

APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX HYPERBOLIC SPACES

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ABSTRACT. This paper investigates the convergence of an iterative process to a fixed point of nonexpansive mappings in uniformly convex hyperbolic spaces. First, we analyze the iteration scheme introduced by Karakaya et al. for such mappings, establishing its key properties. Under specific conditions, we prove both Δ -convergence and strong convergence of the iteration to a fixed point. Additionally, we show that, if the iteration Δ -converges or strongly converges to a fixed point, then every subsequence exhibits the same behavior. These results extend the theory of iterative methods to uniformly convex hyperbolic spaces, broadening their applicability in nonlinear functional analysis.

Keywords: Nonexpansive mapping, uniformly convex, hyperbolic space, fixed point.

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1. INTRODUCTION

Fixed point theory is a cornerstone of nonlinear analysis. Since its foundational development in the mid-20th century, it has flourished through contributions from mathematicians such as Browder [1], Göhde [2], and Kirk [3]. This field addresses two fundamental questions: establishing the existence of fixed points and analyzing the convergence of iterative methods designed to approximate these points. The latter question has spurred extensive research, leading to the development of various iterative processes. Classical examples include the Mann [9] and Ishikawa [10] iterations, which rely on the inherent convexity of vector spaces. Agarwal et al. [12] introduced the S-iteration, which is independent of the Mann and Ishikawa methods, and was further developed by Gursoy [16]. Furthermore, other notable iterations include the Noor iteration, Agarwal iteration [12], Z -iteration [13], M -Iteration [14], and M^* -iteration [15].

Karakaya et al. [5] proposed a new multistep iteration process, demonstrating faster convergence than Mann, Ishikawa, and Picard iterations for weak contraction mappings in Banach spaces. Atalan and Karakaya [6] extended this iteration to hyperbolic spaces with quasi-contractive operators. However, the behavior of this iteration with nonexpansive mappings remains unexplored.

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Assume that for any $x, y \in X$, there exists a unique metric segment $[x, y]$, which is an isometric copy of the real interval $[0, \rho(x, y)]$. If for any $\alpha \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$\rho(x, z) = (1 - \alpha)\rho(x, y) \quad \text{and} \quad \rho(z, y) = \alpha\rho(x, y),$$

then (X, ρ) is said to be a convex metric space. Also, a subset C of a metric space is said to be convex if $[x, y] \subset C$ whenever $x, y \in C$.

A mapping $T : C \rightarrow C$ is said to be *contractive* if there exists a constant $k \in [0, 1)$ such that

$$\rho(T(x), T(y)) \leq k\rho(x, y), \quad \text{for all } x, y \in C. \quad (1)$$

If (1) is valid when $k = 1$, then T is said to be *nonexpansive*. It is clear that every contractive mapping is nonexpansive, but generally, the converse is not true.

A point $x \in X$ is a *fixed point* of T if $x = T(x)$. Moreover, x is called an *end point* (or *stationary point*) of T if $\{x\} = T(x)$. We denote the set of all fixed points and endpoints of T by $F(T)$ and $E(T)$, respectively.

The purpose of this paper is to investigate the convergence of the iteration introduced by Karakaya [5] in hyperbolic spaces endowed with nonexpansive mappings. First, we derive properties of the iteration sequence for nonexpansive mappings in uniformly convex hyperbolic spaces. By these properties, we show that the iteration Δ -converges and strongly converges to a fixed point under suitable conditions. Furthermore, a numerical example is given to illustrate how the iteration process works to find the solution of an equation.

2. PRELIMINARIES

Let (X, ρ) be a metric space and C be a nonempty subset of X . For any $x \in X$, the distance from x to C is defined by

$$\rho(x, C) = \inf\{\rho(x, y) : y \in C\}.$$

For any nonempty subsets $A, B \subseteq X$, the Hausdorff metric on X is defined by

$$H(A, B) = \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(A, b)\}.$$

If $x, y \in X$ and $\alpha \in [0, 1]$, then we use the notation $W(x, y, \alpha)$ for $(1 - \alpha)x + \alpha y$. Also, it is easy to see that

$$\rho(x, (1 - \alpha)x + \alpha y) = \alpha\rho(x, y) \quad \text{and} \quad \rho(y, (1 - \alpha)x + \alpha y) = (1 - \alpha)\rho(x, y). \quad (2)$$

The hyperbolic metric space was first introduced by Reich and Shafrir [8] in 1990. In 2004, Kohlenbach [7] defined the hyperbolic space as follows:

Definition 2.1. [7] *A (X, ρ, W) is called a hyperbolic space, if (X, ρ) is a metric space and $W : X \times X \times [0, 1] \rightarrow X$ is a function satisfying:*

- (i) $\rho(z, W(x, y, \alpha)) \leq (1 - \alpha)\rho(z, x) + \alpha\rho(z, y)$
- (ii) $\rho(W(x, y, \alpha), W(x, y, \tilde{\alpha})) = \|\alpha - \tilde{\alpha}\| \cdot \rho(x, y)$
- (iii) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$
- (iv) $\rho(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)\rho(z, y) + \alpha\rho(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \tilde{\alpha} \in [0, 1]$.

If a metric space (X, ρ) with mapping W only satisfies condition (i), then it is a convex metric space in the sense of Takashi [17]. If (X, ρ) satisfies conditions (i)-(iii), then it is a space of hyperbolic type in Goebel and Kirk [18]. The condition (iv) is applied in [8] to define the hyperbolic space class.

