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PARTIAL CONTROLLABILITY OF SEMILINEAR SYSTEMS

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Abstract. The paper considers semilinear control system in the product of Hilbert spaces $X = H \times G$ driven by densely defined closed linear operator A generating a strongly continuous semigroup. For the linear operator L, projecting X to H, it is proved a sufficient condition for L-partially exact controllability to $L(D(A))$ which means that for every initial state $\xi \in X$ and every $\eta \in D(A)$ there exists a control u such that $Lx^{\xi,u}(T) = L\eta$, where $x^{\xi,u}$ is the state process corresponding the initial state ξ and the control u. The result is demonstrated on examples.

Keywords: Exact controllability, partial controllability, deterministic system, semilinear system, heat equation.

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1. Introduction

Controllable systems play an important role in engineering. These systems are able to move every initial data to every terminal data for a finite time duration. Mathematically, a concept of controllability was defined by Kalman [1] in 1960. Further studies [2, 3] suggested to split it into stronger and weaker versions. Nowadays, they are called exact (or complete) and approximate controllability and form two basic concepts, each having a number of variations.

Most completely, the necessary and sufficient conditions of exact and approximate controllability are investigated for linear systems and discussed in several books, for example [4–7]. For nonlinear systems, the investigations are directed to proving sufficient conditions of controllability for different kinds of systems [8–22] basically by use of fixed-point theorems. An alternative method consisting in a construction of a steering control piecewisely and avoiding fixed-point theorems was suggested in [23]. In this paper, we follow to this method.

For many engineering applications, tasks are defined for outputs [24]. Therefore, the ability of a control system to steer any initial state to any target output becomes important. This leads to the concept of output-controllability which has been extensively developed

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in [25–28] for discrete systems. In particular, outputs may be projections of the terminal value of the state. This type of output-controllability has been defined in [31] under the name of partial controllability. For an infinite dimensional state space, approximate and exact types of partial controllability concepts can be defined. The conditions of partial controllability are similar to those of controllability for linear systems. However, for nonlinear systems, the issue is different.

The conditions of partially approximate controllability for semilinear systems by piecewise construction method are obtained in [32]. In the present paper, we prove a sufficient condition of partially exact controllability for an infinite dimensional semilinear system by the method of piecewise construction of steering control which avoids fixed-point theorems. Besides the restrictions directed to the existence and uniqueness of the solution of the equation under consideration, we make the only restriction to the nonlinear term of the system assuming that it is bounded. Additional restrictions to the partial controllability Gramian include its coercivity at all non-initial instants and a specific rate of convergence to zero operator as time approaches the initial instant.

In the rest of this paper, $L_1(a, b; X)$ and $L_2(a, b; X)$ denote the Lebesgue space of integrable and square integrable, respectively, functions on $[a, b]$ with the values in the space X. If X is the space $\mathbb{R}$ of real numbers, then $L_1(a, b) = L_1(a, b; \mathbb{R})$ and $L_2(a, b) = L_2(a, b; \mathbb{R})$. Scalar products and norms in all considered spaces will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The spaces, to which they correspond, will be clear from the context. In ambiguous cases, they will be shown in the subscript. A^* is the adjoint of the linear operator A. Zero and identity operators will be denoted by 0 and I, respectively. The spaces between which they are operating will follow from the context.

2. Description of System

A deterministic semilinear control system of our study has the form

$$x'(t) = Ax(t) + Bu(t) + f(t, x(t), u(t)),\ 0 < t \le T, \tag{1}$$

with $T > 0$. Here, x and u are the state and control processes. The following conditions on the entries of (1) are assumed throughout this paper:

(A) H, G, and U are separable Hilbert spaces and $X = H \times G$.

(B) A is a densely defined closed linear operator on $D(A) \subseteq X$, generating a strongly continuous semigroup e^{At}, $t \ge 0$.

(C) B is a bounded linear operator from U to X.

(D) f is a function from $[0, T] \times X \times U$ to X, satisfying
- f is Lebesgue measurable in t (the first argument);
- f is Lipschitz continuous in x (the second argument) in the form
$$\|f(t, x, u) - f(t, y, u)\| \le L(t)\|x - y\|,$$
where $L \in L_1(0, T)$;
- f is continuous in u (the third argument);
- f is bounded in the total argument.

