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EXISTENCE OF PERIODIC SOLUTIONS FOR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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Abstract. In this paper, we study the existence of periodic solutions for a kind of first-order impulsive differential equation with a deviating argument by using Mawhin's continuation theorem. Meanwhile, we give an exampleto demonstrate our result.

Keywords: Impulsive differential equations, Existence, periodic solution,Mawhin's continuation theorem.

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1. Introduction

Impulsive differential equations are very important in modern applied mathematics, as they are used to model a wide range of problems in fields such as physics, engineering, economics, and biology.

In recent years, these equations have attracted growing attention from the scientific community not only for the analysis of oscillatory behaviors, as explored in several recent studies (see for e.g., [11–14]) but also for the investigation of the existence of solutions. Numerous studies emphasize that establishing the existence of solutions is a fundamental prerequisite for analyzing the dynamic properties of such systems.

In this context, studying the existence of solutions to impulsive differential equations often requires specialized methods, depending on structural characteristics such as deviating arguments, impulsive effects, or nonlinearities. Among the most commonly used approaches are fixed point theorems (such as those of nonlinear alternative of Leray-Schauder and Krasnoselskii fixed point theorem) (see for e.g. [2, 10]), as well as Mawhin's continuation theorem.

In this article, we focus on Mawhin's continuation theorem, which has proven particularly effective in addressing complex problems, especially in the presence of resonance or nontrivial impulsive effects. Several researchers have applied Mawhin's continuation

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theorem to investigate the existence of periodic solutions for various types of differential equations (see for e.g. [1], [5], and [6]) as well as impulsive differential equations (see, for example, [7] and [8]).
Lu and Ge studied in [6] the existence of single and multiple solutions to a some second order differential equations with a deviating argument:

$$x''(t) = f\left(t, x(t), x(t-\tau(t)), x'(t)\right) + e(t).$$

By using the Mawhin's continuation theorem, among the conditions that must be satisfied for f, are such as
there is a constant $d > 0$ such that

$$f\left(t, x_0, x_1, 0\right) > |e|_0, \quad \forall\left(t, x_0, x_1\right) \in [0, T] \times \mathbb{R}^2 \text{ with } x_0 \geqslant x_1 > d.$$

and

$$f\left(t, x_0, x_1, 0\right) < -|e|_0, \quad \forall\left(t, x_0, x_1\right) \in [0, T] \times \mathbb{R}^2 \text{ with } x_0 \leqslant x_1 < -d;$$

In their work referenced as [7], Juan and José also applied Mawhin's continuation theorem to establish the existence of solutions for an impulsive boundary value problem described by

$$\begin{aligned} & x'(t) = f(x(t)) + e(t), \\ & x(t_k^+) - x(t_k^-) = I_k\big(x(t_k)\big), \quad k = 1, \ldots, q, \\ & x(0) = x(T), \end{aligned}$$

where the function $f : (0, \infty) \to (a, b)$ is continuous (with $a \in [-\infty, +\infty)$ and $b \in (-\infty, +\infty]$, $e : \mathbb{R} \to \mathbb{R}$ is continuous and T -periodic for $T > 0$, and for each $k = 1, 2, \ldots, q$, the impulse functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous. The interval is partitioned as $0 = t_0 < t_1 < t_2 < \cdots < t_q < t_{q+1} = T$.
In [6], the authors studied second-order differential equations with deviating arguments, but without impulses. In contrast, in [7], the authors focused on impulsive differential equations, but without deviating arguments. Additionally, in the problem discussed in [7], the function f depends on only a single variable.
Motivated by the above works, in this paper, and as a generalization of [7], we study the existence of solutions to impulsive differential equations that involve deviating arguments by using Mawhin's continuation theorem. We consider the following problem:

$$\begin{aligned} & x'(t) = f\left(x(t), x(t-\tau(t))\right) + p(t) \quad \text{ for } t \neq t_k, k = 1, \ldots, q \\ & \Delta x\left(t_k\right) = I_k\left(x\left(t_k\right)\right) \quad t = t_k, \\ & x(0) = x(T), \end{aligned} \tag{1}$$