Banach spaces (and more generally, normed vector spaces) are examples of hyperbolic spaces. A subset of a hyperbolic space X is said to be convex if $\{(1 - \alpha)x + \alpha y\} \subseteq C$, whenever $x, y \in X$ and $\alpha \in [0, 1]$. Furthermore, a function $f : X \rightarrow \mathbb{R}$, is said to be convex if $f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y)$, for $\alpha \in [0, 1]$, and $x, y \in X$.

Definition 2.2. [24] *A hyperbolic space (X, ρ, W) is uniformly convex if for $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $a, x, y \in X$,*

$$\rho(W(x, y, \frac{1}{2}), a) \leq (1 - \delta)r, \tag{3}$$

provided $\rho(x, a) \leq r, \rho(y, a) \leq r$, and $\rho(x, y) \geq \varepsilon r$.

For more general, we have the following definition.

Definition 2.3. [24] *Let (X, ρ, W) be hyperbolic space and $2 \leq p < \infty$. X is said to be p -uniformly convex if there exists a constant $C_X > 0$ such that for every $\alpha \in [0, 1]$ and $x, y, z \in X$, we have*

$$\rho^p(W(x, y, \alpha), z) \leq (1 - \alpha)\rho^p(x, z) + \alpha\rho^p(y, z) - C_X^p\alpha(1 - \alpha)\rho^p(x, y). \tag{4}$$

We say X is uniformly convex if it is p -uniformly convex for some $p \geq 2$.

It is easy to see that for any $x \in X$, the function $f : X \rightarrow \mathbb{R}$ defined by $f(x) = \rho^p(x, y)$ for fixed $y \in X$ is convex.

A mapping $\eta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called a modulus of uniform convexity.

The iteration introduced by Karakaya et al. [5] is given as follow:

$$\begin{aligned} x_o &\in C \\ x_{n+1} &= Ty_n \\ y_n &= (1 - \alpha_n)z_n + \alpha_nTz_n \\ z_n &= Tx_n \end{aligned}$$

where α_n is a sequence in $[0, 1]$.

In hyperbolic space, as we apply in this paper, we express this iteration as:

$$\begin{aligned} x_o &\in C \\ x_{n+1} &= Ty_n \\ y_n &= W(z_n, Tz_n, \alpha_n) \\ z_n &= Tx_n \end{aligned} \tag{5}$$

where $\alpha_n \in [0, 1]$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Definition 2.4. [19] *Let (X, ρ) be a complete hyperbolic metric space and $\{x_n\}$ be a bounded sequence in X . Then, the type function $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ is defined by*

$$r(x) = \limsup_{n \rightarrow \infty} \rho(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(X, \{x_n\}) = \inf r(x) : \text{for } x \in X$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as

$$A(\{x_n\}) = \{x \in X : r(x) = r(\{x_n\})\}.$$

A bounded sequence x_n in X is said to be Δ -converges to $x \in X$ if x is unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$ (We write as $x_n \xrightarrow{\Delta} x$).

Lemma 2.1. [20] Let (X, ρ, W) be a uniformly convex hyperbolic space with a monotone modulus of uniform convexity η . For $r > 0, \varepsilon \in (0, 2], \alpha \in [0, 1]$, and $x, y, z \in X$, the inequalities

$$\rho(x, z) \leq r, \rho(y, z) \leq r, \quad \text{and} \quad \rho(x, y) \geq \varepsilon r \quad (6)$$

imply

$$\rho(W(x, y, \alpha), z) \leq (1 - 2\alpha(1 - \alpha)\eta(r, \varepsilon))r. \quad (7)$$

Lemma 2.2. [4] Let (X, ρ, W) be a uniformly convex hyperbolic space with a monotone modulus of uniform convexity η . Let $x \in X$ and a_n be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in x such that $\limsup_{n \rightarrow \infty} \rho(x_n, x) \leq r, \limsup_{n \rightarrow \infty} \rho(y_n, x) \leq r$, and $\lim_{n \rightarrow \infty} \rho(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Lemma 2.3. [20] Let (X, ρ, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η . Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any convex subset C of X .

Lemma 2.4. [23] Let α_n be a real sequence such that $\alpha_n \in [0, 1]$ and $\sum \alpha_n(1 - \alpha_n) = \infty$. Let β_n be a nonnegative sequence of real numbers such that $\sum \alpha_n(1 - \alpha_n)\beta_n$ is bounded. Then, β_n has a subsequence which converges to zero.

Lemma 2.5. [25] Let (X, ρ, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = p$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{\rho(x_n, u)\}$ is convergent, then we have $p = u$.

Theorem 2.1. [21] Let C be a nonempty closed subset of a complete 2-uniformly convex hyperbolic metric space and $\{x_n\}$ be a Fejer monotone sequence with respect to C . Then, $\{x_n\}$ converges strongly to an endpoint of C if and only if $\lim_{n \rightarrow \infty} \rho(x_n, E(T)) = 0$.

Definition 2.5. [26] Suppose that (X, ρ, W) is a hyperbolic space and C is a nonempty closed convex subset of X . A mapping $T : X \rightarrow X$

- Satisfies condition I if $F(T) \neq \emptyset$ and there exists a nondecreasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that φ vanishes only at zero and

$$h(\text{dist}(x, F(T))) \leq \rho(x, Tx) \quad \text{for all } x \in C;$$

- is strongly demiclosed (Δ - demiclosed) if $q \in F(T)$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n \rho(x, Tx) = 0$ and $\lim_n x_n = q \in X$ (Δ - $\lim_n x_n = q$, respectively).