Note that $X = H \times G$ means that X is a separable Hilbert space with the scalar product defined by

$$\langle (h, g), (v, w) \rangle_X = \langle h, v \rangle_H + \langle g, w \rangle_G,\ h, v \in H,\ g, w \in G.$$

Here, X is regarded as a state space and U as a control space. The set $U_{\mathrm{ad}} = L_2(0, T; U)$ is considered as a set of admissible controls. Under the preceding conditions, the equation

$$x(t) = e^{At}\xi + \int_0^t e^{A(t-s)}(Bu(s) + f(s, x(s), u(s)))\, ds,\ 0 \le t \le T, \tag{2}$$

has a unique continuous solution for every control $u \in U_{\rm ad}$ and initial state $x(0) = \xi \in X$ [33]. Moreover, given $x(\tau)$ at some $0 \le \tau < T$, we can express $x(t)$ for $\tau \le t \le T$ as

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-s)}(Bu(s) + f(s, x(s), u(s)))\, ds,\ \tau \le t \le T. \tag{3}$$

The solution x of (2) is called a mild solution of (1). It depends on the initial state $x(0) = \xi$ and the control $u \in U_{\rm ad}$. Therefore, we will denote it by $x = x^{\xi,u}$. Note that the terminal value $x^{\xi,u}(T)$ of this mild solution ranges in X and can take values in $X \setminus D(A)$. However. the most important values are those which are in $D(A)$. This motivates to modify the concept of exact controllability and partially exact controllability for (1) in the following way.

Denote by L the linear operator from X to H defined by $L(h, g) = h$, $h \in H$, $g \in G$. Then $L^*h = (h, 0)$ and $P_L = L^*L$ is a projection operator, projecting $(h, g) \in X = H \times G$ to $(h, 0)$. We denote $\tilde H = P_L(X)$ and $\tilde G = \tilde H^\perp$ (orthogonal complement of $\tilde H$). Then $X = \tilde H \oplus \tilde G$ (the direct sum of $\tilde H$ and $\tilde G$) while $X = H \times G$.

Definition 2.1. System (1) with the set of admissible controls $U_{\rm ad}$ is said to be

(a) *Exactly controllable* on $[0, T]$ if for every $\xi, \eta \in X$, there exists $u \in U_{\rm ad}$ such that $x^{\xi,u}(T) = \eta$.

(b) *L-partially exact controllable* on $[0, T]$ if for every $\xi, \eta \in X$, there exists $u \in U_{\rm ad}$ such that $Lx^{\xi,u}(T) = L\eta$.

(c) *Exactly controllable to* $D(A)$ on $[0, T]$ if for every $\xi \in X$ and $\eta \in D(A)$, there exists $u \in U_{\rm ad}$ such that $x^{\xi,u}(T) = \eta$.

(d) *L-partially exact controllable to* $L(D(A))$ on $[0, T]$ if for every $\xi \in X$ and $\eta \in D(A)$, there exists $u \in U_{\rm ad}$ such that $Lx^{\xi,u}(T) = L\eta$.

The exact controllability is a particular case of the L-partially exact controllability when L is the identity operator on X. In a similar way, the exact controllability to $D(A)$ is a particular case of the L-partially exact controllability to $L(D(A))$. The L-partially exact controllability is motivated from the fact that some systems such as higher order differential equations, delay equations, wave equations, etc. can be written in the standard form if the dimension of the state space is enlarged. This trend is observed for stochastic systems as well. It is known that colored noise driven stochastic systems can be reduced to white noise driven system if the dimension of the state space is enlarged [34]. In [35–40], it is shown that the stochastic systems driven by wide band noises and by shifted white noises can also be handled in a similar way. Then the exact controllability of the enlarged state equation becomes too strong for the original state equation. In this regard, the L-partially exact controllability of the enlarged system becomes the exact controllability of the original system if L is an operator from the enlarged state space to the original one.

The approximate version of Definition 1 can also be defined. The concept of L-partially approximate controllability was studied in [31] for linear and in [32] for semilinear systems. In this paper, we deal with the concept of L-partially exact controllability to $L(D(A))$ only and prove a sufficient condition for the semilinear system in (1) to be controllable in this sense.

