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WEAKER FORMS OF OPEN SETS IN PYTHAGOREAN FUZZY NANO TOPOLOGICAL SPACES AND ITS APPLICATION USING ENTROPY MEASURE

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Abstract. In this paper, we introduce a Pythagorean Fuzzy nano M-open set which is the union of Pythagorean Fuzzy nano $\delta\mathcal{P}$-open sets and Pythagorean Fuzzy nano $\theta\mathcal{S}$-open sets in Pythagorean Fuzzy nano topological spaces. Also, we discuss about near open sets, their properties and examples of a Pythagorean Fuzzy nano M-open set. Moreover, we investigate some of their basic properties and examples of Pythagorean Fuzzy nano M-interior and M-closure in a Pythagorean Fuzzy nano topological spaces. One real life applications, one on better way of shopping, based on this proposed entropy measure are also illustrated.

Keywords: Pythagorean Fuzzy nano M-open sets, Pythagorean Fuzzy nano M-closed sets, Pythagorean Fuzzy nano M-$int(A)$ and Pythagorean Fuzzy nano M-$cl(A)$, Pythagorean Fuzzy Entropy.

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1. Introduction

Zadeh [27] generalized the usual set by using fuzzy set in which every detail is described with a degree of membership function. The fuzzy set has many programs in economic system, decision making, facts mining, commercial enterprise and many others. Fuzzy set has been generalized to greater non-standard fuzzy subsets. As Intuitionistic fuzzy subset become introduced with the aid of Atanassov [3], in which every element had the degree of membership and the degree of non- membership. Yager [24, 25, 26] presented the perception of Pythagorean fuzzy subset that is a typical fuzzy subsets and which has many powerful applications in natural and social sciences. Pythagorean fuzzy subsets can be used appropriately on every instant where intuitionistic fuzzy subsets cannot be

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used. Standard topology has been stepped forward by means of taking its motivation from classical analysis and applied on several sections of research inclusive of system getting to know, statistics evaluation, facts mining. Farther the scrutiny of topology refers the relationship between spatial gadgets and features and it may be used to explain some sure spatial functions and to conceive statistics units which have higher great control and extra statistics integrity. In 1968, Chang [7] described the theory of fuzzy topological space and generalized some fundamental idea of topology inclusive of open set, closed set, continuity and compactness. Following this observation, Lowen gave a different explanation of a fuzzy topological space by way of converting a primary property of topology [13]. In 1995, Coker delivered the notion of intuitionistic fuzzy topological space and studied some equivalent variations of some standards of classical topology together with continuity and compactness [8]. Furthermore, some authors studied the concept of fuzzy soft topological space and its packages in choice-making environment.

Pawlak [16] introduced Rough set theory by handling vagueness and uncertainty. This can be often defined by means of topological operations, interior and closure, called approximations. In 2013, Lellis Thivagar [11] introduced an extension of rough set theory called Nano topology and defined its topological spaces in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it.

S. Saha [17] defined δ-open sets in fuzzy topological spaces, nano topological space by Pankajam et al. [15] and neutrosophic topological space by Vadivel et al. [21]. Recently, Lellis Thivagar et.al [12] explored a new concept of neutrosophic nano topology, intuitionistic nano topology and fuzzy nano topology. El-Maghrabi and Al-Juhani [9] proposed the concept of M-open sets in topological spaces in 2011 and examined some of their features. Padma et al. [14] also found M-open sets in nano topological spaces. Vadivel et al. [19, 20, 22] discussed some open sets in fuzzy nano and neutrosophic nano topological spaces. Kalaiyarsan et al. [10] and Vadivel et al. [23] introduced M-open sets in fuzzy and neutrosophic nano topological spaces.

The remainder of this paper is organized as follows. In section 2, some basic definitions of fs's, IFS's and $\mathcal{PF}s$'s are briefly reviewed. In section 3, We develop the concept of some stronger and weaker forms of Pythagorean fuzzy nano open sets in Pythagorean fuzzy nano topological space and also specialized some of their basic properties with examples. Finally, we presented an entropy measure for $\mathcal{PF}\mathfrak{N}s$'s and one real- world scenarios where this entropy measure can be used are mentioned in section 4. The paper is concluded in section 5.

2. Preliminaries

We recall some basic notions of fuzzy sets, IFS's and $\mathcal{PF}\mathfrak{N}s$'s .

Definition 2.1. [27] Let U be a nonempty set. A fuzzy set A in U is characterized by a membership function $\mu_A : X \to [0,1]$. That is:

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \\ (0,1) & \text{if } x \text{ is partly in } X. \end{cases}$$

Alternatively, a fuzzy set A in U is an object having the form $A = \{< x, \mu_A(x) > | x \in X\}$ or $A = \left\{ \left\langle \frac{\mu_A(x)}{x} \right\rangle | x \in X \right\}$, where the function $\mu_A(x) : X \to [0,1]$ defines the degree of membership of the element, $x \in X$.

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represent full membership and full nonmembership. Fuzzy set is a collection of objects with graded membership, that is, having degree of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in a binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets is special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Fuzzy sets theory permits the gradual assessment of the membership of element in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$.

Let us consider two examples:

(i) all employees of XYZ who are over $1.8m$ in height; (ii) all employees of XYZ who are tall. The first example is a classical set with a universe (all XYZ employees) and a membership rule that divides the universe into members (those over $1.8m$) and nonmembers. The second example is a fuzzy set, because some employees are definitely in the set and some are definitely not in the set, but some are borderline.

This distinction between the ins, the outs, and the borderline is made more exact by the membership function, μ. If we return to our second example and let A represent the fuzzy set of all tall employees and x represent a member of the universe U (i.e. all employees), then $\mu_A(x)$ would be $\mu_A(x) = 1$ if x is definitely tall or $\mu_A(x) = 0$ if x is definitely not tall or $0 < \mu_A(x) < 1$ for borderline cases.

Definition 2.2. [3, 4, 5, 6] Let a nonempty set U be fixed. An IFS A in U is an object having the form: $A = \{< x, \mu_A(x), \lambda_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle | x \in X\right\}$, where the functions $\mu_A(x) : X \to [0, 1]$ and $\lambda_A(x) : X \to [0, 1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X : 0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. For each A in U: $\pi_A(x) = 1 - \mu_A(x) - \lambda_A(x)$ is the intuitionistic fuzzy set index or hesitation margin of x in X. The hesitation margin $\pi_A(x)$ is the degree of nondeterminacy of $x \in X$ to the set A and $\pi_A(x) \in [0, 1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$ or $x \notin X$. Thus: $\mu_A(x) + \lambda_A(x) + \pi_A(x) = 1$.

Example 2.1. Let $X = \{x, y, z\}$ be a fixed universe of discourse and $A = \{\left\langle \frac{0.6, 0.1}{x} \right\rangle$, $\left\langle \frac{0.8, 0.1}{y} \right\rangle$, $\left\langle \frac{0.5, 0.3}{z} \right\rangle\}$, be the intuitionistic fuzzy set in U. The hesitation margins of the elements x, y, z to A are as follows: $\pi_A(x) = 0.3$, $\pi_A(y) = 0.1$ and $\pi_A(z) = 0.2$.

Definition 2.3. [24, 25, 26] Let U be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over U, is defined by the following: $A = \{< x, \mu_A(x), \lambda_A(x) | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle | x \in X\right\}$, where the functions $\mu_A(x) : X \to [0, 1]$ and $\lambda_A(x) : X \to [0, 1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of U, and for every $x \in X$, $0 \leq (\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$. Supposing $(\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\lambda_A(x))^2]}$ and $\pi_A(x) \in [0, 1]$. In what follows, $(\mu_A(x))^2 + (\lambda_A(x))^2 + (\pi_A(x))^2 = 1$. Otherwise, $\pi_A(x) = 0$ whenever $(\mu_A(x))^2 + (\lambda_A(x))^2 = 1$. We denote the set of all Pythagorean fuzzy sets over U by pfs's.