3. MAIN RESULTS

In this section, we investigate the properties of the sequences given by iteration (5) in hyperbolic space. In this paper, we denote the sets of real numbers and natural numbers by \mathbb{R} and \mathbb{N} , respectively. We begin this section with the following lemma.

Lemma 3.1. *Let (X, ρ, W) be a uniformly convex hyperbolic space and C be a bounded, closed, and convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by iteration (5). Then*

- i) $\rho(x_n, x_{n+1}) \leq 3\rho(x_n, Tx_n)$
- ii) $\rho(y_n, y_{n+1}) \leq (5 - 2\alpha_n\alpha_{n+1})\rho(x_n, Tx_n)$.
- iii) $\rho(z_n, z_{n+1}) \leq 3\rho(x_n, Tx_n)$

for all $n \in \mathbb{N}$.

Proof. We will start showing (i) and (iii) all together since they have the same equality.

$$\begin{aligned} \rho(x_n, x_{n+1}) &= \rho(x_n, Ty_n) \\ &\leq \rho(x_n, z_n) + \rho(z_n, Ty_n) \\ &\leq \rho(x_n, Tx_n) + \rho(x_n, y_n) \\ &\leq \rho(x_n, Tx_n) + (1 - \alpha_n)\rho(x_n, z_n) + \alpha_n\rho(x_n, Tz_n) \\ &\leq \alpha_n\rho(x_n, Tx_n) + \alpha_n(\rho(x_n, Tx_n) + \rho(Tx_n, Tz_n)) \\ &\leq 3\rho(x_n, Tx_n). \end{aligned}$$

Part (iii) of this theorem follows this since $\rho(z_n, z_{n+1}) = \rho(Tx_n, Tx_{n+1}) \leq \rho(x_n, x_{n+1})$.

Next, we will see part (ii). By Definition 2.1, we have

$$\rho(y_n, y_{n+1}) \leq (1 - \alpha_n)\rho(z_n, y_{n+1}) + \alpha_n\rho(Tz_n, y_{n+1}). \quad (8)$$

On the first term of inequality (8), we have

$$\begin{aligned} \rho(z_n, y_{n+1}) &= \rho(Tx_n, (1 - \alpha_{n+1})z_{n+1} + \alpha_{n+1}Tz_{n+1}) \\ &\leq (1 - \alpha_{n+1})\rho(Tx_n, Tx_{n+1}) + \alpha_{n+1}\rho(Tx_n, Tz_{n+1}) \\ &\leq (1 - \alpha_{n+1})\rho(x_n, x_{n+1}) + \alpha_{n+1}\rho(x_n, z_{n+1}) \\ &\leq (1 - \alpha_{n+1})\rho(x_n, x_{n+1}) + \alpha_{n+1}[\rho(x_n, z_n) + \rho(z_n, z_{n+1})] \\ &\leq (1 - \alpha_{n+1})\rho(x_n, x_{n+1}) + \alpha_{n+1}[\rho(x_n, Tx_n) + \rho(x_n, x_{n+1})] \\ &= \rho(x_n, x_{n+1}) + \alpha_{n+1}\rho(x_n, Tx_n) \\ &\leq (3 + \alpha_{n+1})\rho(x_n, Tx_n). \end{aligned}$$

From the second term of the inequality (8) we have

$$\begin{aligned} \rho(Tz_n, y_{n+1}) &= \rho(Tz_n, (1 - \alpha_{n+1})z_{n+1} + \alpha_{n+1}Tz_{n+1}) \\ &\leq (1 - \alpha_{n+1})\rho(Tz_n, Tx_{n+1}) + \alpha_{n+1}\rho(Tz_n, Tz_{n+1}) \\ &\leq (1 - \alpha_{n+1})\rho(z_n, x_{n+1}) + \alpha_{n+1}\rho(z_n, z_{n+1}) \\ &\leq (1 - \alpha_{n+1})[\rho(z_n, x_n) + \rho(x_n, x_{n+1})] + \alpha_{n+1}\rho(z_n, z_{n+1}) \\ &\leq (1 - \alpha_{n+1})[\rho(x_n, Tx_n) + \rho(x_n, x_{n+1})] + \alpha_{n+1}\rho(x_n, x_{n+1}) \\ &= (1 - \alpha_{n+1})\rho(x_n, Tx_n) + \rho(x_n, x_{n+1}) \\ &\leq (4 - \alpha_{n+1})\rho(x_n, Tx_n). \end{aligned}$$

As a consequence, we have

$$\begin{aligned}\rho(y_n, y_{n+1}) &\leq (1 - \alpha_n)(3 + \alpha_{n+1})\rho(x_n, Tx_n) + \alpha_n(4 - \alpha_{n+1})\rho(x_n, Tx_n) \\ &= (3 + \alpha_n + \alpha_{n+1} - 2\alpha_n\alpha_{n+1})\rho(x_n, Tx_n) \\ &\leq (5 - 2\alpha_n\alpha_{n+1})\rho(x_n, Tx_n).\end{aligned}$$

□

Theorem 3.1. *Let (X, ρ, W) be a convex hyperbolic space and C be a bounded, closed, and convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $\{x_n\}$ is the sequence defined by iteration (5). Then $\lim_{n \rightarrow \infty} \rho(x_n, Tx_n)$ exist as $n \rightarrow \infty$*

