TWMS J. App. and Eng. Math. V.16, N.3, 2026, pp. 341-348

BERNSTEIN-TYPE INEQUALITIES FOR COMPLEX POLYNOMIALS

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ABSTRACT. Considering a class of polynomials $G(z) = a_n z^n + \sum_{v=t}^{n} a_{n-v} z^{n-v}, 1 \leq t \leq n$ having all it's zeros in the disk $|z| \leq k$, $k \leq 1$, we present a generalization and improvement of results by Malik and Vong[8]. Also a variety of intresting results emerge as special cases of our findings.

Keywords: Complex polynomial, Inequalities, Maximum, zeros.

AMS Subject Classification: 30A10, 30C10, 30C15.

1. INTRODUCTION

Let $p(z)$ be a polynomial of degree n, and $p'(z)$ denote it's derivative. According to the well-known Bernstein's inequality, we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

The result is best possible and equality holds only when $p(z)$ is a constant multiple of z^n. If we restrict ourselves to the class of polynomials whose zeros all lie in $|z| \leq 1$, Turan [10] proved

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

The Inequality (2) is sharp, and equality holds for a polynomial whose zeros are all located on $|z| = 1$.

As an extension of (2) Malik [7] showed that if $p(z)$ has all zeros in $|z| \leq k$ where $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \Big\{ \max_{|z|=1} |p(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |p(z)| \Big\}. \tag{3}$$

The estimate is sharp, and equality is attained for $p(z) = (z+k)^n$.
While seeking for the inequality analogous to (3), Aziz and Shah [4] studied the class of

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§ Manuscript received: February 14, 2025; accepted: June 23, 2025.
TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.3;

polynomials $p(z) = a_n z^n + \sum_{v=t}^{n} a_{n-v} z^{n-v}, \quad 1 \le t \le n$ of degree n, where all zeros lie in $|z| \le k$, $k \le 1$, they demonstrated

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k^t}\Big\{ \max_{|z|=1} |p(z)| + \frac{1}{k^{n-t}} \min_{|z|=k} |p(z)| \Big\}. \tag{4}$$

Bernstein [5] proved the following result from which Inequality (1) can be deduced for $Q(z) = Mz^n$, where $M = \max\limits_{|z|=1}|p(z)|$.

Theorem 1.1. *Let $p(z)$ and $Q(z)$ be two polynomials, with the degree of $p(z)$ not exceeding that of $Q(z)$. If $Q(z)$ has all it's zeros in $|z| \le 1$, and*

$$|p(z)| \le |Q(z)|, \quad \textit{for} \quad |z| = 1,$$

then

$$|p'(z)| \le |Q'(z)|, \ \textit{for}\ |z| = 1. \tag{5}$$

More generally, Malik and Vong [8] proved that for every real or complex number β with $|\beta| \le 1$, Inequality (5) can be replaced by ,

$$\Big| zp'(z) + \frac{n\beta}{2} p(z) \Big| \le \Big| zQ'(z) + \frac{n\beta}{2} Q(z) \Big|, \ \textit{for}\ |z| = 1. \tag{6}$$

Concerning the minimum modulus of the polynomial $p(z)$, the following result is attributed to Dewan and Hans [6].

Theorem 1.2. *If $p(z)$ is a polynomial of degree n, having all it's zeros in $|z| < 1$, then for every real or complex number β with $|\beta| \le 1$,*

$$\min_{|z|=1} \Big| zp'(z) + \frac{n\beta}{2} p(z) \Big| \ge n \Big| 1 + \frac{\beta}{2} \Big| \min_{|z|=1} |p(z)|. \tag{7}$$

As an application of Theorem 1.2, Dewan and Hans [6] established the following result.

Theorem 1.3. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for every real or complex number β with $|\beta| \le 1$,*

$$\begin{aligned} \max_{|z|=1} \Big| zp'(z) + \frac{n\beta}{2} p(z) \Big| \le \frac{n}{2} \Bigg[& \Big\{ \Big|\frac{\beta}{2}\Big| + \Big|1 + \frac{\beta}{2}\Big| \Big\} \max_{|z|=1} |p(z)| \\ & - \Big\{ \Big|1 + \frac{\beta}{2}\Big| - \Big|\frac{\beta}{2}\Big| \Big\} \min_{|z|=1} |p(z)| \Bigg]. \end{aligned} \tag{8}$$

2. Main Results

In this section, we present our first result by considering a class of polynomials $G(z) = a_n z^n + \sum_{v=t}^{n} a_{n-v} z^{n-v}, 1 \le t \le n$ having all their zeros within the disc $|z| \le k$, $k \le 1$. The result generalizes Inequalities (1.5),(1.6) and leads to several interesting special cases.