where $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, $\tau, p \in C(\mathbb{R}, \mathbb{R})$ are T-periodic functions with $\tau(0) = 0$, and $\Delta x\left(t_k\right) = x(t_k^+) - x(t_k^-)$. $I_k \in C(\mathbb{R}, \mathbb{R}), k = 1, \ldots, q$, and $0 = t_0 < t_1 < \cdots < t_q < t_{q+1} = T$.
The structure of this paper is organized as follows: In Section 2, we present some preliminary results and necessary preparations. Section 3 is devoted to establishing sufficient conditions for the existence of a T-periodic solution of system (1), by using Mawhin's continuation theorem, in Section 4, we present an example to illustrate the application of the main results. Finally, in Section 5, conclusion are presented.

2. Preliminaries

In what follows, we present some results that will be used in section 3 to prove the existence of a solution for system (1).

In the following, we first introduce a series of notations, definitions, and auxiliary results that form the foundation for the analysis throughout this paper.

Definition 2.1. *[3] Let X and Y be normed vector spaces, and let $L : Dom(L) \subset X \to Y$ be a linear mapping. We classify L as a* Fredholm mapping of index zero *if it satisfies the following properties:*

1. *The kernel of L, denoted* $\mathrm{Ker}(L)$*, has a finite dimension, i.e.,* $\dim(\mathrm{Ker}(L)) < \infty$.
2. *The image of L, denoted* $\mathrm{Im}(L)$*, this codimension equals the dimension of the kernel, i.e.,* $\mathrm{codim}(\mathrm{Im}(L)) = \dim(\mathrm{Ker}(L))$.
3. *The image* $\mathrm{Im}(L)$ *forms a closed subspace of Y.*

Remark 2.1. *[3] Suppose L is a Fredholm mapping of index zero. Then, there exist continuous projection operators $P : X \to X$ and $Q : Y \to Y$ with the following characteristics:*

- $\mathrm{Im}(P) = \mathrm{Ker}(L)$,
- $\mathrm{Im}(L) = \mathrm{Ker}(Q) = \mathrm{Im}(I - Q)$*, where I denotes the identity operator on Y.*

Furthermore, the restriction of L to the intersection $Dom(L) \cap \mathrm{Ker}(P)$, with values in $\mathrm{Im}(L)$*, is a bijective linear mapping. The inverse of this restriction is denoted by K_P.*

Definition 2.2. *[3] Consider normed vector spaces X and Y, and a continuous mapping $N : X \to Y$. Let Ω be an open and bounded subset of X, with $\overline{\Omega}$ representing its closure. Suppose L is a Fredholm mapping of index zero, equipped with the associated projectors P and Q and the inverse K_P as described in the previous remark.*
We say that N is L-compact on $\overline{\Omega}$ *if it satisfies the following conditions:*

1. *The set $QN(\overline{\Omega})$ is bounded in Y.*
2. *The composite mapping $K_P(I - Q)N : \overline{\Omega} \to X$ is compact, meaning that the set $K_P(I - Q)N(\overline{\Omega})$ is relatively compact in X.*

Lemma 2.1. *[3] Consider two Banach spaces X and Y, and let Ω be an open, bounded subset of X. Suppose $L : Dom(L) \subseteq X \to Y$ is a Fredholm operator with index zero, and $N : \overline{\Omega} \to Y$ is an L-compact mapping on $\overline{\Omega}$. Assume the following conditions hold:*

1. *The equation $Lx = \lambda Nx$ has no solutions for any $x \in \partial\Omega \cap Dom(L)$ and any $\lambda \in (0, 1)$.*
2. *For all $x \in \partial\Omega \cap \mathrm{Ker}(L)$, $QNx \neq 0$.*
3. *The Brouwer degree of JQN on $\Omega \cap \mathrm{Ker}(L)$ with respect to the origin is non-zero, i.e., $\deg(JQN, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, where $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$ is an isomorphism.*

Then, there exists at least one $x \in Dom(L) \cap \overline{\Omega}$ satisfying $Lx = Nx$.