Proof. Let x_n be the sequence as defined by iteration (5). Then

$$\begin{aligned}\rho(x_{n+1}, Tx_{n+1}) &= \rho(Ty_n, Tx_{n+1}) \\ &\leq \rho(y_n, x_{n+1}) \\ &= \rho((1 - \alpha_n)z_n + \alpha_n Tz_n, Ty_n) \\ &\leq (1 - \alpha_n)\rho(z_n, Ty_n) + \alpha_n\rho(Tz_n, Ty_n) \\ &\leq (1 - \alpha_n)\rho(x_n, y_n) + \alpha_n\rho(z_n, y_n) \\ &\leq (1 - \alpha_n)[\rho(x_n, z_n) + \rho(z_n, y_n)] + \alpha_n\rho(z_n, y_n) \\ &= (1 - \alpha_n)\rho(x_n, z_n) + \rho(z_n, y_n) \\ &= (1 - \alpha_n)\rho(x_n, z_n) + \rho(z_n, (1 - \alpha_n)z_n + \alpha_n Tz_n) \\ &\leq (1 - \alpha_n)\rho(x_n, z_n) + \alpha_n\rho(z_n, Tz_n) \\ &= (1 - \alpha_n)\rho(x_n, z_n) + \alpha_n\rho(Tx_n, Tz_n) \\ &= (1 - \alpha_n)\rho(x_n, z_n) + \alpha_n\rho(x_n, z_n) \\ &= (1 - \alpha_n)\rho(x_n, z_n) + \alpha_n\rho(x_n, Tx_n) \\ &= \rho(x_n, Tx_n).\end{aligned}$$

Since $\rho(x_n, Tx_n)$ is bounded below and decreasing, hence $\lim_{n \rightarrow \infty} \rho(x_n, Tx_n)$ exists as $n \rightarrow \infty$. □

From this theorem, we have the following corollary.

Corollary 3.1. *Let (X, ρ, W) be a uniformly convex hyperbolic space and C be a bounded, closed, and convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by iteration (5). Then the following statements hold.*

- i) $\rho(x_n, x_{n+k}) \leq 3(k - 1)\rho(x_n, Tx_n)$
- ii) $\rho(y_n, y_{n+k}) \leq \sum_{i=0}^{k-1} (5 - \alpha_{n+i}\alpha_{n+i+1})\rho(x_n, Tx_n)$
- iii) $\rho(z_n, z_{n+k}) \leq 3(k - 1)\rho(x_n, Tx_n)$

for $k \in \mathbb{N}$.

Proof. From the Lemma 3.1 and Theorem 3.2, we have

$$\rho(x_n, x_{n+k}) \leq \sum_{i=0}^{k-1} \rho(x_{n+k}, x_{n+k+1}) \leq 3(k - 1)\rho(x_n, Tx_n).$$

The statement (iii) follows the part (i). Next, we have

$$\rho(y_n, y_{n+k}) \leq \sum_{i=0}^{k-1} \rho(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{k-1} (5 - \alpha_{n+i}\alpha_{n+i+1})\rho(x_n, Tx_n).$$

□

Lemma 3.2. *Let (X, ρ, W) be a uniformly convex hyperbolic space and C be a bounded, closed, and convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by iteration (5). Then $\lim_{n \rightarrow \infty} \rho(x_n, p)$ exists for every $p \in F(T)$ and $n \in \mathbf{N}$*

Proof. Let $p \in F(T)$. Then

$$\begin{aligned}
 \rho(x_{n+1}, p) &= \rho(Ty_n, p) \\
 &= \rho(Ty_n, Tp) \\
 &\leq \rho(y_n, p) \\
 &= \rho((1 - \alpha_n)z_n + \alpha_n Tz_n, p) \\
 &\leq (1 - \alpha_n)\rho(z_n, p) + \alpha_n\rho(Tz_n, p) \\
 &= (1 - \alpha_n)\rho(Tx_n, Tp) + \alpha_n\rho(Tz_n, Tp) \\
 &\leq (1 - \alpha_n)\rho(x_n, p) + \alpha_n\rho(z_n, p) \\
 &= (1 - \alpha_n)\rho(x_n, p) + \alpha_n\rho(Tx_n, Tp) \\
 &\leq (1 - \alpha_n)\rho(x_n, p) + \alpha_n\rho(x_n, p) \\
 &= \rho(x_n, p).
 \end{aligned} \tag{9}$$

As $\rho(x_n, p)$ is decreasing and bounded, $\lim_{n \rightarrow \infty} \rho(x_n, p)$ exists for every $p \in F(T)$ and $n \in \mathbf{N}$. □

Lemma 3.3. *Let (X, ρ, W) be a uniformly convex hyperbolic space and C be a bounded, closed, and convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by iteration (5). Then*

$$\lim_{n \rightarrow \infty} \rho(x_n, p) = \lim_{n \rightarrow \infty} \rho(y_n, p) = \lim_{n \rightarrow \infty} \rho(z_n, p) \tag{10}$$

Proof. From Lemma 3.2, we see that $\lim_{n \rightarrow \infty} \rho(x_n, p)$ exists for $p \in F(T)$. We suppose that $\lim_{n \rightarrow \infty} \rho(x_n, p) = c$ for some $c \geq 0$.