Definition 2.4. [26] Let A and B be pfs's of the forms $A = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$ and $B = \{< a, \mu_B(a), \lambda_B(a) > | a \in X\}$. Then

(i) $A \subseteq B$ if and only if $\mu_A(a) \leq \mu_B(a)$ and $\lambda_A(a) \geq \lambda_B(a)$ for all $a \in X$.

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
(iii) $\bar{A} = \{< a, \lambda_A(a), \mu_A(a) > |a \in X\}$.
(iv) $A \cap B = \{< a, \mu_A(a) \wedge \mu_B(a), \lambda_A(a) \vee \lambda_B(a) > |a \in X\}$.
(v) $A \cup B = \{< a, \mu_A(a) \vee \mu_B(a), \lambda_A(a) \wedge \lambda_B(a) > |a \in X\}$.
(vi) $0_P = \{< a, 0, 1 > |a \in X\}$ and $1_P = \{< a, 1, 0 > |a \in X\}$.
(vii) $\bar{1_P} = 0_P$ and $\bar{0_P} = 1_P$.

Definition 2.5. [1] Let U be a non-empty set and R be an equivalence relation on U. Let A be a Pythagorean fuzzy set in U with the membership function $\mu_A(x)$ and non membership function $\lambda_A(x)$, $\forall\ x \in U$. The Pythagorean fuzzy nano lower, Pythagorean fuzzy nano upper approximation and Pythagorean fuzzy nano boundary of A in the approximation (U, R) denoted by $\underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A)$ and $B_{\mathcal{PFN}}(A)$ are respectively defined as follows:

(i) $\underline{\mathcal{PFN}}(A) = \left\{\langle x, \mu_{\underline{R}(A)}(x), \lambda_{\overline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
(ii) $\overline{\mathcal{PFN}}(F) = \left\{\langle x, \mu_{\overline{R}(A)}(x), \lambda_{\underline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
(iii) $B_{\mathcal{PFN}}(F) = \overline{\mathcal{PFN}}(F) - \underline{\mathcal{PFN}}(F)$

where $\mu_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \mu_A(y)$
$\lambda_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} \lambda_A(y)$,
$\mu_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \mu_A(y)$,
$\lambda_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} \lambda_A(y)$.

Definition 2.6. [1] Let U be an universe of discourse, R be an equivalence relation on U and A be a Pythagorean fuzzy set in U and if the collection $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A), B_{\mathcal{PFN}}(A)\}$ forms a topology then it is said to be a Pythagorean fuzzy nano topology. We call $(U, \tau_{\mathcal{R}}(A))$ (or simply U) as the Pythagorean fuzzy nano topological space. The elements of $\tau_{\mathcal{R}}(A)$ are called Pythagorean fuzzy nano open (briefly, $\mathcal{PFN}o$) sets.

Remark 2.1. [1] $[\tau_{\mathcal{R}}(A)]^c$ is called the dual fuzzy nano topology of $\tau_{\mathcal{R}}(A)$. Elements of $[\tau_{\mathcal{R}}(A)]^c$ are called Pythagorean fuzzy nano closed (briefly, $\mathcal{PFN}c$) sets. Thus, we note that a Pythagorean fuzzy set G of U is Pythagorean fuzzy nano closed in $\tau_{\mathcal{R}}(A)$ if and only if $1_P - G$ is Pythagorean fuzzy nano open in $\tau_{\mathcal{R}}(A)$.

Definition 2.7. [1, 2] Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PFN}ts$ with respect to A where A is a Pythagorean fuzzy subset of U. Let S be a Pythagorean fuzzy subset of U. Then Pythagorean fuzzy nano

(i) interior of S (briefly, $\mathcal{PFN}int(S)$) is defined by $\mathcal{PFN}int(S) = \cup\{I : I \subseteq S\ \&\ I$ is a $\mathcal{PFN}o$ set in $U\}$.
(ii) closure of S (briefly, $\mathcal{PFN}cl(S)$) is defined by $\mathcal{PFN}cl(S) = \cap\{A : S \subseteq A\ \&\ A$ is a $\mathcal{PFN}c$ set in $U\}$.
(iii) regular open (briefly, $\mathcal{PFN}ro$) set if $S = \mathcal{PFN}int(\mathcal{PFN}cl(S))$.
(iv) regular closed (briefly, $\mathcal{PFN}rc$) set if $S = \mathcal{PFN}cl(\mathcal{PFN}int(S))$.

3. Pythagorean fuzzy nano M-open sets

Definition 3.1. Let $(U, \tau_R(A))$ be an $\mathcal{PF}\mathfrak{N}ts$ and $S = \{< s, \mu_S(s), \lambda_S(s) > |s \in U\}$ be an $\mathcal{PF}\mathfrak{N}s$ in U. Then the $\mathcal{PF}\mathfrak{N}\delta$-interior and the $\mathcal{PF}\mathfrak{N}\delta$-closure of S are denoted by $\mathcal{PF}\mathfrak{N}\delta int(S)$ and $\mathcal{PF}\mathfrak{N}\delta cl(S)$ and are defined as follows. $\mathcal{PF}\mathfrak{N}\delta int(S) = \cup\{G|G$ is an $\mathcal{PF}\mathfrak{N}ros$ and $G \subseteq S\}$, $\mathcal{PF}\mathfrak{N}\delta cl(S) = \cap\{K|K$ is an $\mathcal{PF}\mathfrak{N}rcs$ and $S \subseteq K\}$.

Definition 3.2. Let $(U, \tau_R(A))$ be a $\mathcal{PF}\mathfrak{N}ts$ and $S = \{< s, \mu_S(s), \lambda_S(s) > |s \in U\}$ be an $\mathcal{PF}\mathfrak{N}s$ in U. A set S is said to be $\mathcal{PF}\mathfrak{N}$

(i) δ-open set (briefly, $\mathcal{PFN}\delta os$) if $S = \mathcal{PFN}\delta int(S)$,
(ii) δ-pre open set (briefly, $\mathcal{PFN}\delta\mathcal{P}os$) if $S \subseteq \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.
(iii) δ-semi open set (briefly, $\mathcal{PFN}\delta\mathcal{S}os$) if $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S))$.
(iv) e open set (briefly, $\mathcal{PFN}eos$) if $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.
(v) δ (resp. δ-pre, δ-semi and e) dense if $\mathcal{PFN}\delta cl(S)$ (resp. $\mathcal{PFN}\delta\mathcal{P}cl(S)$, $\mathcal{PFN}\delta\mathcal{S}cl(S)$ and $\mathcal{PFN}ecl(S)) = 1_P$.

The complement of an $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta\mathcal{P}os$, $\mathcal{PFN}\delta\mathcal{S}os$ and $\mathcal{PFN}eos$) is called an $\mathcal{PFN}\delta$ (resp. $\mathcal{PFN}\delta\mathcal{P}$, $\mathcal{PFN}\ \delta\mathcal{S}$ and $\mathcal{PFN}e$) closed set (briefly, $\mathcal{PFN}\delta cs$ (resp. $\mathcal{PFN}\delta\mathcal{P}cs$, $\mathcal{PFN}\delta\mathcal{S}cs$ and $\mathcal{PFN}ecs$)) in U.

The family of all $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta cs$, $\mathcal{PFN}\delta\mathcal{P}os$, $\mathcal{PFN}\delta\mathcal{P}cs$, $\mathcal{PFN}\delta\mathcal{S}os$, $\mathcal{PFN}\delta\mathcal{S}cs$, $\mathcal{PFN}eos$ and $\mathcal{PFN}ecs$) of U is denoted by $\mathcal{PFN}\delta OS(U)$, (resp. $\mathcal{PFN}\delta CS(U)$, $\mathcal{PFN}\delta\mathcal{P}OS(U)$, $\mathcal{PFN}\delta\mathcal{P}CS(U)$, $\mathcal{PFN}\delta\mathcal{S}OS(U)$, $\mathcal{PFN}\delta\mathcal{S}CS(U)$, $\mathcal{PFN}eOS(U)$ and $\mathcal{PFN}eCS(U)$).

