.1 in the above equation (58), we get

$$LHS = \sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\phi+n)\Gamma(\phi+n+\eta-\epsilon)}{\Gamma(\phi+n-\epsilon)\Gamma(\phi+n+\gamma+\eta)}x^{\phi+n-\epsilon-1}$$
$$= \frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}\sum_{n=0}^{\infty}\frac{B_{p_1,p_2}^{F_1}(\varrho_2+n, \varrho_3-\varrho_2)}{B(\varrho_2, \varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}$$
$$\times x^{\phi+n-\epsilon-1}\frac{(\phi)_n(\phi+\eta-\epsilon)_n}{(\phi-\epsilon)_n(\phi+\gamma+\eta)_n}. \tag{59}$$

Using Hadamard Product in (59), we get

$$
\begin{aligned}
&LHS \\
&= \frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)} x^{\phi-\epsilon-1} \sum_{n=0}^{\infty} \frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!}(\alpha x)^n \frac{(\phi)_n(\phi+\eta-\epsilon)_n}{(\phi-\epsilon)_n(\phi+\gamma+\eta)_n}\frac{(1)_n}{n!} \\
&= x^{\phi-\epsilon-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}\Psi^{F_1}_{p_1,p_2}(\varrho_2;\varrho_3;\alpha x) * {}_3F_2\begin{bmatrix} \phi & \phi+\eta-\epsilon & 1 \\ & & ;\alpha x \\ \phi-\epsilon & \phi+\gamma+\eta & \end{bmatrix}.
\end{aligned}
$$

Hence, the desired result (57). $\square$

Theorem 4.4. *Let $\gamma,\epsilon,\eta \in \mathbb{C}$ and $\Re(\gamma) > 0, \Re(\xi) < 1+\min[\Re(\epsilon),\Re(\eta)]$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. Then*

$$
\begin{aligned}
&\left(\mathfrak{I}^{\gamma,\epsilon,\eta}_{0^-}\left[t^{\phi-1}\Psi^{F_1}_{p_1,p_2}\left(\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x) \\
&= x^{\phi-\epsilon-1}\frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+1)} \\
&\Psi^{F_1}_{p_1,p_2}\left(\varrho_2;\varrho_3;\frac{\alpha}{x}\right) * {}_3F_2\begin{bmatrix} \epsilon-\phi+1 & \eta-\phi+1 & 1 \\ & & ;\frac{\alpha}{x} \\ 1-\phi & \gamma+\epsilon+\eta-\phi+1 & \end{bmatrix}.
\end{aligned}
\tag{60}
$$

Proof. Consider

$$
\begin{aligned}
LHS &= \left(\mathfrak{I}^{\gamma,\epsilon,\eta}_{0^-}\left[t^{\phi-1}\Psi^{F_1}_{p_1,p_2}\left(\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x) \\
&= \left(\mathfrak{I}^{\gamma,\epsilon,\eta}_{0^-}\left[t^{\phi-1}\sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{\alpha^n}{t^n n!}\right]\right)(x)
\end{aligned}
$$

Switching the order of integral and sum based on the valid conditions mentioned, we get

$$
LHS = \sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\left[\mathfrak{I}^{\gamma,\epsilon,\eta}_{0^+}t^{\phi-n-1}\right](x). \tag{61}
$$

Using Lemma 4.2 in the above equation (61), we get

$$
\begin{aligned}
LHS &= \sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!}\frac{\Gamma(\epsilon-\phi+n+1)\Gamma(\eta-\phi+n+1)}{\Gamma(1-\phi+n)\Gamma(\gamma+\epsilon+\eta-\phi+n+1)}x^{\phi-n-\epsilon-1} \\
&= \frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+n+1)}\sum_{n=0}^{\infty}\frac{B^{F_1}_{p_1,p_2}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)}\frac{(\alpha)^n}{n!} \\
&\times x^{\phi-n-\epsilon-1}\frac{(\epsilon-\phi+1)_n(\eta-\phi+1)_n}{(1-\phi)_n(\gamma+\epsilon+\eta-\phi+n+1)_n}.
\end{aligned}
\tag{62}
$$

Using Hadamard Product in (62), we get

$$\begin{aligned} LHS &= \frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+n+1)} x^{\phi-\epsilon-1} \\ &\times \sum_{n=0}^{\infty} \frac{B_{p_1,p_2}^{F_1}(\varrho_2+n,\varrho_3-\varrho_2)}{B(\varrho_2,\varrho_3-\varrho_2)n!} \left(\frac{\alpha}{x}\right)^n \frac{(\epsilon-\phi+1)_n(\eta-\phi+1)_n}{(1-\phi)_n(\gamma+\epsilon+\eta-\phi+n+1)_n} \frac{(1)_n}{n!} \\ &= x^{\phi-\epsilon-1} \frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)} \Psi_{p_1,p_2}^{F_1}\left(\varrho_2;\varrho_3;\frac{\alpha}{x}\right) \\ &* {}_3F_2 \left[\begin{matrix} \epsilon-\phi+1 & \eta-\phi+1 & 1 \\ & & ;\frac{\alpha}{x} \\ 1-\phi & \gamma+\epsilon+\eta-\phi+n+1 & \end{matrix}\right]. \end{aligned}$$

Hence, the desired result (60). □

Special Cases

When $p_2 = 0$ and, if $p_1 = 1, r = 0, a_1, b'_1, d = 1$, theorems 4.3 and 4.4 give the results for the confluent hypergeometric function.