Let

$$X = \{x : [0, T] \to \mathbb{R} \mid x(0) = x(T), \quad x \text{ is continuous except at } t_k, \\ x\left(t_k^-\right) \text{ and } x\left(t_k^+\right) \text{ exist, and } x(t_k) = x\left(t_k^-\right)\},$$

and

$$\mathbb{Y} = X \times \mathbb{R}^q$$

be two Banach spaces with their respective norms

$$\|x\|_X = \sup\{|x(t)| : t \in [0, T]\}$$

and

$$\|y\| = \|(x, a_1, \ldots, a_q)\| = \|x\| + |(a_1, \ldots, a_q)|, \quad \text{for all } y \in Y,\ x \in X,$$

in which $|.|$ is any norm of $\mathbb{R}^q$. Now, we set
$L : \mathrm{Dom}\, L \subset X \to Y$ by

$$L(x) = \left(x'(t), \Delta x\left(t_1\right), \ldots, \Delta x\left(t_q\right)\right) \tag{2}$$

$N : X \to Y$ by

$$N(x) = \left(f\left(x(t), x(t-\tau(t)),\right) + p(t), I_1(x(t_1)), \ldots, I_q(x(t_q))\right). \tag{3}$$

It is easy to check that

$$\operatorname{Ker}(L) = \{x \in X : \exists c \in \mathbb{R} \text{ with } x(t) = c \quad \forall t \in [0, T]\}$$

$$\operatorname{Im}(L) = \left\{(y, a_1, \ldots, a_q) \in Y : \int_0^T y(t)dt + \sum_{k=1}^{q} a_k = 0\right\}.$$

It is easy to see that $\operatorname{Im}(L)$ is closed on Y and $\dim(\operatorname{Ker}(L)) = \operatorname{codim}(\operatorname{Im}(L)) = 1$. So, by Definition 2.1, L is a Fredholm operator with index zero. Also Consider the linear operators $P : X \to X$ and $Q : Y \to Y$ defined by

$$(Px)(t) = \frac{1}{T}\int_0^T x(s)ds, \quad t \in [0, T],$$

$$Q\left(x, a_1, \ldots, a_q\right)(t) = \left(\frac{1}{T}\int_0^T x(s)ds + \frac{1}{T}\sum_{i=1}^{q} a_i, 0, \ldots, 0\right).$$

We have

$$P(Px)(t) = \frac{1}{T}\int_0^T \left(\frac{1}{T}\int_0^T (Px)(t)dt\right) ds = \frac{1}{T}\frac{1}{T}\int_0^T (Px)(s)T = \frac{1}{T}\int_0^T (Px)(s) = (Px)(t).$$

And

$$\begin{aligned}
Q\left(Q\left(x, a_1, \ldots, a_q\right)\right)(t) =& Q\left(\frac{1}{T}\int_0^T x(s)\,ds + \frac{1}{T}\sum_{i=1}^{q} a_i, 0, \ldots, 0\right), \\
=& \left(\frac{1}{T}\int_0^T \left(\frac{1}{T}\int_0^T x(s)\,ds + \frac{1}{T}\sum_{i=1}^{q} a_i\right) ds + 0, 0, \ldots, 0\right) \\
=& \left(\frac{1}{T}\cdot T\left(\frac{1}{T}\int_0^T x(s)\,ds + \frac{1}{T}\sum_{i=1}^{q} a_i\right), 0, \ldots, 0\right), \\
=& \left(\frac{1}{T}\int_0^T x(s)\,ds + \frac{1}{T}\sum_{i=1}^{q} a_i, 0, \ldots, 0\right) \\
=& Q\left(x, a_1, \ldots, a_q\right)(t).
\end{aligned}$$

Since P and Q are linear operators and satisfy $P^2 = P$ and $Q^2 = Q$, it follows that P and Q are projection operators.