Example 3.1. Assume $U = \{s_1, s_2, s_3, s_4\}$ be the universe set and the equivalence relation is $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. Let $A = \left\{\left\langle\frac{s_1}{0.6,0.4}\right\rangle, \left\langle\frac{s_2}{0.4,0.8}\right\rangle, \left\langle\frac{s_3}{0.5,0.75}\right\rangle, \left\langle\frac{s_4}{0.7,0.55}\right\rangle\right\}$ be a Pythagorean fuzzy subset of U.

$$\underline{\mathcal{PFN}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.6, 0.55}\right\rangle, \left\langle\frac{s_2}{0.4, 0.8}\right\rangle, \left\langle\frac{s_3}{0.5, 0.75}\right\rangle\right\},$$
$$\overline{\mathcal{PFN}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.7, 0.4}\right\rangle, \left\langle\frac{s_2}{0.4, 0.8}\right\rangle, \left\langle\frac{s_3}{0.5, 0.75}\right\rangle\right\},$$
$$B_{\mathcal{PFN}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.55, 0.6}\right\rangle, \left\langle\frac{s_2}{0.4, 0.8}\right\rangle, \left\langle\frac{s_3}{0.5, 0.75}\right\rangle\right\}.$$

Thus $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A), B_{\mathcal{PFN}}(A)\}$. Then $\left\{\left\langle\frac{s_1,s_4}{0.6,0.55}\right\rangle, \left\langle\frac{s_2}{0.4,0.8}\right\rangle, \left\langle\frac{s_3}{0.5,0.75}\right\rangle\right\}$ is a $\mathcal{PFN}o$ (resp. $\mathcal{PFN}\delta\mathcal{P}o$, $\mathcal{PFN}\delta\mathcal{S}o$, $\mathcal{PFN}\delta\alpha o$, $\mathcal{PFN}\delta\beta o$ and $\mathcal{PFN}eo$) set.

Definition 3.3. Let $(U, \tau_R(A))$ be an $\mathcal{PFN}ts$ and $S = \{< s, \mu_S(s), \lambda_S(s) > | s \in U\}$ be an $\mathcal{PFN}s$ in U. Then the $\mathcal{PFN}\delta$-pre (resp. $\mathcal{PFN}\delta$-semi and $\mathcal{PFN}\delta\beta$)-interior and the $\mathcal{PFN}\delta$-pre (resp. $\mathcal{PFN}\delta$-semi and $\mathcal{PFN}e$)-closure of S are denoted by $\mathcal{PFN}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}int(S)$ and $\mathcal{PFN}eint(S)$) and the $\mathcal{PFN}ecl(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}cl(S)$ and $\mathcal{PFN}ecl(S)$) and are defined as follows:

$\mathcal{PFN}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}int(S)$ and $\mathcal{PFN}eint(S)) = \cup\{G|G$ is a $\mathcal{PFN}\delta\mathcal{P}os$ (resp. $\mathcal{PFN}\delta\mathcal{S}os$ and $\mathcal{PFN}eos$)

and $G \subseteq S\}$ and $\mathcal{PFN}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{PFN}\delta\mathcal{S}cl(S)$ and $\mathcal{PFN}ecl(S)) = \cap\{K|K$ is an $\mathcal{PFN}\delta\mathcal{P}cs$ (resp. $\mathcal{PFN}\delta\mathcal{S}cs$ *and* $\mathcal{PFN}ecs$) and $S \subseteq K\}$.

Example 3.2. In Example 3.1, (i) $\mathcal{PFN}\delta\mathcal{P}int(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$, (ii) $\mathcal{PFN}\delta\mathcal{S}int(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$, (iii) $\mathcal{PFN}\delta\beta int(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$, (iv) $\mathcal{PFN}eint(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$,

(v) $\mathcal{PFN}\delta\mathcal{P}cl((B_{\mathcal{PFN}}(A))^c) = (B_{\mathcal{PFN}}(A))^c$, (vi) $\mathcal{PFN}\delta\mathcal{S}cl((B_{\mathcal{PFN}}(A))^c) = (B_{\mathcal{PFN}}(A))^c$, (vii) $\mathcal{PFN}\delta\beta cl((B_{\mathcal{PFN}}(A))^c) = (B_{\mathcal{PFN}}(A))^c$, (viii) $\mathcal{PFN}eint((B_{\mathcal{PFN}}(A))^c) = (B_{\mathcal{PFN}}(A))^c$,

Definition 3.4. Let $(U, \tau_R(A))$ be a $\mathcal{PFN}ts$ and S be a $\mathcal{PFN}s$ in U. A set S is said to be $\mathcal{PFN}$

(i) θ-interior of S (briefly, $\mathcal{PFN}\theta int(S)$) is defined by $\mathcal{PFN}\theta int(S) = \bigcup\{\mathcal{PFN}int(T) : T \subseteq S \ \&\ T$ is a $\mathcal{PFN}cs$ in $U\}$.
(ii) θ-open set (briefly, $\mathcal{PFN}\theta os$) if $S = \mathcal{PFN}\theta int(S)$.

(iii) θ -semi open set (briefly, $\mathcal{PFN}\theta\mathcal{S}os$) if $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))$.
(iv) M-open set (briefly, $\mathcal{PFN}Mos$) if $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.

The complement of a $\mathcal{PFN}Mos$ (resp. $\mathcal{PFN}\theta os$ & $\mathcal{PFN}\theta\mathcal{S}os$) is called an $\mathcal{PFN}M$ (resp. $\mathcal{PFN}\theta$ & $\mathcal{PFN}\theta\mathcal{S}$) closed set (briefly, $\mathcal{PFN}Mcs$ (resp. $\mathcal{PFN}\theta cs$ & $\mathcal{PFN}\theta\mathcal{S}cs$)) in U.

The family of all $\mathcal{PFN}\theta os$ (resp. $\mathcal{PFN}\theta cs$, $\mathcal{PFN}\theta\mathcal{S}os$, $\mathcal{PFN}\theta\mathcal{S}cs$, $\mathcal{PFN}Mos$ and $\mathcal{PFN}Mcs$) of U is denoted by $\mathcal{PFN}\theta OS(U)$ (resp. $\mathcal{PFN}\theta CS(U)$, $\mathcal{PFN}\theta\mathcal{S}OS(U)$, $\mathcal{PFN}\theta\mathcal{S}CS(U)$, $\mathcal{PFN}\ MOS(U)$ and $\mathcal{PFN}MCS(U)$).

Example 3.3. Assume $U = \{s_1, s_2, s_3, s_4\}$ be the universe set and the equivalence relation is $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. Let $A = \left\{\left\langle\frac{s_1}{0.8,0.4}\right\rangle, \left\langle\frac{s_2}{0.6,0.6}\right\rangle, \left\langle\frac{s_3}{0.7,0.7}\right\rangle, \left\langle\frac{s_4}{0.5,0.7}\right\rangle\right\}$ be a Pythagorean fuzzy subset of U.

$$\underline{\mathcal{PFN}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.5, 0.7}\right\rangle, \left\langle\frac{s_2}{0.6, 0.6}\right\rangle, \left\langle\frac{s_3}{0.7, 0.7}\right\rangle\right\},$$
$$\overline{\mathcal{PFN}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.8, 0.4}\right\rangle, \left\langle\frac{s_2}{0.6, 0.6}\right\rangle, \left\langle\frac{s_3}{0.7, 0.7}\right\rangle\right\},$$
$$B_{\mathcal{PFN}}(A) = \left\{\left\langle\frac{s_1, s_4}{0.7, 0.5}\right\rangle, \left\langle\frac{s_2}{0.6, 0.6}\right\rangle, \left\langle\frac{s_3}{0.7, 0.7}\right\rangle\right\}.$$

Thus $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A), B_{\mathcal{PFN}}(A)\}$. Then $\left\{\left\langle\frac{s_1,s_4}{0.5,0.7}\right\rangle, \left\langle\frac{s_2}{0.6,0.6}\right\rangle, \left\langle\frac{s_3}{0.7,0.7}\right\rangle\right\}$ is a $\mathcal{PFN}\theta o$ (resp. $\mathcal{PFN}\theta\mathcal{S}o$ and $\mathcal{PFN}Mo$) set; $\mathcal{PFN}\theta int(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$.