For $f(t, x, u) \equiv 0$, the system in (1) takes the linear form:

$$y'(t) = Ay(t) + Bu(t),\ 0 \le \tau < t \le T. \tag{4}$$

The controllability Gramian $Q(t)$ associated with this linear system is defined by

$$Q(t) = \int_0^t e^{As}BB^*e^{A^*s}ds,\ 0 < t \le T.$$

We define also the L-partial controllability Gramian by

$$Q_L(t) = LQ(t)L^*.$$

Clearly, for all $0 \le t \le T$, $Q_L(t)$ is nonnegative (symbolically, $Q_L(t) \ge 0$), i.e., $Q_L(t)^* = Q_L(t)$ and $\langle Q_L(t)h, h\rangle \ge 0$ for all $h \in H$. We will say that $Q_L(t)$ is coercive if there exists $\gamma > 0$ such that $\langle Q_L(t)h, h\rangle \ge \gamma \|h\|^2$ for all $h \in H$. Here, $Q_L(0) = 0$ and has no chance to be coercive while $Q_L(t)$ may be coercive for $0 < t \le T$. Note that the coercivity of $Q_L(t)$ with the coercivity constant γ implies the existence of $Q_L(t)^{-1}$ as a bounded linear operator and $\|Q_L(t)^{-1}\| \le \gamma^{-1}$. Respectively, the following theorem holds.

Theorem 2.1. *Under the conditions* (A)–(C)*, the linear system in* (4) *is L-partially exact controllable on the interval $[\tau, T]$ if and only if $Q_L(T-\tau)$ is coercive. Moreover, for any given $\xi, \eta \in X$, a control steering $y(\tau) = \xi$ to $y(T)$ with $Ly(T) = L\eta$ can be defined by*

$$u(t) = B^* e^{A^*(T-t)} L^* Q_L(T-\tau)^{-1} L\big(\eta - e^{A(T-\tau)}\xi\big),\ \tau \le t \le T. \tag{5}$$

Proof. It can be straightforwardly verified that for the control u, defined by (5), $Ly(T) = L\eta$. □

According to this theorem, for linear systems, the coercivity of $Q_L(t)$ is required at some fixed $t > 0$, which equals to the length of the time interval under consideration. However, for semilinear systems, the coercivity of $Q_L(t)$ is needed at all positive t up to the length of the interval. In particular, if $A = 0$ and $B = L = I$, then $Q_L(t) = tI$ and, therefore, $Q_L(t)$ is coercive for all $0 < t \le T$ with $Q_L^{-1}(t) = t^{-1}I$. This means that $\lim_{t\to 0^+} \|Q_L^{-1}(t)\| = \infty$. Therefore, we will adjust the possible unboundedness of the function $\|Q_L^{-1}(t)\|$ and add technical conditions in the following form:

(E) $Q_L(t)$ is coercive for every $0 < t \le T$ and there is $0 \le \alpha < 1$ such that $t^{1+\alpha}\|Q_L^{-1}(t)\| \le W$ for all $0 < t \le T$ and for some $W \ge 0$.

(F) $e^{At}\big(\tilde{G}\big) \subseteq \tilde{G}$, $t \ge 0$.

Remark 2.1. Condition (F) holds, for example, for the following large class of semigroups. Let $\{e_n\}$ be a sequence of complete orthonormal vectors in X and let $\{\lambda_n\}$ be a strictly decreasing sequence of real numbers. Define

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n,$$

where $x \in X$ is so that the series in the right side converges. Here, $\{\lambda_n\}$ is a sequence of eigenvalues and $\{e_n\}$ is a respective sequence of eigenvectors of A. Then A generates the strongly continuous semigroup of the form

$$e^{At}x = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle x, e_n\rangle e_n,\ x \in X,\ t \ge 0.$$

Divide $\{e_1, e_2, \ldots\}$ into two disjoint nonempty finite or infinite sets $\{e_n : n \in J_1\}$ and $\{e_n : n \in J_2\}$, where $J_1 \cup J_2 = \{1, 2, \ldots\}$ and $J_1 \cap J_2 = \emptyset$. Let H and G be the closed spaces spanned by them, respectively. Then $X = H \times G$ and for every $x \in \tilde{G}$,

$$e^{At}x = \sum_{n\in J_2} e^{\lambda_n t}\langle x, e_n\rangle e_n \in \tilde{G},\ x \in X,\ t \ge 0.$$

This construction and its generalization to the case when the eigenvalues of A have finite multiplicity covers a large class of second order partial differential equations which can be solved by the Fourier method of separation of variables, including heat and wave equations as well.