Note that

$$\limsup_{n \rightarrow \infty} \rho(z_n, p) = \limsup_{n \rightarrow \infty} \rho(Tx_n, Tp) \leq \limsup_{n \rightarrow \infty} \rho(x_n, p) = c.$$

On the other hand, we also have

$$\rho(x_{n+1}, p) \leq \rho(y_n, p) \leq \rho(x_n, p). \tag{11}$$

Taking the $\limsup_{n \rightarrow \infty}$ of the equation above yields

$$\limsup_{n \rightarrow \infty} \rho(y_n, p) = \limsup_{n \rightarrow \infty} \rho(x_n, p) = c. \tag{12}$$

Also, from the proof of Lemma 3.2, we have

$$\rho(x_{n+1}, p) \leq (1 - \alpha_n)\rho(x_n, p) + \alpha_n\rho(z_n, p),$$

which implies

$$\frac{\rho(x_{n+1}, p) - (1 - \alpha_n)\rho(x_n, p)}{\alpha_n} \leq \rho(z_n, p).$$

As a consequence, taking $\liminf_{n \rightarrow \infty}$ for the both sides yields

$$c \leq \liminf_{n \rightarrow \infty} \rho(z_n, p). \tag{13}$$

It follows that

$$\lim_{n \rightarrow \infty} \rho(z_n, p) = c. \quad (14)$$

This completes the proof. \square

The theorem above implies the following corollary.

Theorem 3.2. *Let (X, ρ, W) be a uniformly convex hyperbolic space and C be a bounded, closed, and convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by iteration (5). Then*

$$\lim_{n \rightarrow \infty} \rho(x_n, Tx_n) = 0. \quad (15)$$

Proof. Taking the $n \rightarrow \infty$ of (5) gives

$$\lim_{n \rightarrow \infty} \rho(y_n, p) \leq \lim_{n \rightarrow \infty} (1 - \alpha_n)\rho(x_n, p) + \alpha_n\rho(z_n, p) \leq \lim_{n \rightarrow \infty} \rho(x_n, p)$$

Applying Lemma 3.3 yields

$$\lim_{n \rightarrow \infty} \rho(W(x_n, z_n, \alpha_n), p) = \lim_{n \rightarrow \infty} (1 - \alpha_n)\rho(x_n, p) + \alpha_n\rho(z_n, p) = c$$

As a consequence, by Lemma 2.2, we have

$$\lim_{n \rightarrow \infty} \rho(x_n, z_n) = 0. \quad (16)$$

It implies that

$$\lim_{n \rightarrow \infty} \rho(x_n, Tx_n) = \lim_{n \rightarrow \infty} \rho(x_n, z_n) = 0. \quad (17)$$

\square

This theorem results the following corollary.

Corollary 3.2. *Let (X, ρ, W) be a uniformly convex hyperbolic space and C be a bounded, closed, and convex subset of X . Let $T : C \rightarrow C$ be a nonexpansive mapping. Assume that $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by iteration (5). Then*

$$\lim_{n \rightarrow \infty} \rho(y_n, Ty_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \rho(z_n, Tz_n) = 0 \quad (18)$$

Proof. It is clear that

$$\lim_{n \rightarrow \infty} \rho(z_n, Tz_n) \leq \lim_{n \rightarrow \infty} \rho(x_n, Tx_n) = 0.$$

Also, we obtain

$$\begin{aligned} \rho(y_n, Ty_n) &= \rho((1 - \alpha_n)z_n + \alpha_nTz_n, Ty_n) \\ &\leq (1 - \alpha_n)\rho(z_n, Ty_n) + \alpha_n\rho(Tz_n, Ty_n) \\ &\leq (1 - \alpha_n)\rho(x_n, y_n) + \alpha_n\rho(z_n, y_n) \\ &\leq (1 - \alpha_n)[\rho(x_n, z_n) + \rho(z_n, y_n)] + \alpha_n\rho(y_n, z_n) \\ &= (1 - \alpha_n)\rho(x_n, z_n) + \rho(z_n, y_n) \\ &\leq (1 - \alpha_n)\rho(x_n, z_n) + \rho(z_n, (1 - \alpha_n)z_n + \alpha_nTz_n) \\ &\leq (1 - \alpha_n)\rho(x_n, z_n) + \alpha_n\rho(z_n, Tz_n) \\ &\leq (1 - \alpha_n)\rho(x_n, Tx_n) + \alpha_n\rho(x_n, Tx_n) \\ &= \rho(x_n, Tx_n). \end{aligned}$$

As a consequence,

$$\lim_{n \rightarrow \infty} \rho(y_n, Ty_n) \leq \lim_{n \rightarrow \infty} \rho(x_n, Tx_n) = 0.$$

□

Theorem 3.3. *Let (X, ρ, W) be a complete uniformly convex hyperbolic space and C be a nonempty bounded, closed, and convex subset of X . Assume that $T : C \rightarrow C$ is a nonexpansive mapping which is Δ -demiclosed with $F(T) \neq \emptyset$. Let x_n be the sequence in C defined by iteration (5). Then $\{x_n\}$ Δ -converges to a fixed point of T .*