Definition 3.5. Let $(U, \tau_R(A))$ be a $\mathcal{PFN}ts$ and S be a $\mathcal{PFN}s$ in U. Then the $\mathcal{PFN}$

(i) M-interior (resp. $\mathcal{PFN}\theta$-interior and $\mathcal{PFN}\theta$-semi interior) of S (briefly, $\mathcal{PFN}Mint(S)$ (resp. $\mathcal{PFN}\theta int(S)$, $\mathcal{PFN}\theta\ \mathcal{S}int(S)$) is defined by $\mathcal{PFN}Mint(S)$ (resp. $\mathcal{PFN}\theta int(S)$ and $\mathcal{PFN}\theta\mathcal{S}int(S)) = \cup\{T : T \subseteq S$ and T is a $\mathcal{PFN}Mos$ (resp. $\mathcal{PFN}\theta os$ and $\mathcal{PFN}\theta\mathcal{S}\ os)\}$ in U.
(ii) M-closure (resp. θ-closure and θ-semi closure) of S (briefly, $\mathcal{PFN}Mcl(S)$ (resp. $\mathcal{PFN}\theta cl(S)$ & $\mathcal{PFN}\theta\mathcal{S}cl(S)$) is defined by $\mathcal{PFN}Mcl(S)$ (resp. $\mathcal{PFN}\theta cl(S)$ and $\mathcal{PFN}\theta\mathcal{S}cl(S)) = \cap\{T : S \subseteq T$ and T is a $\mathcal{PFN}Mcs$ (resp. $\mathcal{PFN}\theta cs$ and $\mathcal{PFN}\theta\mathcal{S}cs)\}$ in U.

Example 3.4. In Example 3.3, (i) $\mathcal{PFN}\theta int(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$,
(ii) $\mathcal{PFN}\theta\mathcal{S}int(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$, (iii) $\mathcal{PFN}Mint(B_{\mathcal{PFN}}(A)) = B_{\mathcal{PFN}}(A)$, (iv) $\mathcal{PFN}\theta cl((B_{\mathcal{PFN}}(A))^c) = (B_{\mathcal{PFN}}(A))^c$, (v) $\mathcal{PFN}\theta\mathcal{S}int((B_{\mathcal{PFN}}(A))^c) = (B_{\mathcal{PFN}}(A))^c$, (vi1) $\mathcal{PFN}Mint((B_{\mathcal{PFN}}(A))^c) = (B_{\mathcal{PFN}}(A))^c$,

Proposition 3.1. Let $(U, \tau_R(A))$ be a $\mathcal{PFN}ts$. Then the following statements are hold but the converse does not true.

(i) Every $\mathcal{PFN}\theta os$ (resp. $\mathcal{PFN}\theta cs$) is a $\mathcal{PFN}os$ (resp. $\mathcal{PFN}cs$).
(ii) Every $\mathcal{PFN}\theta os$ (resp. $\mathcal{PFN}\theta cs$) is a $\mathcal{PFN}\theta\mathcal{S}os$ (resp. $\mathcal{PFN}\theta\mathcal{S}cs$).
(iii) Every $\mathcal{PFN}\theta\mathcal{S}os$ (resp. $\mathcal{PFN}\theta\mathcal{S}cs$) is a $\mathcal{PFN}Mos$ (resp. $\mathcal{PFN}Mcs$).
(iv) Every $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta cs$) is a $\mathcal{PFN}\delta\mathcal{S}os$ (resp. $\mathcal{PFN}\delta\mathcal{S}cs$).
(v) Every $\mathcal{PFN}\delta os$ (resp. $\mathcal{PFN}\delta cs$) is a $\mathcal{PFN}\delta\mathcal{P}os$ (resp. $\mathcal{PFN}\delta\mathcal{P}cs$).
(vi) Every $\mathcal{PFN}\delta\mathcal{S}os$ (resp. $\mathcal{PFN}\delta\mathcal{S}cs$) is a $\mathcal{PFN}eos$ (resp. $\mathcal{PFN}ecs$).
(vii) Every $\mathcal{PFN}\delta\mathcal{P}os$ (resp. $\mathcal{PFN}\delta\mathcal{P}cs$) is a $\mathcal{PFN}Mos$ (resp. $\mathcal{PFN}Mcs$).
(viii) Every $\mathcal{PFN}Mos$ (resp. $\mathcal{PFN}Mcs$) is a $\mathcal{PFN}eos$ (resp. $\mathcal{PFN}ecs$).

Proof.

(i) If S is a $\mathcal{PFN}\theta os$ in U, then $S = \mathcal{PFN}\theta int(S) \subseteq \mathcal{PFN}int(S)$. Therefore, S is a $\mathcal{PFN}os$.

(ii) If S is a $\mathcal{PFN}\theta os$ in U, then $S = \mathcal{PFN}\theta int(S)$. So, $S = \mathcal{PFN}\theta int(S) \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))$. Therefore, S is a $\mathcal{PFN}\theta Sos$.

(iii) If S is a $\mathcal{PFN}\theta Sos$ in U, then $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))$. So, $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$. Therefore, S is a $\mathcal{PFN}Mos$.

(iv) If S is a $\mathcal{PFN}\delta os$ in U, then $S = \mathcal{PFN}\delta int(S)$. So, $S = \mathcal{PFN}\delta int(S) \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S))$. Therefore, S is a $\mathcal{PFN}\delta Sos$.

(v) If S is a $\mathcal{PFN}\delta os$ in U, then $S = \mathcal{PFN}\delta int(S)$. So, $S = \mathcal{PFN}\delta int(S) \subseteq \mathcal{PFN}int(S) \subseteq \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$. Therefore, S is a $\mathcal{PFN}\delta \mathcal{P}os$.

(vi) If S is a $\mathcal{PFN}\delta Sos$ in U, then $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S)) \subseteq \mathcal{PFN}cl((\mathcal{PFN}\delta int(S))) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$. Therefore, S is a $\mathcal{PFN}eos$.

(vii) If S is a $\mathcal{PFN}\delta \mathcal{P}os$ in U, then $S \subseteq \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$. Therefore, S is a $\mathcal{PFN}Mos$.

(viii) If S is a $\mathcal{PFN}Mos$ then $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$. So, $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \subseteq \mathcal{PFN}int(\mathcal{PFN}\delta cl(S) \subseteq \mathcal{PFN}cl(\mathcal{PFN}\delta\ int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)))$. Therefore S is a $\mathcal{PFN}eos$. It is also true for their respective closed sets.