Remark 2.2. The function $Q_L(t)$, $0 \le t \le T$, is continuous by definition. Then under the inequality in (E), the function $Q_L^{-1}(t)$, $0 < t \le T$, is also continuous since $0 < \tau < s < t \le T$ implies

$$\|Q_L^{-1}(t) - Q_L^{-1}(s)\| \le \|Q_L^{-1}(s)(Q_L(t) - Q_L(s))Q_L^{-1}(t)\| \le \frac{W^2}{\tau^{2(1+\alpha)}}\|Q_L(t) - Q_L(s)\|.$$

Therefore, we can consider the proper or improper integral $\int_0^T t\|Q_L^{-1}(t)\|\,dt$. Then

$$t\|Q_L^{-1}(t)\| \le \frac{W}{t^\alpha} \;\Rightarrow\; \int_0^T t\|Q_L^{-1}(t)\|\,dt \le W\int_0^T \frac{dt}{t^\alpha} = \frac{WT^{1-\alpha}}{1-\alpha} < \infty.$$

At the same time, if we additionally assume that $\|Q_L^{-1}(t)\|$ is a decreasing function, then

$$t\|Q_L^{-1}(t)\| \le \int_0^T \|Q_L^{-1}(s)\|\,ds,\; 0 < t \le T.$$

Therefore,

$$\int_0^T \|Q_L^{-1}(t)\|\,dt < \infty \;\Rightarrow\; \text{(E)} \;\Rightarrow\; \int_0^T t\|Q_L^{-1}(t)\|\,dt < \infty.$$

This determines the place of condition (E) which is weaker than $\int_0^T \|Q_L^{-1}(t)\|\,dt < \infty$ while stronger than $\int_0^T t\|Q_L^{-1}(t)\|\,dt < \infty$. Note that $\int_0^T \|Q_L^{-1}(t)\|\,dt < \infty$ is a required condition if the generalized Banach fixed-point theorem is applied to investigate the exact controllability [22].

3. Main Result

The following is the main result of this paper.

Theorem 3.1. *Under conditions* (A)–(F)*, the system in* (1) *is L-partially exact controllable to $L(D(A))$ on the interval* $[0,T]$.

Proof. We fix the initial state $x_0 \in X$ and $\eta \in D(A)$ and construct a control u in the following way.

Let $\tau_n = T/2^n$ for $n = 0, 1, \dots$ Clearly, $\sum_{n=1}^\infty \tau_n = T$. Define

$$t_0 = 0,\; t_1 = \tau_1, \dots,\; t_n = \sum_{k=1}^n \tau_k, \dots$$

Then

$$\lim_{n\to\infty} t_n = \sum_{k=1}^\infty \tau_k = T.$$

According to Theorem 1, the linear system in (4) is L-partially exact controllable on the interval $[t_0, t_1]$ and the control

$$u_1(t) = B^* e^{A^*(t_1 - t)} L^* Q_L^{-1}(\tau_1) L e^{A\tau_1}(\eta - x_0),\; t_0 \le t \le t_1,$$

steers x_0 to $y(t_1)$ with $Ly(t_1) = Le^{A\tau_1}\eta$, that is,

$$Le^{A\tau_1}\eta = Le^{A\tau_1}x_0 + \int_{t_0}^{t_1} Le^{A(t_1 - s)}Bu_1(s)\,ds.$$

Letting $u(t) = u_1(t)$ for $t_0 \le t \le t_1$, from (3), we obtain

$$Lx^{x_0,u}(t_1) = Le^{A\tau_1}\eta + \int_{t_0}^{t_1} Le^{A(t_1-s)} f\big(s, x^{x_0,u}(s), u(s)\big)\, ds.$$

For brevity, we denote $x^{x_0,u}(t_1) = x_1$.