Proof. Suppose that $A(\{x_n\}) = \{p\}$, where $p \in X$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Since $\{x_n\}$ is bounded, by Lemma 2.3, there exists unique $q \in X$ such that $x_{n_k} \xrightarrow{\Delta} q$. Since T is Δ -demiclosed and $\lim_{k \rightarrow \infty} \rho(x_{n_k}, Tx_{n_k}) = 0$, then $q \in F(T)$. Note that $\lim_{n \rightarrow \infty} \rho(x_n, q)$ exists for $q \in F(T)$ by Lemma 3.2. Next, to complete the proof, we show that $p = q$. Since $\{x_{n_k}\}$ is bounded, there exist a subsequence $\{y_k\}$ of $\{x_{n_k}\}$ such that $\{y_k\} \xrightarrow{\Delta} q'$ for some $q' \in X$. If $q \neq q'$, then $\limsup_{k \rightarrow \infty} \rho(x_{n_k}, q) < \lim_{n \rightarrow \infty} \rho(x_n, q') = \lim_{k \rightarrow \infty} \rho(y_k, q') < \limsup_{k \rightarrow \infty} \rho(y_k, q) \leq \limsup_{k \rightarrow \infty} \rho(x_{n_k}, q)$. If $p \neq q'$, then $\limsup_{n \rightarrow \infty} \rho(x_n, p) < \lim_{n \rightarrow \infty} \rho(x_n, q') = \lim_{k \rightarrow \infty} \rho(y_k, q') < \limsup_{k \rightarrow \infty} \rho(y_k, p) \leq \limsup_{n \rightarrow \infty} \rho(x_n, p)$ which is impossible. As a consequence, we have $p = q$ and this completes the proof.

□

From this theorem, as a consequence of Lemma 3.3, we have the following corollary.

Corollary 3.3. *Let (X, ρ, W) be a complete uniformly convex hyperbolic space and C be a nonempty bounded, closed, and convex subset of X . Assume that $T : C \rightarrow C$ is a nonexpansive mapping which is Δ -demiclosed with $F(T) \neq \emptyset$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ defined by iteration (5) are Δ -converge to a fixed point of T .*

Theorem 3.4. *Let (X, ρ, W) be a complete uniformly convex hyperbolic space and C be a nonempty closed convex subset of X . Assume that $T : C \rightarrow C$ is a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in X defined by iteration (5) with $\alpha_n \in [0, 1]$. $\{x_n\}$ converges strongly to an endpoint of T if one of the following conditions is satisfied:*

- i) T satisfies condition I
- ii) T is strongly demiclosed.

Proof. It follows from Theorem 3.3 that $\lim_n T(x_n, Tx_n) = 0$,

If T satisfies the condition I, by Theorem 3.3, we have

$$\text{dist}(x_n, F(T)) \leq \rho(x_n, Tx_n), \quad \text{where } n \geq 1.$$

As $\lim_n \rho(x_n, Tx_n) = 0$, we have $\text{dist}(x_n, F(T)) = 0$. Therefore, we get that x_n converges to a point in $F(T)$.

If $\{x_n\}$ is demiclosed, we $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$. Since $\{x_{n_j}\}$ is bounded, $x_{n_j} \xrightarrow{P} p$ for some $p \in C$. As $\lim_n(x_n, p)$ exists and $\lim_j \rho(x_{n_j}, p) = 0$, we conclude that $\lim_n x_n = p$. By demiclosedness of T and $\lim_n \rho(x_n, Tx_n) = 0$ that $p \in F(T)$. □

From this theorem, as a consequence of Lemma 3.3, we have the following corollary.

Corollary 3.4. *Let (X, ρ, W) be a complete uniformly convex hyperbolic space and C be a nonempty closed convex subset of X . Assume that $T : C \rightarrow C$ is a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in X as defined by iteration (5) with $\alpha_n \in [0, 1]$. If one of the following conditions is satisfied:*

- i) T satisfies condition I
- ii) T is strongly demiclosed,

the sequences $\{y_n\}$ and $\{z_n\}$ as defined by iteration (5) strongly converge to a fixed point of T .

To support the findings, we provide the following numerical example.

Example 3.1. Let $X = \mathbb{R}$ and $C = [2, 4]$. Let $T : C \rightarrow C$ be a mapping defined by $T(x) = 1 + \sqrt{x} + \sin \frac{x}{25}$. It is easy to show that T is a nonexpansive mapping with a fixed point $p = 2.77743171643522$. Choose $\alpha_n = \beta_n = \frac{1}{n}$, with the initial value $x_0 = 2$. Therefore, we have

- i) Mann iteration

$$x_{n+1} = Tx_n;$$

- ii) Ishikawa iteration

$$\begin{aligned} x_{n+1} &= \frac{1}{n}Ty_n + \left(\frac{n-1}{n}\right)Tx_n \\ y_n &= \left(\frac{n-1}{n}\right)x_n + \frac{1}{n}Tx_n; \end{aligned}$$

- iii) Pichard S-Iteration

$$\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= \left(\frac{n-1}{n}\right)Tx_n + \frac{1}{n}Tz_n \\ z_n &= \left(\frac{n-1}{n}\right)x_n + \frac{1}{n}Tx_n; \end{aligned}$$

- iv) Karakaya Iteration

$$\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= \left(\frac{n-1}{n}\right)z_n + \frac{1}{n}Tz_n \\ z_n &= Tx_n; \end{aligned}$$

- v) Noor Iteration

$$\begin{aligned} x_1 &\in X \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)x_n + \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n; \end{aligned}$$