Corollary 4.3. *Let* $\gamma,\epsilon,\eta \in \mathbb{C}$, $\Re(\gamma) > 0, \Re(\xi) > \max\{0, \Re(\epsilon-\eta)\}$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\alpha t| < 1$. *Then*

$$\begin{aligned} &\left(\mathfrak{I}_{0^+}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}{}_1\Psi_1(\varrho_2;\varrho_3;\alpha t)\right]\right)(x) \\ &= x^{\phi-\epsilon-1}\frac{\Gamma(\phi)\Gamma(\phi+\eta-\epsilon)}{\Gamma(\phi-\epsilon)\Gamma(\phi+\gamma+\eta)}{}_1\Psi_1(\varrho_2;\varrho_3;\alpha x) * {}_3F_2\left[\begin{matrix} \phi & \phi+\eta-\epsilon & 1 \\ & & ;\alpha x \\ \phi-\epsilon & \phi+\gamma+\eta & \end{matrix}\right]. \end{aligned} \tag{63}$$

Corollary 4.4. *Let* $\gamma,\epsilon,\eta \in \mathbb{C}$, $\Re(\gamma) > 0, \Re(\xi) < 1 + \min[\Re(\epsilon), \Re(\eta)]$, $\Re(\varrho_3) > \Re(\varrho_2) > 0, \Re(p_1), \Re(p_2) > 0$, $|\frac{\alpha}{t}| < 1$. *Then*

$$\begin{aligned} &\left(\mathfrak{I}_{0^-}^{\gamma,\epsilon,\eta}\left[t^{\phi-1}{}_1\Psi_1\left(\varrho_2;\varrho_3;\frac{\alpha}{t}\right)\right]\right)(x) \\ &= x^{\phi-\epsilon-1}\frac{\Gamma(\epsilon-\phi+1)\Gamma(\eta-\phi+1)}{\Gamma(1-\phi)\Gamma(\gamma+\epsilon+\eta-\phi+1)} \\ &{}_1\Psi_1\left(\varrho_2;\varrho_3;\frac{\alpha}{x}\right) * {}_3F_2\left[\begin{matrix} \epsilon-\phi+1 & \eta-\phi+1 & 1 \\ & & ;\frac{\alpha}{x} \\ 1-\phi & \gamma+\epsilon+\eta-\phi+1 & \end{matrix}\right]. \end{aligned} \tag{64}$$

5. Comparative Study and Conclusion

The use of fractional calculus operators such as Appell function kernels, Saigo operators on extended and confluent hypergeometric functions is a big step forward in mathematical research. Unlike earlier studies, which have mainly focused on particular operators or specific functions, this research incorporates numerous operators to build a comprehensive framework. This integration enables a more versatile and resilient technique to solving fractional differential equations, which has not been thoroughly investigated in the existing literature.

One of the most important contributions of this study is its capacity to generalize and unify distinct mathematical notions. By applying these various fractional operators to hypergeometric functions, the study presents a cohesive theoretical framework that bridges the gap between distinct techniques in fractional calculus. This unification simplifies the theoretical foundations while simultaneously increasing the applicability of these mathematical tools to a broader range of problems and disciplines.

This study, uses the generalized properties of hypergeometric functions to provide more efficient algorithms and analytical tools. These innovations greatly lower the computing burden and improve solution accuracy, making them more suitable for real-world applications.

In this paper, we have considered various fractional operators to investigate the image formulas corresponding to the extended hypergeometric and confluent hypergeometric function involving Appell series and Lauricella function. The different fractional operators used here include fractional integral operator involving Appell series $F_3(.)$ and Saigo operator. These fractional operators have wide range of applications in solving kinetic equations, fractional diffusion, fractional reaction diffusion equations and many more. The results obtained for extended hypergeometric and confluent hypergeometric function are reducible to Gauss hypergeometric and confluent hypergeometric function under certain particular conditions.

6. Applications and Future Scope

This research is new in addition to having a wide range of possible practical applications. In physics, engineering, finance, biology, and other domains, the study opens up new possibilities for tackling challenging issues by applying fractional calculus to hypergeometric functions. These sophisticated mathematical tools, for instance, enable improved modeling of quantum mechanics, viscoelastic materials, and anomalous diffusion, providing solutions which were previously unattainable.

We can use the above results to discuss the fractional diffusion equations, which will help in enhancing the understanding of particle transport in heterogenous materials. We can also apply these operators to the stress-strain relationship to solve the viscoelastic model, improving simulations of material behavior under different loading conditions.These results can be used to solve fractional Schrodinger equation to find the wave functions of particles in complex potentials. This inturn is useful in quantum transport and nanoscale systems. Furthermore, these can be used to design fractional filters to enhance signal processing algorithms, improving the resolution of medical images and seismic signals, in fractional-order controllers to improve the robustness and precision of robotic arms and automated systems in manufacturing. Also, these can be used to model the spread of diseases or the growth of populations with memory effects.

Finally, this work opens up new avenues for theoretical and practical mathematics research. Future study will benefit greatly from the innovative combination of numerous fractional operators with hypergeometric functions. It promotes the discovery of new hypergeometric functions, the investigation of other fractional operators, and the use of these tools to address novel issues in a range of scientific fields. This research's unique and progressive nature is further demonstrated by its potential for expansion and discovery in the future. Various other fractional operators such as Erdélyi-Kober, Caputo-type, Riemann Liouville and Weyl can be used in future to obtain various other results corresponding to the extended hypergeometric and confluent hypergeometric function involving Appell series and Lauricella function.

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