Thus, we have $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Let $K_P : \operatorname{Im} L \to \operatorname{Dom}(L) \cap \operatorname{Ker} P$ be the inverse of L restricted to $\operatorname{Dom}(L) \cap \operatorname{Ker} P$. Then, we obtain:

$$K_P\left(x, a_1, \ldots, a_q\right)(t) = \int_0^t x(s)ds + \sum_{0<t_i<t} a_i - \frac{1}{T}\int_0^T \int_0^s x(u)duds - \sum_{k=1}^{q} a_k \frac{T - t_k}{T},$$

$$(QNx)(t) = \left(\frac{1}{T}\int_0^T [f\left(x(t), x(t-\tau(t))\right) + p(t)]ds + \frac{1}{T}\sum_{k=1}^{q} I_k\left(x\left(t_k\right)\right), 0, \ldots, 0\right).$$

We will show that $QN(\bar{\Omega})$ is relatively compact. The proof will be given in several steps.

Step 1. QN is continuous.

The operator QN is continuous, because all the functions f, τ, p, and I_k are continuous,

and the operations used (composition, integration, summation) preserve continuity.
Step 2: QN Maps bounded sets into bounded sets
For any open, bounded subset Ω of X, $QN(\bar{\Omega})$ is bounded (by the continuity of QN).
Step 3: QN maps bounded sets into equicontinuous sets. Let $l_1, l_2 \in J$, $l_1 < l_2$ and Ω be a bounded subset of X. Let $x \in \Omega$.
From the expression of QN, it is clear that

$$|(QN)(x)(l_2) - (QN)(x)(l_1)| \to 0$$

as $l_2 \to l_1$.
As a consequence of Steps 1, 2 and 3 together with the Ascoli-Arzela theorem, we can conclude that $QN(\bar{\Omega})$ is relatively compact.
Similarly, one can show that $K_P(I-Q)N(\bar{\Omega})$ is relatively compact.
Consequently, it follows that N is L-compact on $\bar{\Omega}$ for every open, bounded set $\Omega \subset X$.

3. Main result

(H1) There exist $m_k, M_k \in \mathbb{R}$ such that $m_k \leq I_k(x) \leq M_k$ for all $x \in \mathbb{R}$.
(H2) There is a constant $d > 0$ such that:
(i)

$$f(x_0, x_1) > c_1, \quad \textit{for all} \quad x_i > d, (i = 0, 1);$$

where $c_1 = \frac{-m_1 - \cdots - m_q}{T} - \frac{1}{T}\int_0^T p(t)dt$.
(ii)

$$f(x_0, x_1) < c_2, \quad \textit{for all} \quad x_i < -d, (i = 0, 1);$$

where $c_2 = \frac{-M_1 - \cdots - M_q}{T} - \frac{1}{T}\int_0^T p(t)dt$.

Theorem 3.1. *Assume that conditions (H1) and (H2) are satisfied. Then, problem (1) has at least one T-periodic solution.*

Proof. To prove this theorem, we will use Lemma 2.1. Initially, we will construct a subset Ω on which the conditions of the lemma will be verified. In order to do this, let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in]0,1]\}$ if $x(\cdot) \in \Omega_1$, then we have from (2) and (3) that

$$\begin{cases} x'(t) = \lambda f(x(t), x(t-\tau(t))) + \lambda p(t), t \neq t_k, \\ \Delta x_1(t_k) = \lambda I_k(x_1(t_k)), k = 1, \ldots, q. \end{cases} \tag{4}$$

Integrating (4) over $[0, T]$, we obtain

$$\int_0^T x'(t) = \lambda \int_0^T f(x(t), x(t-\tau(t)))\, dt + \lambda \int_0^T p(t)dt. \tag{5}$$

We have

$$\int_0^T x'(t)dt = \sum_{k=1}^{q+1} \int_{t_{k-1}^+}^{t_k^-} x'(t)dt = -x(0) - \sum_{k=1}^{q} x\left(t_k^+\right) - x\left(t_k^-\right) + x(T) = -\lambda \sum_{k=1}^{q} I_k(x(t_k)) \tag{6}$$