Example 3.5. Assume $U = \{s_1, s_2, s_3, s_4\}$ be the universe set and the equivalence relation is $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. Let $A = \left\{\left\langle \frac{s_1}{0.3,0.1} \right\rangle, \left\langle \frac{s_2}{0.1,0.5} \right\rangle, \left\langle \frac{s_3}{0.2,0.45} \right\rangle, \left\langle \frac{s_4}{0.4,0.25} \right\rangle\right\}$ be a Pythagorean fuzzy subset of U.

$$\underline{\mathcal{PFN}}(A) = \left\{\left\langle \frac{s_1, s_4}{0.3, 0.25} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle\right\},$$

$$\overline{\mathcal{PFN}}(A) = \left\{\left\langle \frac{s_1, s_4}{0.4, 0.1} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle\right\},$$

$$B_{\mathcal{PFN}}(A) = \left\{\left\langle \frac{s_1, s_4}{0.25, 0.3} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle\right\}.$$

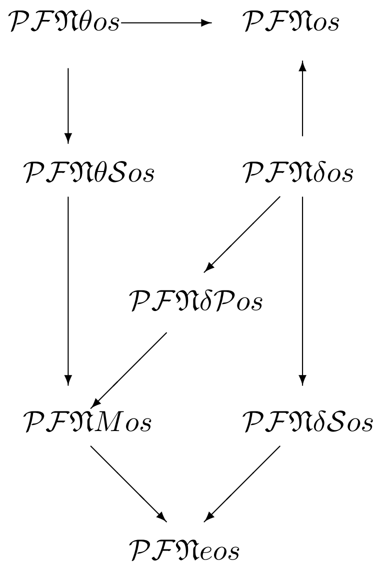
Thus $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A), \overline{\mathcal{PFN}}(A), B_{\mathcal{PFN}}(A)\}$. Then

(i) $\left\{\left\langle \frac{s_1,s_4}{0.4,0.1} \right\rangle, \left\langle \frac{s_2}{0.1,0.5} \right\rangle, \left\langle \frac{s_3}{0.2,0.45} \right\rangle\right\}$ is a $\mathcal{PFN}o$ (resp. $\mathcal{PFN}\delta\mathcal{P}o$, $\mathcal{PFN}eo$ and $\mathcal{PFN}eo$) set but not $\mathcal{PFN}\delta o$ (resp. $\mathcal{PFN}\delta o$, $\mathcal{PFN}Mo$ and $\mathcal{PFN}\delta So$) set.

(ii) $\left\{\left\langle \frac{s_1,s_4}{0.25,0.3} \right\rangle, \left\langle \frac{s_2}{0.5,0.1} \right\rangle, \left\langle \frac{s_3}{0.45,0.2} \right\rangle\right\}$ is a $\mathcal{PFN}\delta So$ (resp. $\mathcal{PFN}\theta So$ and $\mathcal{PFN}Mo$) set but not $\mathcal{PFN}\delta o$ (resp. $\mathcal{PFN}\theta o$ and $\mathcal{PFN}\delta\mathcal{P}o$) set.

(iii) $\left\{\left\langle \frac{s_1,s_4}{0.3,0.25} \right\rangle, \left\langle \frac{s_2}{0.1,0.5} \right\rangle, \left\langle \frac{s_3}{0.2,0.45} \right\rangle\right\}$ is a $\mathcal{PFN}o$ (resp. $\mathcal{PFN}Mo$) set but not $\mathcal{PFN}\theta o$ (resp. $\mathcal{PFN}\theta So$) set.

Remark 3.1. Form the above proposition and the examples, the following implications are hold.



Note: $A \to B$ denotes A implies B, but not conversely.

Proposition 3.2. The statements are true.

(i) $\mathcal{PFN}\theta\mathcal{S}cl(S) = S \cup \mathcal{PFN}int(\mathcal{PFN}\theta cl(S))$.
(ii) $\mathcal{PFN}\theta\mathcal{S}int(S) = S \cap \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))$.
(iii) $\mathcal{PFN}\delta\mathcal{P}cl(S) = S \cup \mathcal{PFN}cl(\mathcal{PFN}\delta int(S))$.
(iv) $\mathcal{PFN}\delta\mathcal{P}int(S) = S \cap \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.
(v) $\mathcal{PFN}\delta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S)) = \mathcal{PFN}\delta Pint(S) \cup \mathcal{PFN}cl(\mathcal{PFN}\delta int(S))$.
(vi) $\mathcal{PFN}\delta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S)) = \mathcal{PFN}\delta\mathcal{P}cl(S) \cap \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.
(vii) $\mathcal{PFN}\delta\mathcal{S}int(S) = S \cap \mathcal{PFN}cl(\mathcal{PFN}\delta int(S))$.
(viii) $\mathcal{PFN}\delta\mathcal{S}cl(S) = S \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.
(ix) $\mathcal{PFN}cl(\mathcal{PFN}\delta int(S)) = \mathcal{PFN}\delta cl(\mathcal{PFN}\delta int(S))$.
(x) $\mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) = \mathcal{PFN}\delta int(\mathcal{PFN}\delta cl(S))$.

Proof. Let S be any pfs, using Definition 3.4 we have $\mathcal{PFN}int(\theta cl(S)) \subseteq S \subseteq \theta\mathcal{S}cl(S) \subseteq \theta\mathcal{S}cl(S) \cup S \subseteq int(\theta cl(S)) \cup S \subseteq S$. Others are similar.

Theorem 3.1. S is a $\mathcal{PFN}Mos$ iff $S = \mathcal{PFN}\delta\mathcal{P}int(S) \cup \mathcal{PFN}\theta\mathcal{S}int(S)$.

Proof. Let S be a $\mathcal{PFN}Mos$. Then $S \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$. By Theorem 3.2, we have

$$\begin{aligned}\mathcal{PFN}\delta\mathcal{P}int(S) \cup \mathcal{PFN}\theta\mathcal{S}int(S) &= (S \cap (\mathcal{PFN}int(\mathcal{PFN}\delta cl(S)))) \cup (S \cap (\mathcal{PFN}cl(\mathcal{PFN}\theta \\ &\quad int(S)))) \\ &= S \cap (\mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \cup \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))) \\ &= S\end{aligned}$$

Conversely, if $S = \mathcal{PFN}\delta\mathcal{P}int(S) \cup \mathcal{PFN}\theta\mathcal{S}int(S)$ then, by Theorem 3.2

$$\begin{aligned} S &= \mathcal{PFN}\delta\mathcal{P}int(S) \cup \mathcal{PFN}\theta\mathcal{S}int(S) \\ &= (S \cap \mathcal{PFN}int(\mathcal{PFN}\delta cl(S))) \cup (S \cap \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))) \\ &= S \cap (\mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \cup \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))) \\ &\subseteq \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \cup \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)). \end{aligned}$$

and hence S is a $\mathcal{PFN}Mos$.

Theorem 3.2. S is a $\mathcal{PFN}Mcs$ iff $S = \mathcal{PFN}\delta\mathcal{P}cl(S) \cap \mathcal{PFN}\theta\mathcal{S}cl(S)$.

Proof. Obvious.

Theorem 3.3. The union (resp. intersection) of any family of $\mathcal{PFN}Mos$ (resp. $\mathcal{PFN}Mcs$) of U is a $\mathcal{PFN}Mos$ (resp. $\mathcal{PFN}Mcs$).

Proof. Let $\{S_a : a \in \tau_R(A)\}$ be a family of $\mathcal{PFN}Mos's$. For each $a \in \tau_R(A)$, $S_a \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S_a) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S_a))$.

$$\begin{aligned} \bigcup_{a\in\tau_R(A)} (S_a) &\subseteq \bigcup_{a\in\tau_R(A)} \mathcal{PFN}cl(\mathcal{PFN}\theta int(S_a)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S_a)) \\ &\subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(\cup(S_a))) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(\cup(S_a))). \end{aligned}$$

The other case is similar.

Remark 3.2. The intersection of two $\mathcal{PFN}Mos$'s need not be $\mathcal{PFN}Mos$.

Example 3.6. In Example 3.1, $B_1 = \left\{\left\langle \frac{s_1,s_4}{0.1,0.2}\right\rangle, \left\langle \frac{s_2}{0.5,0.1}\right\rangle, \left\langle \frac{s_3}{0.45,0.1}\right\rangle\right\}$ and $B_2 = \{\left\langle \frac{s_1,s_4}{0.2,0.4}\right\rangle,$ $\left\langle \frac{s_2}{0.7,0.1}\right\rangle, \left\langle \frac{s_3}{0.5,0.2}\right\rangle\}$ are $\mathcal{PFN}Mos$ but $B_1 \cap B_2$ is not $\mathcal{PFN}Mos$.