- vi) Agarwal Iteration

$$\begin{aligned} x_1 &\in X \\ x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n; \end{aligned}$$

vii) *Z-Iteration*

$$\begin{aligned}
 x_1 &\in X \\
 x_{n+1} &= Tz_n \\
 z_n &= (1 - \alpha_n)y_n + \alpha_nTy_n \\
 y_n &= (1 - \beta_n)x_n + \beta_nTx_n;
 \end{aligned}$$

viii) *M-Iteration*

$$\begin{aligned}
 x_1 &\in X \\
 x_{n+1} &= T((1 - \alpha_n)Ty_n + \alpha_ny_n) \\
 y_n &= T((1 - \beta_n)z_n + \beta_nTz_n) \\
 z_n &= T((1 - \gamma_n)x_n + \gamma_nTx_n);
 \end{aligned}$$

The following table compares the convergence of the iteration of the nonexpansive mapping T

TABLE 1. Comparison rate of convergence among Mann, Ishikawa, Pichard-s, and Karakaya et al.

x_n	Mann	Ishikawa	Pichard-s	Karakaya et al.
x_1	2	2	2	2
x_2	2.49412825634227	2.67888064767166	2.85457528945787	2.74367892192976
x_3	2.67888064767166	2.71685789704066	2.87895580339568	2.77481215369389
\vdots	\vdots	\vdots	\vdots	\vdots
x_{14}	2.77743102146650	2.76045446778587	2.80012320132297	2.77743171643518
x_{15}	2.77743148030423	2.76127400955341	2.79856536372570	2.77743171643522
x_{16}	2.77743163620450	2.76200086063692	2.79720731868539	2.77743171643522
x_{17}	2.777431689175060	2.762650744741310	2.79601301313625	2.77743171643522
\vdots	\vdots	\vdots	\vdots	\vdots
x_{32}	2.77743171643522	2.76790967161467	2.78717745452064	2.77743171643522
x_{33}	2.77743171643522	2.76810814302114	2.78687789959859	2.77743171643522
x_{34}	2.77743171643522	2.76829653008458	2.78659620476692	2.77743171643522
x_{35}	2.77743171643522	2.76847562941954	2.78633081946415	2.77743171643522
\vdots	\vdots	\vdots	\vdots	\vdots
x_{2000}	2.77743171643522	2.77758521126543	2.77744566616279	2.77743171643522

Table 1 shows that iteration method by Karakaya et al. reach the fixed point slower than Z-iteration but faster than any other iteration methods . The following are graphical presentation of the above results:

TABLE 2. Comparison rate of convergence among Noor, Agarwal, Z, and M iteration

x_n	Noor	Agarwal	Z-Iteration	M-Iteration
x_1	2	2	2	2
x_2	2.30318585348554	2.54145545097579	2.73387766309634	2.80012109942148
x_3	2.48927816381421	2.70170685596896	2.77516641962731	2.81000010831355
\vdots	\vdots	\vdots	\vdots	\vdots
x_{14}	2.69676814678025	2.77743125525611	2.77743171643521	2.78048125215594
x_{15}	2.70065487614469	2.77743156019922	2.77743171643522	2.78023821646454
x_{16}	2.70410243312391	2.77743166348749	2.77743171643522	2.78003071605244
x_{17}	2.70718528459846	2.77743169848616	2.77743171643522	2.77985153884344
\vdots	\vdots	\vdots	\vdots	\vdots
x_{32}	2.73214702099954	2.77743171643522	2.77743171643522	2.77861787338840
x_{33}	2.73308967005399	2.77743171643522	2.77743171643522	2.77857877633490
x_{34}	2.73398446718862	2.77743171643522	2.77743171643522	2.77854216794945
x_{35}	2.73483518983521	2.77743171643522	2.77743171643522	2.77850781845043
\vdots	\vdots	\vdots	\vdots	\vdots
x_{2000}	2.77454429615162	2.77743171643522	2.77743171643522	2.77744908631892

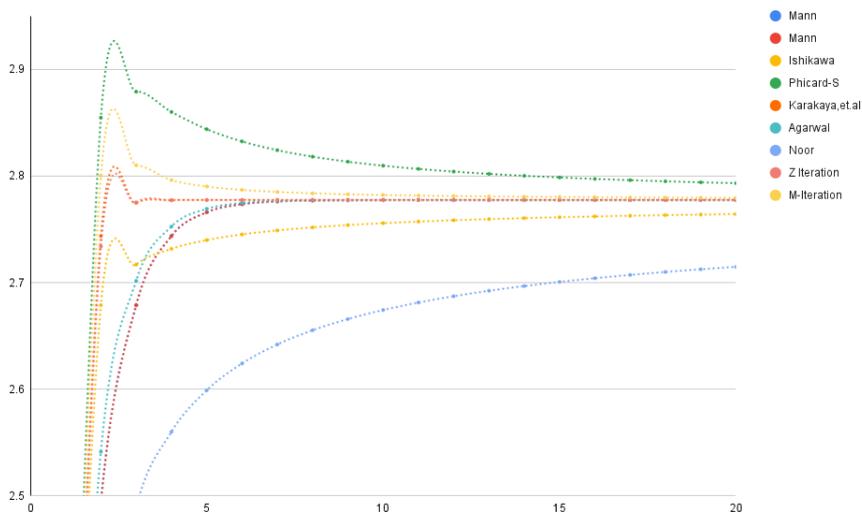


FIGURE 1. Comparison of rate of convergence among iteration methods

4. CONCLUSION

In this work, we show the convergence theorem to a fixed point of nonexpansive mappings in uniformly convex hyperbolic spaces. If C is a nonempty bounded, closed, and convex subset of a complete uniformly convex hyperbolic space X and $T : C \rightarrow C$ is a nonexpansive mapping which is Δ -demiclosed with $F(T) \neq \emptyset$, then the sequences $\{x_n\}$ in C defined by iteration (5) Δ -converges to a fixed point of T . In addition, $\{x_n\}$ strongly converges to $E(T)$ if T satisfies condition I or is strongly demiclosed.