It follows from (5) and (6) that

$$-\int_0^T f(x(t), x(t-\tau(t)))\, dt = \sum_{k=1}^{q} I_k(x(t_k)) + \int_0^T p(t)dt. \tag{7}$$

Thus, by applying the mean value theorem for definite integrals, there exists a constant $\eta \in [0, T]$ such that

$$f\left(x\left(\eta\right), x\left(\eta-\tau\left(\eta\right)\right)\right)=-\frac{1}{T}\int_0^T p(t)dt-\frac{1}{T}\sum_{k=1}^{q} I_k\left(x\left(t_k\right)\right).$$

In view of (7), we have from $(H1)$ that

$$m_1+\cdots+m_q+\int_0^T p(t)dt \leq -\int_0^T f\left(x(t), x(t-\tau(t))\right)dt \leq M_1+\cdots+M_q+\int_0^T p(t)dt.$$

Then

$$f\left(x\left(\eta\right), x\left(\eta-\tau\left(\eta\right)\right)\right) \leq c_1 \tag{8}$$

$$f\left(x\left(\eta\right), x\left(\eta-\tau\left(\eta\right)\right)\right) \geq c_2, \tag{9}$$

where c_1 and c_2 are given as in hypothesis $H2$.
We now proceed to demonstrate that there exists a constant $\xi \in [0, T] \backslash \{t_1, \ldots, t_q\}$ such that

$$|x(\xi)| \leqslant d. \tag{10}$$

(1) If $|x(\eta)| \leqslant d$, by choosing $\xi = \eta$, it immediately follows that $|x(\xi)| \leq d$.
(2) If $|x(\eta)| > d$ we have from assumption (i) in (H_2) and (8) that

$$x\left(\eta-\tau\left(\eta\right)\right) \leq d \tag{11}$$

and from assumption (ii) in $(H2)$ and (9) that

$$x\left(\eta-\tau\left(\eta\right)\right) \geq -d \tag{12}$$

then

$$\left|x\left(\eta-\tau\left(\eta\right)\right)\right| \leq d.$$

In both cases (1) or (2) we conclude that equation (10) is satisfied. Moreover, for any two points $t, s \in [0, T] \setminus \{t_1, \ldots, t_q\}$,

$$\int_s^t x'(u)du = x(t)-x(s)-\sum_{s<t_k<t}\Delta x\left(t_k\right).$$

For $s = \xi$, we have

$$\begin{aligned}
&\int_\xi^t x'(u)du = x(t)-x(\xi)-\sum_{\xi<t_k<t}\Delta x\left(t_k\right)\\
&\Longrightarrow x(t) = x(\xi)+\sum_{\xi<t_k<t}\Delta x\left(t_k\right)+\int_\xi^t x'(u)du\\
&\Longrightarrow |x(t)| \leq |x(\xi)|+\sum_{k=1}^{q}\left|\Delta x\left(t_k\right)\right|+\int_0^T \left|x'(u)\right|du,
\end{aligned}$$

it follows from (H1) and (7) that

$$\begin{aligned}
\int_0^T |x'(t)|dt &\leqslant \int_0^T \left|f\left(x(t), x(t-\tau(t))\right)\right|dt+\int_0^T |p(t)|dt\\
&\leqslant \widetilde{M}_1+\cdots+\widetilde{M}_q+\int_0^T p(t)dt+\int_0^T |p(t)|dt,
\end{aligned}$$

where $\widetilde{M_k} = \max\{|M_k|, |m_k|\}$.
Hence

$$
\begin{aligned}
|x(t)| &\leq |x(\xi)| + \sum_{k=1}^{q} |\Delta x(t_k)| + \int_0^T |x'(u)|\, du \\
&\leqslant d + 2\sum_{k=1}^{m} \widetilde{M_k} + \int_0^T p(t)dt + \int_0^T |p(t)|dt := M.
\end{aligned} \tag{13}
$$