Proposition 3.3. If S is a

(i) $\mathcal{PFN}Mos$ and $\mathcal{PFN}\theta int(S) = 0_P$, then S is a $\mathcal{PFN}\delta\mathcal{P}os$.
(ii) $\mathcal{PFN}Mos$ and $\mathcal{PFN}\delta cl(S) = 0_P$, then S is a $\mathcal{PFN}\theta\mathcal{S}os$.
(iii) $\mathcal{PFN}Mos$ and $\mathcal{PFN}\delta cs$, then S is a $\mathcal{PFN}\theta\mathcal{S}os$.
(iv) $\mathcal{PFN}\theta\mathcal{S}os$ and $\mathcal{PFN}\delta cs$, then S is a $\mathcal{PFN}Mos$.

Proof.

(i) Let S be a $\mathcal{PFN}Mos$ and $\mathcal{PFN}\theta int(S) = 0_P$, that is

$$\begin{aligned} S &\subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \\ &= 0_P \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \\ &= \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)). \end{aligned}$$

Hence, S is a $\mathcal{PFN}\delta\mathcal{P}os$.

(ii) Let S be a $\mathcal{PFN}Mos$ and $\mathcal{PFN}\delta cs$, that is

$$\begin{aligned} S &\subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \\ &= \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)). \end{aligned}$$

Hence S is a $\mathcal{PFN}\theta\mathcal{S}os$.

(iii) Let S be a $\mathcal{PFN}Mos$ and $\mathcal{PFN}\delta cl(S) = 0_P$, that is

$$\begin{aligned} S &\subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)) \\ &= \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)). \end{aligned}$$

Hence S is a $\mathcal{PFN}\theta\mathcal{S}os$.

(iv) Let S be a $\mathcal{PFN}\theta\mathcal{S}os$ and $\mathcal{PFN}\delta cs$, that is

$$\begin{aligned} S &\subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \\ &\subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(S)). \end{aligned}$$

Hence S is a $\mathcal{PFN}Mos$.

Remark 3.3. The converse of the above Proposition 3.3 need not be true as shown in the following example.

Example 3.7. In Example 3.3, (i) $\underline{\mathcal{PFN}}(A)$ is $\mathcal{PFN}\delta\mathcal{P}o$ set but $\mathcal{PFN}\theta int(\underline{\mathcal{PFN}}(A)) \neq 0_P$, (ii) $\underline{\mathcal{PFN}}(A)$ is $\mathcal{PFN}\theta\mathcal{S}o$ set but $\mathcal{PFN}\delta cl(\underline{\mathcal{PFN}}(A)) \neq 0_P$, (iii) $\underline{\mathcal{PFN}}(A)$ is $\mathcal{PFN}\theta\mathcal{S}o$ set but not $\mathcal{PFN}\delta cs$, (iv) $\underline{\mathcal{PFN}}(A)$ is $\mathcal{PFN}Mo$ set but not $\mathcal{PFN}\delta cs$.

Theorem 3.4. S is a $\mathcal{PFN}Mcs$ (resp. $\mathcal{PFN}Mos$) iff $S = \mathcal{PFN}Mcl(S)$ (resp. $S = \mathcal{PFN}Mint(S)$).

Proof. Suppose $S = \mathcal{PFN}Mcl(S) = \bigcap\{T : S \subseteq T \;\&\; T \text{ is a } \mathcal{PFN}Mcs\}$. This means $S \in \bigcap\{T : S \subseteq T \;\&\; T \text{ is a } \mathcal{PFN}Mcs\}$ and hence S is $\mathcal{PFN}Mcs$.

Conversely, suppose S be a $\mathcal{PFN}Mcs$ in U. Then, we have $S \in \bigcap\{T : S \subseteq T \;\&\; T \text{ is a } \mathcal{PFN}Mcs\}$. Hence, $S \subseteq T$ implies $S = \bigcap\{T : S \subseteq T \;\&\; T \text{ is a } \mathcal{PFN}Mcs\} = \mathcal{PFN}Mcl(S)$. Hence $S = \mathcal{PFN}Mint(S)$.

Theorem 3.5. Let S and T in U, then the $\mathcal{PFN}Mcl$ have,

(i) $\mathcal{PFN}Mcl(0_P) = 0_P$, $\mathcal{PFN}Mcl(1_P) = 1_P$.
(ii) $\mathcal{PFN}Mcl(S)$ is a $\mathcal{PFN}Mcs$ in U.
(iii) $\mathcal{PFN}Mcl(S) \subseteq \mathcal{PFN}Mcl(T)$ if $S \subseteq T$.
(iv) $\mathcal{PFN}Mcl(\mathcal{PFN}Mcl(S)) = \mathcal{PFN}Mcl(S)$.

Proof. The proofs are directly from definition 3.4 of $\mathcal{PFN}Mc$ set.

Theorem 3.6. Let S and T in U, then the $\mathcal{PFN}Mint$ have,

(i) $\mathcal{PFN}Mint(0_P) = 0_P$, $\mathcal{PFN}Mint(1_P) = 1_P$.
(ii) $\mathcal{PFN}Mint(S)$ is a $\mathcal{PFN}Mos$ in U.
(iii) $\mathcal{PFN}Mint(S) \subseteq \mathcal{PFN}Mint(T)$ if $S \subseteq T$.
(iv) $\mathcal{PFN}Mint(\mathcal{PFN}Mint(S)) = \mathcal{PFN}Mint(S)$.

Proof. The proofs are directly from definition 3.4 of $\mathcal{PFN}Mo$ set.

Proposition 3.4. Let S and T are in U, then

(i) $\mathcal{PFN}Mcl(S^c) = [\mathcal{PFN}Mint(S)]^c$, $\mathcal{PFN}Mint(S^c) = [\mathcal{PFN}Mcl(S)]^c$.
(ii) $\mathcal{PFN}Mcl(S \cup T) \supseteq \mathcal{PFN}Mcl(S) \cup \mathcal{PFN}Mcl(T)$, $\mathcal{PFN}Mcl(S \cap T) \subseteq \mathcal{PFN}Mcl(S)$ $\cap \mathcal{PFN}Mcl(T)$.
(iii) $\mathcal{PFN}Mint(S \cup T) \supseteq \mathcal{PFN}Mint(S) \cup \mathcal{PFN}Mint(T)$, $\mathcal{PFN}Mint(S \cap T) \subseteq \mathcal{PFN}M$ $int(S) \cap \mathcal{PFN}Mint(T)$.

Proof.

(i) The proof is directly from definition 3.5.
(ii) $S \subseteq S \cup T$ or $T \subseteq S \cup T$. Hence, $\mathcal{PFN}Mcl(S) \subseteq \mathcal{PFN}Mcl(S \cup T)$ or $\mathcal{PFN}Mcl(T) \subseteq \mathcal{PFN}Mcl(S \cup T)$. Therefore, $\mathcal{PFN}Mcl(S \cup T) \supseteq \mathcal{PFN}Mcl(S) \cup \mathcal{PFN}Mcl(T)$. The other one is similar.
(iii) $S \subseteq S \cup T$ or $T \subseteq S \cup T$. Hence, $\mathcal{PFN}Mint(S) \subseteq \mathcal{PFN}Mint(S \cup T)$ or $\mathcal{PFN}Mint(T)$ $\subseteq \mathcal{PFN}Mint(S \cup T)$. Therefore, $\mathcal{PFN}Mint(S \cup T) \supseteq \mathcal{PFN}Mint(S) \cup \mathcal{PFN}Mint(T)$. The other one is similar.

Proposition 3.5. If S is in U, then

(i) $\mathcal{PFN}Mcl(S) \supseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S)) \cap \mathcal{PFN}int(\mathcal{PFN}\theta cl(S))$.

(ii) $\mathcal{PFN}Mint(S) \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta int(S)) \cup \mathcal{PFN}int(\delta cl(S))$.

Proof.