Considering (4) on the interval $[t_1, t_2]$, we let

$$u_2(t) = B^* e^{A^*(t_2-t)} L^* Q_L^{-1}(\tau_2) Le^{A\tau_2}(\eta - x_1),\ t_1 \le t \le t_2.$$

By Theorem 1, the control u_2 steers x_1 to $y(t_2)$ with $Ly(t_2) = Le^{A\tau_2}\eta$, that is,

$$Le^{A\tau_2}\eta = Le^{A\tau_2}x_1 + \int_{t_1}^{t_2} Le^{A(t_2-s)} Bu_2(s)\, ds.$$

Letting $u(t) = u_2(t)$ for $(t_1, t_2]$, we obtain

$$Lx^{x_0,u}(t_2) = Le^{A\tau_2}\eta + \int_{t_1}^{t_2} Le^{A(t_2-s)} f\big(s, x^{x_0,u}(s), u(s)\big)\, ds.$$

For brevity, let $x^{x_0,u}(t_2) = x_2$. Continuing this procedure, we recursively obtain the sequence of controls

$$u_n(t) = B^* e^{A^*(t_n-t)} L^* Q_L^{-1}(\tau_n) Le^{A\tau_n}(\eta - x_{n-1}),\ t_{n-1} \le t \le t_n. \tag{6}$$

Using these controls as the steps of the total control, we define

$$u(t) = \begin{cases} u_1(t), & \text{if } t_0 \le t \le t_1, \\ u_2(t), & \text{if } t_1 < t \le t_2, \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ u_n(t), & \text{if } t_{n-1} < t \le t_n, \\ \ldots\ldots & \ldots\ldots\ldots\ldots \end{cases}$$

Then

$$Le^{A\tau_n}\eta = Le^{A\tau_n}x_{n-1} + \int_{t_{n-1}}^{t_n} Le^{A(t_n-s)} Bu(s)\, ds, \tag{7}$$

implying

$$Lx^{x_0,u}(t_n) = Le^{A\tau_n}\eta + \int_{t_{n-1}}^{t_n} Le^{A(t_n-s)} f\big(s, x^{x_0,u}(s), u(s)\big)\, ds. \tag{8}$$

Denote $x^{x_0,u}(t_n) = x_n$. To estimate $\|Lx_n - L\eta\|$, let

$$K = \sup_{[0,T]} \big\|e^{At}\big\| \text{ and } M = \sup_{[0,T]\times X\times U} \|f(t,x,u)\|.$$

Using $\|L\| \le 1$ and (8), we have

$$\begin{aligned} \|Lx_n - L\eta\| &\le \big\|L\big(x_n - e^{A\tau_n}\eta\big)\big\| + \big\|L\big(e^{A\tau_n}\eta - \eta\big)\big\| \\ &\le \big\|L\big(e^{A\tau_n}\eta - \eta\big)\big\| + \int_{t_{n-1}}^{t_n} \big\|Le^{A(t_n-s)} f\big(s, x^{x_0,u}(s), u(s)\big)\big\|\, ds \\ &\le \big\|e^{A\tau_n}\eta - \eta\big\| + \int_{t_{n-1}}^{t_n} \big\|e^{A(t_n-s)}\big\| \big\|f\big(s, x^{x_0,u}(s), u(s)\big)\big\|\, ds \\ &\le \big\|e^{A\tau_n}\eta - \eta\big\| + MK\tau_n,\ n = 1, 2, \ldots \end{aligned} \tag{9}$$

Therefore, $\lim_{n\to\infty} Lx_n = L\eta$ since e^{At} is strongly continuous and $\lim_{n\to\infty} \tau_n = 0$.

Note that up to this point we did not use the restriction $\eta \in D(A)$. This part of the proof is valid for all $\eta \in X$. Now, we are going to prove that $u \in U_{\text{ad}}$ and use this restriction. For this, first of all, note that each piece u_n of u is continuous on the corresponding interval $(t_{n-1}, t_n]$. Therefore, u is measurable. Moreover,