Let $\Omega_2 = \{x \in \ker L, QNx = 0\}$. If $x \in \Omega_2$ then $x(t) = x(0)$ for all $t \in [0, T]$, and we have

$$\frac{1}{T}\int_0^T [f(x(t), x(t-\tau(t)),) + p(t)]ds + \frac{1}{T}\sum_{k=1}^{q} I_k(x(t_k)) = 0.$$

Then

$$f(x(0), x(-\tau(0))) = -\frac{1}{T}\int_0^T p(t)dt - \frac{1}{T}\sum_{k=1}^{q} I_k(x(0)).$$

Furthermore,

$$m_1 + \cdots + m_q + \int_0^T p(t)dt \leq -\int_0^T f(x(0), x(-\tau(0)))\, dt \leq M_1 + \cdots + M_q + \int_0^T p(t)dt$$

thus

$$m_1 + \cdots + m_q + \int_0^T p(t)dt \leq -\int_0^T f(x(0), x(0))\, dt \leq M_1 + \cdots + M_q + \int_0^T p(t)dt$$

This implies that

$$c_2 = \frac{-M_1 - \cdots - M_q}{T} - \frac{1}{T}\int_0^T p(t)dt \leq f(x, x) \leq \frac{-m_1 - \cdots - m_q}{T} - \frac{1}{T}\int_0^T p(t)dt = c_1$$

.
In view of $(H2)$, we see that

$$\|x\|_X \leq d. \tag{14}$$

Now, We define

$$\Omega = \{x \in X, \|x\|_X < M + 1\},$$

then $\Omega_1 \cup \Omega_2 \subset \Omega$.
Verification of the Conditions of Lemma 2.1 in Ω.
Assume that there exists $x \in \partial\Omega \cap \mathrm{Dom}(L)$ such that $Lx = \lambda N(x)$. From equation (13), we have:

$$\|x\|_X \leqslant M := d + 2\sum_{k=1}^{m} \widetilde{M_k} + \int_0^T p(t)dt + \int_0^T |p(t)|dt,$$

since $x \in \partial\Omega \cap \mathrm{Dom}(L)$, it follows that

$$\|x\|_X = M + 1,$$

which is a contradiction. Therefore, for all $x \in \partial\Omega \cap \mathrm{Dom}(L)$, we must have $Lx \neq \lambda N(x)$, and thus condition (1) is satisfied.

Now let us verify condition (2). Assume that there exists $x \in \partial\Omega \cap \mathrm{Dom}(L)$ such that

$$QN(x) = 0.$$

Since $x \in \partial\Omega \cap \mathrm{Dom}(L)$, we have

$$\|x\|_X = M + 1.$$

On the other hand, $QN(x) = 0$ implies, from equation (14), that

$$\|x\|_X \leq d.$$

Thus,

$$M + 1 \leq d,$$

which contradicts the fact that, by the definition of M in equation (13), we have $M+1 > d$. Next, we verify the condition (3) in Lemma 2.1. To do this, We define $J : (x, 0, \ldots, 0) \in \mathrm{Im}(Q) \longrightarrow x \in \mathrm{Ker}(L)$ an isomorphism. it follows that

$$JQNx(t) = \frac{1}{T}\int_0^T [f(x(t), x(t-\tau(t))) + p(t)]ds + \frac{1}{T}\sum_{k=1}^{q} I_k(x(t_k)),$$

$x \in \overline{\ker L \cap \Omega}$. We identify now $\mathrm{Ker}(L) \cap \Omega$ with the interval $(-M-1, M+1)$ of $\mathbb{R}$. Then the degree of JQN in $\Omega \cap \mathrm{Ker}(L)$ with respect to 0 is

$$\deg(JQN, \Omega \cap \mathrm{Ker}(L), 0) = \deg(g, (a, b), 0),$$

where $(a, b) = (-M-1, M+1)$ and the function $g : [a, b] \longrightarrow \mathbb{R}$ is given by

$$g(x) = \frac{1}{T}\int_0^T [f(x(t), x(t-\tau(t))) + p(t)]ds + \frac{1}{T}\sum_{k=1}^{q} I_k(x(t_k)).$$