(i) $\mathcal{PFN}Mcl(S)$ is a $\mathcal{PFN}Mcs$ and $S \subseteq \mathcal{PFN}Mcl(S)$, then $\mathcal{PFN}Mcl(S) \supseteq \mathcal{PFN}cl$ $(\mathcal{PFN}\delta int\ (\mathcal{PFN}Mcl(S))) \cap \mathcal{PFN}int(\mathcal{PFN}\theta cl(\mathcal{PFN}Mcl(S)) \supseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int$ $(S)) \cap \mathcal{PFN}\ (\mathcal{PFN}\ \theta cl(S))$.

(ii) $\mathcal{PFN}Mint(S)$ is a $\mathcal{PFN}Mos$ and $S \supseteq \mathcal{PFN}Mint(S)$, then $\mathcal{PFN}Mint(S) \subseteq \mathcal{PFN}cl$ $(\mathcal{PFN}\theta int\ (\mathcal{PFN}Mint(S)) \cup \mathcal{PFN}int(\mathcal{PFN}\delta cl(\mathcal{PFN}Mint(S))) \subseteq \mathcal{PFN}cl(\mathcal{PFN}\theta$ $int(S)) \cup \mathcal{PFN}int\ (\mathcal{PFN}\delta cl(S))$.

Theorem 3.7. Let S be in U, then

(i) $\mathcal{PFN}Mcl(S) = \mathcal{PFN}\delta\mathcal{P}cl(S) \cap \mathcal{PFN}\theta\mathcal{S}cl(S)$,

(ii) $\mathcal{PFN}Mint(S) = \mathcal{PFN}\delta\mathcal{P}int(S) \cup \mathcal{PFN}\theta\mathcal{S}int(S)$.

Proof.

(i) It is obvious that, $\mathcal{PFN}Mcl(S) \subseteq \mathcal{PFN}\delta\mathcal{P}cl(S) \cap \mathcal{PFN}\theta\mathcal{S}cl(S)$. Conversely, from Definition 3.4, we have $\mathcal{PFN}\ Mcl(S) \supseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(\mathcal{PFN}Mcl(S))) \cap \mathcal{PFN}int$ $(\mathcal{PFN}\theta cl\ (\mathcal{PFN}Mcl(S))) \supseteq \mathcal{PFN}cl(\mathcal{PFN}\delta int(S)) \cap \mathcal{PFN}int\ (\mathcal{PFN}\theta cl(S))$. Since, $\mathcal{PFN}Mcl(S)$ is $\mathcal{PFN}Mcs$, by Theorem 3.6, we have $\mathcal{PFN}\delta\mathcal{P}cl(S) \cap \mathcal{PFN}\theta\mathcal{S}cl(S) =$ $S \cup \mathcal{PFN}cl(\mathcal{PFN}\ \delta int(S)) \cap (S \cup \mathcal{PFN}int(\mathcal{PFN}\theta cl(S))) = S \cup (\mathcal{PFN}cl(\mathcal{PFN}\delta int(S)) \cap$ $\mathcal{PFN}int(\mathcal{PFN}\theta cl(S))) = S \subseteq \mathcal{PFN}Mcl(S)$. Therefore, $\mathcal{PFN}Mcl(S) = \mathcal{PFN}\delta\mathcal{P}$ $cl(S) \cap \mathcal{PFN}\theta\mathcal{S}cl(S)$.

(ii) is similar from (i).

Theorem 3.8. Let S be in U. Then

(i) $\mathcal{PFN}Mcl(1_P - S) = 1_P - \mathcal{PFN}Mint(S)$,

(ii) $\mathcal{PFN}Mint(1_P - S) = 1_P - \mathcal{PFN}Mcl(S)$.

Proof.

(i) Let T be $\mathcal{PFN}Mcs$ in U and S be any set in U. Then $\mathcal{PFN}Mint(S) = \bigcup\{1_P - T :$ $1_P - T \subseteq S,\ 1_P - T$ is a $\mathcal{PFN}Mos$ in$U\} = 1_P - \bigcap\{T : T \supseteq 1_P - S,\ T$ is a $\mathcal{PFN}Mcs$ in $U\} = 1_P - \mathcal{PFN}Mcl(S)$. Thus, $\mathcal{PFN}Mcl(1_P - S) = 1_P - \mathcal{PFN}Mint(S)$.

(ii) Let T be $\mathcal{PFN}Mos$ in U and S be any set in U. Then $\mathcal{PFN}Mcl(S) = \bigcap\{1_P - T : 1_P -$ $T \supseteq S,\ 1_P - T$ is a $\mathcal{PFN}Mcs$ in $U\} = 1_P - \cup\{T : T \subseteq 1_P - S, T$ is a $\mathcal{PFN}Mos$ in $U\}$ $= 1_P - \mathcal{PFN}Mint(S)$. Thus, $\mathcal{PFN}Mint(1_P - S) = 1_P - \mathcal{PFN}Mcl(S)$.

Lemma 3.1. Let $(U, \tau_R(A))$ be $\mathcal{PFN}ts$ and S be a pfs on U. Then the following statements are hold.

(i) $\mathcal{PFN}\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S)) = \mathcal{PFN}\delta\mathcal{P}cl(S) \cap \mathcal{PFN}int(\mathcal{PFN}cl(S))$ and $\mathcal{PFN}\mathcal{P}cl(\mathcal{PF}$ $\mathcal{N}\delta\mathcal{P}int(S)) = \mathcal{PFN}\delta\mathcal{P}int(S) \cup \mathcal{PFN}cl(\mathcal{PFN}int(S))$.

(ii) $\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S)) = \mathcal{PFN}\delta\mathcal{P}cl(S) \cap \mathcal{PFN}int(\mathcal{PFN}\theta cl(S))$ and $\mathcal{PFN}\theta\mathcal{P}cl$ $(\mathcal{PFN}\delta\mathcal{P}int(S)) = \mathcal{PFN}\delta\mathcal{P}\ int(S) \cup \mathcal{PFN}cl(\mathcal{PFN}\theta int(S))$.

(iii) $\mathcal{PFN}\theta\mathcal{S}cl(\mathcal{PFN}\theta int(S)) = \mathcal{PFN}\mathcal{S}cl(\mathcal{PFN}\theta int(S)) = \mathcal{PFN}int(\mathcal{PFN}cl(\mathcal{PFN}\theta int$ $(S)))$.

Proof. Obvious.

Proposition 3.6. Let S be in U, then

(i) $\mathcal{PFN}Mcl(S) = S \cup \mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$.

(ii) $\mathcal{PFN}Mint(S) = S \cap \mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$.

Proof.

(i) By Lemma 3.1

$$\begin{aligned} S\cup\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S)) &= S\cup(\mathcal{PFN}\delta\mathcal{P}cl(S)\cap\mathcal{PFN}int(\mathcal{PFN}\theta cl(S))) \\ &= (S\cup\mathcal{PFN}\delta\mathcal{P}cl(S))\cap(S\cup\mathcal{PFN}int(\mathcal{PFN}\theta cl(S))) \\ &= \mathcal{PFN}\delta\mathcal{P}cl(S)\cap\mathcal{PFN}\theta Scl(S) \\ &= \mathcal{PFN}Mcl(S). \end{aligned}$$

(ii) Obvious.

Theorem 3.9. Let S be in U, then the following are equivalent.

(i) S is an $\mathcal{PFN}Mos$.
(ii) $S\subseteq\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$.
(iii) $\mathcal{PFN}\theta\mathcal{P}cl(S)=\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$.

Proof. (i) $\to$ (ii): Let S be an $\mathcal{PFN}Mos$. Then by Theorem 3.6, $S=\mathcal{PFN}Mint(S)$ and by Proposition 3.6, $S=S\cap\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$ and hence, $S\subseteq\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$.

(ii) $\to$ (i): Let $S\subseteq\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$. Then by Proposition 3.6, $S\subseteq S\cap\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))=\mathcal{PFN}Mint(S)$. So, $S\subseteq\mathcal{PFN}Mint(S)$ and hence, S is an $\mathcal{PFN}Mos$.