$$\begin{aligned}\int_{t_{n-1}}^{t_n} \|u_n(t)\|^2\,dt &= \int_{t_{n-1}}^{t_n} \big\|B^* e^{A^*(t_n-t)} L^* Q_L^{-1}(\tau_n) L e^{A\tau_n}(\eta - x_{n-1})\big\|^2\,dt \\ &= \int_{t_{n-1}}^{t_n} \big\langle L e^{A(t_n-t)} B B^* e^{A^*(t_n-t)} L^* Q_L^{-1}(\tau_n)\varphi_n, Q_L^{-1}(\tau_n)\varphi_n\big\rangle\,dt \\ &= \big\langle \varphi_n, Q_L^{-1}(\tau_n)\varphi_n\big\rangle \le \big\|Q_L^{-1}(\tau_n)\big\|\|\varphi_n\|^2,\ n = 1, 2, \dots \end{aligned} \tag{10}$$

where $\varphi_n = L e^{A\tau_n}(\eta - x_{n-1})$. To estimate $\|\varphi_n\|$, write

$$\eta - x_{n-1} = P_L(\eta - x_{n-1}) + (I - P_L)(\eta - x_{n-1}).$$

Here, $e^{A\tau_n}(I - P_L)(\eta - x_{n-1}) \in \tilde{G}$ by condition (F) and, therefore, $P_L e^{A\tau_n}(I - P_L)(\eta - x_{n-1}) = 0$. By (9), this implies

$$\begin{aligned}\|\varphi_n\| &= \big\|L e^{A\tau_n}(\eta - x_{n-1})\big\| = \big\|L e^{A\tau_n} P_L(\eta - x_{n-1})\big\| \\ &= \big\|L e^{A\tau_n} L^* L(\eta - x_{n-1})\big\| \le K\big\|e^{A\tau_{n-1}}\eta - \eta\big\| + MK^2\tau_{n-1},\ n = 2, 3, \dots\end{aligned}$$

and by (10),

$$\int_{t_{n-1}}^{t_n} \|u_n(t)\|^2\,dt \le K\big\|Q_L^{-1}(\tau_n)\big\|\big(\big\|e^{A\tau_{n-1}}\eta - \eta\big\| + MK\tau_{n-1}\big)^2,\ n = 2, 3, \dots$$

Thus, by (8),

$$\begin{aligned}\int_{t_1}^{T} \|u(t)\|^2\,dt &= \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \|u_{n+1}(t)\|^2\,dt \\ &\le K\sum_{n=1}^{\infty} \big\|Q_L^{-1}(\tau_{n+1})\big\|\big(\big\|e^{A\tau_n}\eta - \eta\big\| + MK\tau_n\big)^2 \\ &\le 2K\sum_{n=1}^{\infty} \big\|Q_L^{-1}(\tau_{n+1})\big\|\big(\big\|e^{A\tau_n}\eta - \eta\big\|^2 + M^2K^2\tau_n^2\big) \\ &= 8K\sum_{n=1}^{\infty} \tau_{n+1}^2\big\|Q_L^{-1}(\tau_{n+1})\big\|\bigg(\bigg\|\frac{e^{A\tau_n}\eta - \eta}{\tau_n}\bigg\|^2 + M^2K^2\bigg).\end{aligned}$$

Here,

$$\lim_{n\to\infty} \frac{e^{A\tau_n}\eta - \eta}{\tau_n} = A\eta$$

since $\eta \in D(A)$ and $\lim_{n\to\infty} \tau_n = 0$. Therefore,

$$\bigg\|\frac{e^{A\tau_n}\eta - \eta}{\tau_n}\bigg\| \le N$$

for some $N \ge 0$. Then by geometric test,

$$\begin{aligned}\int_{t_1}^{T} \|u(t)\|^2\,dt &\le 8KW\big(N^2 + M^2K^2\big)\sum_{n=1}^{\infty} \tau_n^{1-\alpha} \\ &\le 8KWT\big(N^2 + M^2K^2\big)\frac{2^{1-\alpha}}{2^{1-\alpha} - 1} < \infty.\end{aligned}$$

Thus,

$$\int_0^T \|u(t)\|^2\,dt \le \int_0^{t_1} \|u(t)\|^2\,dt + 8WT\big(N^2+M^2K^2\big)\frac{2^{1-\alpha}}{2^{1-\alpha}-1} < \infty,$$

implying $u \in U_{\text{ad}}$. For this control, $x^{x_0,u}(t)$ is continuous and, therefore,

$$Lx^{x_0,u}(T) = \lim_{n\to\infty} Lx^{x_0,u}(t_n) = \lim_{n\to\infty} Lx_n = L\eta.$$

This completes the proof. □

4. Examples

In this section, we demonstrate Theorem 3.1 in examples. Among the conditions of this theorem, (A) to (D) are of general nature. Therefore, we concentrate on conditions (E) and (F).