REFERENCES

- [1] Browder, F. E., (1965), Nonexpansive nolinear operators in a Banach space, Proc. Natl. Acad. Sci. USA, 54(4), pp. 1041–1044.
- [2] Göhde, D., (1965), Zum Prinzip der kontraktiven abildung, Math. Nachr., 30(3-4), pp. 251–258.
- [3] Kirk, W. A., (1965), A Fixed point theorem for mapping which do not increase distance, Amer. Math. Monthly, 72(9), pp. 1004–1006.
- [4] Khan, A. R., Fukharuddin, H., and Khan, M. A. A., (2012), An implicit algorithm for two finite families of nonexpansive maps in hyperbolic space, J. Fixed Point Theory Appl., 54. <https://doi.org/10.1186/1687-1812-2012-54>.
- [5] Karakaya, V., Atalan, Y., Dogan, K., and Bouzara, N. E. H., (2017), Some fixed point results for a new three steps iteration process in Banach spaces, Fixed Point Theory, 18, pp. 625–640.
- [6] Atalan, Y. and Karakaya, V., (2019), Investigation of some fixed point theorems in hyperbolic spaces for a three steps iteration process, Korean J. Math., 27(4), pp. 929–947.
- [7] Kohlenbach, U., (2004), Some logical metatheorem with applications in functional analysis, Trans. Amer. Math. Soc., 534, pp. 89–128.
- [8] Reich, S. and Shafrir, I., (1990), Nonexpansive iterations in hyperbolic spaces, Nonlinear Anal., 15, pp. 537–558.
- [9] Mann, W. R., (1953), Mean Valued Methods in Iteration, Proc. Amer. Math. Soc. , 4, pp. 506–610.
- [10] Ishikawa, S., (1974), Fixed points by A new iteration method, Proc. Amer. Math. Soc. , 44, pp. 147–150.
- [11] Noor, M. A., (2000), New Approximation Schemes for General Variational Inequalities, J. Math. Anal. Appl., 251, pp. 217–229. <https://doi.org/10.1006/jmaa.2000.7042>.
- [12] Agarwal, R. P., O'Regan, D. , and Sahu, D. R., (2007), Iterative contraction of fixed points of nearly asymptotically nonexpansive mappings, J. Convex Anal., 8, pp. 61–79.
- [13] Zaheer, S., Chanda, A., Nashine, H. K., (2024), Reckoning applications of Z-iteration: Data dependence and solution to a delay Caputo fractional differential equation, Nonlinear Anal. Model. Control, 29(5), pp. 833–857.
- [14] Zaheer, S., Chanda, A., Nashine, H. K., (2024), Investigation of data dependence and convergence behavior of M-iteration on Banach spaces with an application, Math. Notes, 116(4), pp. 627–645.
- [15] Bera A., Chanda, A., Dey, L. K., Ali J., (2022), Iterative approximation of fixed points of a general class of non-expansive mappings in hyperbolic metric spaces, J. Appl. Math. Comput., 68(3), pp. 1817–1839.
- [16] Gürsoy, F., (2016), A Pichard-S iterative method for approximating fixed point of weak construction mappings, Filomat, 30, pp. 2829–2845.
- [17] Takashi, W. A., (1970), Convexity in metric space and nonexpansive mappings, Kodai Math. Sem. Rep., 22, pp. 142–149.
- [18] Goebel, K. and Kirk, W. A., (1983), Iteration processes for nonexpansive mappings, Topol. Methods Nonlinear Anal., 21, pp. 115–123.
- [19] Goebel, K. and Reich, S., (1984), Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York.
- [20] Leustean, L. (2007), A quadratic rate of asymptotically regularity for CAT (0) spaces, J. Math. Anal. Appl., 325, pp. 386–399.
- [21] Chuadchawna, P., Farajzadeh, A., and Kawcharoen, A., (2020), Convergence theorems and approximating endpoints for multivalued Suzuki mappings in hyperbolic spaces, J. Comput. Anal. Appl., 20, pp. 903–916.
- [22] Pandey, R., Pant, R., Rakocevic, V., and Shukla, R., (2019), Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications, Results Math, 74, pp. 1–24.
- [23] Uddin, I., Aggarwal, S., and Abdou, A. A. N., (2021), Approximation of endpoints for α -Reich-Suzuki nonexpansive mapping in hyperbolic metric spaces, Mathematics, 9. <https://doi.org/10.3390/math9141692>.
- [24] Naor A. and Silberman L., (2011), *Poincaré inequalities, embeddings, and wild groups*, Compos. Math., 147(5) ,1546–1572. <https://doi.org/10.1112/S0010437X11005343>.
- [25] Dompongsa, S. and Panyanak, B., (2008), On Δ -convergence theorem in CAT(0) space, Comput. Math. Appl., 56(10), pp. 2572–2579.
- [26] Senter, H. F., Dotson, W. G., (1974), Approximating fixed points of nonexpansive mappings. Proc. Amer. Math. Soc., 44, pp. 375–380.



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