(ii) $\to$ (iii): Let $S\subseteq\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$. Then $\mathcal{PFN}\theta\mathcal{P}cl(S)\subseteq\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$ and hence, $\mathcal{PFN}\theta\mathcal{P}cl(S)=\mathcal{PFN}\theta\mathcal{P}cl(\mathcal{PFN}\delta\mathcal{P}int(S))$.

(iii) $\to$ (ii): Obvious

Theorem 3.10. Let S be in U, then the following are equivalent.

(i) S is an $\mathcal{PFN}Mcs$.
(ii) $\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))\subseteq S$.
(iii) $\mathcal{PFN}\theta\mathcal{P}int(S)=\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$.

Proof. (i) $\to$ (ii): Let S be an $\mathcal{PFN}Mcs$. Then by Theorem 3.5, $S=\mathcal{PFN}Mcl(S)$ and by Proposition 3.6, $S=S\cap\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$ and hence, $S\supseteq\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$.

(ii) $\to$ (i): Let $S\supseteq\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$. Then by Proposition 3.6, $S\supseteq S\cup\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))=\mathcal{PFN}Mcl(S)$. So, $S\supseteq\mathcal{PFN}Mcl(S)$ and hence, S is an $\mathcal{PFN}Mcs$.

(ii) $\to$ (iii): Let $S\supseteq\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$. Then $\mathcal{PFN}\theta\mathcal{P}int(S)\supseteq\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$ and hence, $\mathcal{PFN}\theta\mathcal{P}int(S)=\mathcal{PFN}\theta\mathcal{P}int(\mathcal{PFN}\delta\mathcal{P}cl(S))$.

(iii) $\to$ (ii): Obvious

4. Application

Entropy as a measure of fuzziness was first proposed by Zadeh [28]. Later many mathematicians defined several entropy measures. In this section, we focus on defining an entropy measure for pfs that connects the degree of membership and non-membership. Ss an example, we have applied the proposed entropy measure in the field of decision making.

Definition 4.1. Let $A=\{<x,\mu_A(x),\lambda_A(x)|x\in X\}$ be a pfs in U. The new entropy measure for A denoted by $\varepsilon_{pfs}(A)$, is a function, $\varepsilon_{pfs}:\tau_{pfs}(U)\to[0,1]$ and is defined as $\varepsilon_{pfs}(A)=1-\frac{1}{n}\sum_{i=1}^{n}(\mu_A-\lambda_A)^2$; for every'$x_i\in A$, where $\tau_{pfs}(U)$ denote the family of all pfs's on U.

Example 4.1. To select the most suitable educational institution for specific courses, we incorporate both qualitative and quantitative decision-making criteria. In particular, we consider alumni feedback as well as online ratings and rankings provided by various professional and educational organizations. These sources offer valuable insights into the performance and reputation of institutions from multiple perspectives.

In this study, we focus on five educational institutions, denoted as I1, I2, I3, I4 and I5, and evaluate them with respect to five different courses, labeled C1, C2, C3, C4 and C5. The online ratings for each institution in each course category are collected and then transformed into Pythagorean fuzzy sets, which allow for a more nuanced representation of uncertainty, hesitation, and partial truth associated with subjective evaluations.

To facilitate decision-making and identify the optimal institution for each course, we apply an entropy measure to assess the degree of fuzziness in the collected data. The entropy measure serves as a tool to quantify the uncertainty inherent in the evaluations and helps in selecting the institution that offers the highest clarity or confidence in terms of course quality. Ultimately, the institution with the minimum fuzziness across the evaluated criteria is identified as the most appropriate choice for each respective course.

Table 1. Ratings of the educational institutions based on the courses.

	Course 1 (C1)	Course 2 (C2)	Course 3 (C3)	Course 4 (C4)	Course 5 (C5)
I1	$< I1, C1; 0.4, 0.6 >$	$< I1, C2; 0.3, 0.2 >$	$< I1, C3; 0.1, 0.2 >$	$< I1, C4; 0.4, 0.3 >$	$< I1, C5; 0.1, 0.2 >$
I2	$< I2, C1; 0.7, 0.3 >$	$< I2, C2; 0.2, 0.2 >$	$< I2, C3; 0.0, 0.1 >$	$< I2, C4; 0.7, 0.3 >$	$< I2, C5; 0.1, 0.1 >$
I3	$< I3, C1; 0.3, 0.4 >$	$< I3, C2; 0.6, 0.3 >$	$< I3, C3; 0.2, 0.1 >$	$< I3, C4; 0.2, 0.2 >$	$< I3, C5; 0.1, 0.0 >$
I4	$< I4, C1; 0.1, 0.2 >$	$< I4, C2; 0.2, 0.4 >$	$< I4, C3; 0.8, 0.2 >$	$< I4, C4; 0.2, 0.1 >$	$< I4, C5; 0.2, 0.1 >$
I5	$< I5, C1; 0.1, 0.1 >$	$< I5, C2; 0.0, 0.2 >$	$< I5, C3; 0.2, 0.0 >$	$< I5, C4; 0.2, 0.0 >$	$< I5, C5; 0.8, 0.1 >$

Clearly, all values in the Table 1 are $\mathcal{PF}s$'s. Now we calculate the $\varepsilon_{\mathcal{PF}_s}$ of each value.

Table 2. Entropy measure of each institutions for the different courses.

	Course 1 (C1)	Course 2 (C2)	Course 3 (C3)	Course 4 (C4)	Course 5 (C5)
Institution 1 (I1)	**0.96**	0.99	0.99	0.99	0.99
Institution 2 (I2)	**0.84**	1	0.99	0.84	1
Institution 3 (I3)	0.99	**0.91**	0.99	1	0.99
Institution 4 (I4)	0.99	0.96	**0.64**	0.99	0.99
Institution 5 (I5)	1	0.96	0.96	0.96	**0.51**

From Table 2, it is clear that $\varepsilon_{\mathcal{PF}\mathfrak{N}s}(I2, C1) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I1, C1) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I3, C1) \leq \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I4, C1) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I5, C1)$

Similarly

$\varepsilon_{\mathcal{PF}\mathfrak{N}s}(I3, C2) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I4, C2) \leq \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I5, C2) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I1, C2) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I2, C2)$

$\varepsilon_{\mathcal{PF}\mathfrak{N}s}(I4, C3) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I5, C3) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I1, C3) \leq \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I2, C3) \leq \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I3, C3)$

$\varepsilon_{\mathcal{PF}\mathfrak{N}s}(I2, C4) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I5, C4) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I1, C4) \leq \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I4, C4) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I3, C4)$

$\varepsilon_{\mathcal{PF}\mathfrak{N}s}(I5, C5) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I1, C5) \leq \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I3, C5) \leq \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I4, C5) < \varepsilon_{\mathcal{PF}\mathfrak{N}s}(I2, C5)$.

It is clear that Institution 2 is best for the course C1 and C4, Institution 3 is best for the course C2, Institution 4 is best for the course C3 and Institution 5 is best for the course C5.

5. Conclusion

We have studied about Pythagorean fuzzy M-open set and Pythagorean fuzzy M-closed set and their respective interior and closure operators in Pythagorean fuzzy Topological Space in this paper. Also we have studied some of their fundamental properties along with examples in $\mathcal{PF}\mathfrak{N}ts$. Moreover, we have discussed about near open sets of Pythagorean fuzzy M-open sets in $\mathcal{PF}\mathfrak{N}ts$. In future, we can extend these results to Pythagorean fuzzy M- continuous mappings, Pythagorean fuzzy M-open mappings and Pythagorean fuzzy M-closed mappings in $\mathcal{PFN}ts$. We present ameasure of entropy and one application related to it. This measure is consistent with similar considerations for other sets like fuzzy sets and Pythagorean fuzzy sets etc. Hence the proposed entropy measure can be used to measure the uncertainty factor in related problems.

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