Example 4.1. Consider system (1) with $X = U = l_2$ (the space of square summable sequences). Define $A = 0$ and

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $e^{At} = I$. For $\{x_n\} \in l_2$, we have

$$B\{x_n\} = \{x_n/n\},$$

implying

$$\|B\{x_n\}\|_{l_2}^2 = \sum_{n=1}^{\infty} \frac{x_n^2}{n^2} \le \sum_{n=1}^{\infty} x_n^2 = \|\{x_n\}\|_{l_2}^2.$$

Therefore, B is a bounded linear operator on l_2 with $\|B\| \le 1$ and, moreover, $B^* = B$.

We calculate

$$Q(t) = \int_0^t e^{As}BB^*e^{A^*s}\,ds = \int_0^t B^2\,ds = tB^2.$$

For the basis

$$e_1 = (1,0,0,\ldots),\ e_2 = (0,1,0,\ldots),\ e_3 = (0,0,1,\ldots),\ \ldots$$

in l_2, we have

$$\langle Q(t)e_n, e_n\rangle_{l_2} = \big\langle tB^2 e_n, e_n\big\rangle_{l_2} = \frac{t}{n^2} \to 0 \text{ as } n \to \infty.$$

Therefore, $Q(t)$ is non-coercive.

Consider the finite dimensional space H generated by $e_1, \ldots, e_k$. Let L be an operator from l_2 to H consisting in removing the terms containing $e_{k+1}, e_{k+2}, \ldots$ in the expansion of elements $x \in l_2$. Then

$$Q_L(t) = LQ(t)L^* = tLB^2L^* = t\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1/2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/k^2 \end{bmatrix},$$

implying

$$Q_L^{-1}(t) = \frac{1}{t}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k^2 \end{bmatrix}.$$

So, $t\|Q_L^{-1}(t)\| = k^2 < \infty$. Thus, although $Q(t)$ is non-coercive, any partial controllability Gramian $Q_L(t)$ obtained by projection to a finite dimensional subspace becomes coercive and satisfies condition (E) with $\alpha = 0$. Condition (F) holds as well because $e^{At} = I$.

Example 4.2. It is known that the Gramian of the heat equation is not coercive. Respectively, the heat equation is not exactly controllable even in the linear form. In [41] the null controllability of finite-dimensional projections of the heat equation has been proved. Below, we demonstrate that the exact controllability of finite-dimensional projections of the heat equation can be obtained from Theorem 3.1 as well.

Consider the semilinear heat equation

$$\frac{\partial y(t,\theta)}{\partial t} = \frac{\partial^2 y(t,\theta)}{\partial \theta^2} + u(t,\theta) + f(t, y(t,\theta), u(t,\theta)) \tag{11}$$

for $(t,\theta) \in (0,T) \times (0,1)$ with the initial and boundary conditions

$$y(0,\theta) = y(t,0) = y(t,1) = 0. \tag{12}$$

Let $X = L_2(0,1)$. Here, $A = d^2/d\theta^2$ is a densely defined closed linear operator with domain

$$D(A) = \{h \in X : h'' \in X,\ h(0) = h(1) = 0\}.$$

By use of Fourier method of separation of variables, it can be proved that the strongly continuous semigroup generated by A has the form

$$e^{At}h = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\sin(n\pi\theta)\int_0^1 h(\alpha)\sin(n\pi\alpha)\,d\alpha. \tag{13}$$

In the Fourier expansion, the weights of higher order terms are negligible. If the terms of order higher than k are regarded as negligible, then tasks can be defined with regard of the first k terms. Motivated from this scenario, define Hilbert spaces H and G as

$$H = \left\{\sum_{n=1}^{k} \alpha_n \sin n\pi\theta : \alpha_1, \ldots, \alpha_k \in \mathbb{R}\right\}$$

and

$$G = \left\{\sum_{n=k+1}^{\infty} \alpha_n \sin n\pi\theta : \alpha_{k+1}, \alpha_{k+2}, \ldots \in \mathbb{R},\ \sum_{n=k+1}^{\infty} \alpha_n^2 < \infty\right\}$$

with $\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle_G = \langle\cdot,\cdot\rangle_{L_2(0,1)}$. Then $X = H \times G$ and we can define the subspaces $\tilde{H}$ and $\tilde{G}$ of X making $X = \tilde{H} \oplus \tilde{G}$. Let $P_L = L^*L$ be the projection operator from X to $\tilde{H}$. Here, L is a linear operator from X to H assigning to $h \in X$ the sum of the first k terms in the Fourier sine expansion of h.