We compute now $\deg(g, (a, b), 0)$. We have

$$\begin{aligned} g(a) &= f(a, a) + \frac{1}{T}\int_0^T p(t)dt + \frac{1}{T}\sum_{k=1}^{q} I_k(a) \\ &\leq f(a, a) + \frac{1}{T}\int_0^T p(t)dt + \frac{1}{T}(M_1 + \cdots + M_q) \\ &= f(-M-1, -M-1) + \frac{1}{T}\int_0^T p(t)dt + \frac{1}{T}(M_1 + \cdots + M_q). \end{aligned}$$

by (13) we have $M \geq d$, then $x_0 = x_1 = -M - 1 < -d$.
From condition (ii) of $(H2)$, we obtain $g(a) < 0$.
On the other hand,

$$\begin{aligned} g(b) &= f(b, b) + \frac{1}{T}\int_0^T p(t)dt + \frac{1}{T}\sum_{k=1}^{q} I_k(b) \\ &\geq f(b, b) + \frac{1}{T}\int_0^T p(t)dt + \frac{1}{T}(m_1 + \cdots + m_q) \\ &= f(M+1, M+1) + \frac{1}{T}\int_0^T p(t)dt + \frac{1}{T}(m_1 + \cdots + m_q) > 0. \end{aligned}$$

By the definition of M and assumption (i) of $(H2)$, it follows that $g(b) > 0$.
We deduce that $\deg(g, (a, b), 0) \neq 0$ using properties of the Brouwer's degree. Consequently, condition (3) of Lemma 2.1 holds. Therefore, by applying Lemma 2.1, we deduce that the equation $Lx = Nx$ possesses at least one T-periodic solution within $\bar{\Omega}$. It follows that system (1) has at least one T-periodic solution. □

4. Example

In this section, we present an example to demonstrate the application of Theorem 3.1. Let us consider the impulsive differential equations:

$$\begin{aligned} &x' = f(x(t), x(t-\tau)) + p(t), t \neq t_1, \\ &\Delta x'(t_1) = I_1\left(x\left(t_1\right)\right), \end{aligned} \tag{15}$$

where $f(x,y) = x^3(t) + sgn(y)y^2(t)$, $\tau(t) = \frac{|\sin(5t)|}{10}$, $I_1(x) = 2\cos(x(t))$, $p(t) = \sin(2t)$.

We have

$$-2 \leq I_1(x) \leq 2.$$

Thus, the hypothesis (H1) in Theorem 3.1 is verified, with $m_1 = -2$ and $M_1 = 2$. For $T = 2\pi$ and $d = 1$, We have

$$\begin{aligned} c_1 &= -\frac{m_1}{T} - \frac{1}{T}\int_0^T p(t)\,dt \\ &= \frac{2}{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \sin(2t)\,dt = \frac{1}{\pi}, \end{aligned}$$

and

$$\begin{aligned} c_2 &= -\frac{M_1}{T} - \frac{1}{T}\int_0^T p(t)\,dt \\ &= -\frac{2}{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \sin(2t)\,dt = -\frac{1}{\pi}. \end{aligned}$$

It follows that

$$f(x,y) > c_1 \quad \text{for all } x > 1 \text{ and } y > 1,$$

and

$$f(x,y) < c_2 \quad \text{for all } x < -1 \text{ and } y < -1.$$

Thus, the condition (H2) is satisfied.
Hence, by Theorem 3.1, system (15) has at least one $2\pi-$ periodic solution.

5. conclusion

The study of the existence of solutions for impulsive differential equations is a key element for a deeper understanding of the problem, as well as for selecting appropriate solution methods. In this paper, under certain conditions, we have examined a type of impulsive differential equation with a deviating argument, using the Mawhin continuation method. In the future, we plan to proposed method could be extended to deal with second-order impulsive differential equations with a deviating argument and stochastic impulsive systems, which are relevant in many real-world applications involving randomness and abrupt changes.

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