The semigroup in (13) definitely satisfies condition (F) with regard of the space $\tilde{G}$ because this falls to the case discussed in Remark 2.1. To verify condition (E), calculate

$Q_L(t)$. Since $e^{At} = e^{A^*t}$ and $B = I$, we have

$$[Q_L(t)h]_\theta = \int_0^t \left[Le^{2As}L^*h\right]_\theta ds = \sum_{n=1}^k \int_0^t 2e^{-2n^2\pi^2 s}\sin(n\pi\theta)\int_0^1 h_\alpha \sin(n\pi\alpha)\,d\alpha\,ds$$
$$= \sum_{n=1}^k \frac{1-e^{-2n^2\pi^2 t}}{n^2\pi^2}\sin(n\pi\theta)\int_0^1 h_\alpha \sin(n\pi\alpha)\,d\alpha. \tag{14}$$

Here, the sequence $\{a_n\} = \left\{\frac{1-e^{-2n^2\pi^2 t}}{n^2\pi^2}\right\}$ is decreasing. To prove, let $u = n^2\pi^2$ and consider the function

$$f(u) = \frac{1-e^{-2ut}}{u},\ u > 0.$$

Its derivative equals to

$$f'(u) = \frac{(2ut+1)e^{-2ut}-1}{u^2},\ u > 0.$$

From the inequality $1 + x \le e^x$, it follows that $f'(u) \le 0$. Therefore, $f(u)$ is a decreasing function, implying that $\{a_n\}$ is decreasing too. We conclude that

$$\min\left\{\frac{1-e^{-2n^2\pi^2 t}}{n^2\pi^2} : n = 1,\dots,k\right\} = \frac{1-e^{-2k^2\pi^2 t}}{k^2\pi^2} = c(t) > 0 \text{ if } t > 0.$$

Using this in (14), we obtain

$$\langle Q_L(t)h, h\rangle \ge c(t)\sum_{n=1}^k \left(\int_0^1 h_\alpha \sin(n\pi\alpha)d\alpha\right)^2 = \frac{c(t)}{2}\|h\|^2_{\tilde{H}}.$$

This means that $Q_L(t)$ is coercive for all $t > 0$. Furthermore, this implies that

$$\left\|Q_L^{-1}(t)\right\| \le \frac{2}{c(t)} = \frac{2k^2\pi^2}{1-e^{-2k^2\pi^2 t}},$$

implying

$$t\left\|Q_L(t)^{-1}\right\| \le \frac{2tk^2\pi^2}{1-e^{-2k^2\pi^2 t}}.$$

Here, by L'Hopital's theorem, the limit of the right side when $t \to 0^+$ equals to 1. So, the left side is bounded and, consequently, condition (E) holds with $\alpha = 0$. It remains to assume that f satisfies (D) and obtain that the system described by (11) and (12) is L-partially exact controllable to $D(A)$.

Resuming, it is seen that the non-coercivity of $Q(t)$ in the above examples is due to the infinite dimension of the state space. Every finite dimensional projection removes the non-coercivity and makes valid condition (E). However, the non-coercivity may be sourced within each dimension. In such cases, condition (F) fails.

5. Conclusion

In this paper a sufficient condition for the partially exact controllability of a semilinear system is proved. The specific feature of the proof method is that it does not use a fixed point theorem but constructs a steering control by use of a piecewise construction method. This result is demonstrated on two examples one of which is a heat equation. It is known that a linear heat equation is approximately controllable but not exactly. However, it is demonstrated that finitely many orthogonal components of the heat equation are exactly controllable.

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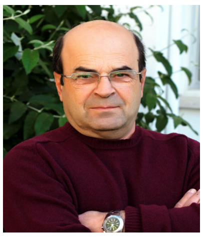

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