Theorem 2.1. *Let $G(z) = a_n z^n + \sum_{v=t}^{n} a_{n-v} z^{n-v}, 1 \le t \le n$ be a polynomial of degree n having all it's zeros in $|z| \le k$, $|k| \le 1$ and $g(z) = b_m z^m + \sum_{v=t}^{m} b_{m-v} z^{m-v}, 1 \le t \le m$ be a polynomial of degree not exceeding that of $G(z)$. If $|g(z)| \le |G(z)|$, for $|z| = k$, then for every real or complex number β with $|\beta| \le 1$,*

$$\Big| zg'(z) + \frac{n\beta}{1+s_t} g(z) \Big| \le \Big| zG'(z) + \frac{n\beta}{1+s_t} G(z) \Big|, \quad \textit{for} \quad |z| = 1. \tag{9}$$

Where

$$S_t = \frac{n|a_n - \alpha b_n| k^{2t} + t|a_{n-t} - \alpha b_{n-t}| k^{t-1}}{n|a_n - \alpha b_n| k^{t-1} + t|a_{n-t} - \alpha b_{n-t}|},$$

for $|\alpha| < 1$, $\alpha \in \mathbb{C}$.

For $\beta = 0$, $t = 1$, we have the following result which is a generalization of Inequality(5).

Corollary 2.1. *Let $G(z) = \sum_{i=1}^{n} a_i z^i \quad 1 \leq i \leq n$ be a polynomial of degree n, having all it's zeros in $|z| \leq k$, $k \leq 1$ and $g(z)$ be a polynomial of degree at most n, such that $|g(z)| \leq |G(z)|$ for $|z| = k$, then $|g'(z)| \leq |G'(z)|$ for $|z| = 1$.*

For $t = k = 1$, we have $S_t = 1$, and the Inequality(2.1) reduces to Inequality (6). Applying Theorem 2.1 to the polynomials $p(z) = \sum_{i=0}^{n} a_i z^i$ and $m\dfrac{z^n}{s_t^n}$, where $m = \min_{|z|=k} |p(z)|$, we obtain the following result.

Corollary 2.2. *If $p(z) = a_n z^n + \sum_{v=t}^{n} a_{n-v} z^{n-v}$, $1 \leq t \leq n$ is a polynomial of degree n, having all it's s zeros in $|z| \leq k$, $k \leq 1$, then for any β with $|\beta| \leq 1$,*

$$\min_{|z|=1} \left| zp'(z) + \frac{n\beta}{1+s_t} p(z) \right| \geq \frac{n}{s_t^n} \left| 1 + \frac{\beta}{1+s_t} \right| \min_{|z|=k} |p(z)|, \text{ for } |z| = 1. \tag{10}$$

The result is best possible and equality holds for $p(z) = \alpha z^n$.

Remark 2.1. *If we take $\beta = -1$ Inequality (10), the following inequality holds,*

$$. \min_{|z|=1} \left| zp'(z) - \frac{n}{1+s_t} p(z) \right| \geq \frac{n}{s_t^{n-1} + s_t^n} \min_{|z|=k} |p(z)|, \text{ for } |z| = 1. \tag{11}$$

Also if $k = t = 1$, in Corollary 2.2, Inequality (10) reduces to Inequality (7). For $\beta = 0$, Corollary 2.2 reduces to the following result which is an improvement of result due to Mezrji [9].

Corollary 2.3. *If $p(z) = a_n z^n + \sum_{v=t}^{n} a_{n-v} z^{n-v}$,$1 \leq t \leq n$ be a polynomial of degree n having all it's s zeros in $|z| \leq k$, $k \leq 1$, then*

$$\frac{n}{s_t^n} \min_{|z|=k} |p(z)| \leq \min_{|z|=1} |p'(z)|. \tag{12}$$

For $k = t = 1$, Corollary 2.3 reduces of the result due to Aziz [1].

For every real or complex number β with $|\beta| \leq 1$ and $k > 0$ as $s_t > 0$, we have

$$s_t|\beta| \leq |s_t + 1 + \beta| \text{ or } \left| 1 + \frac{\beta}{1+s_t} \right| \geq s_t \left| \frac{\beta}{1+s_t} \right|.$$

Therefore, from Inequality (10) and $|\beta| \leq 1$, we have for —z—=1,

$$\begin{aligned} \left| zp'(z) + \frac{n\beta}{1+s_t} p(z) \right| &\geq n s_t^{-n} \left| 1 + \frac{\beta}{1+s_t} \right| \min_{|z|=k} |p(z)| \\ &\geq n s_t^{-n+1} \left| \frac{\beta}{1+s_t} \right| \min_{|z|=k} |p(z)|. \end{aligned} \tag{13}$$

By choosing a suitable argument of β, we get

$$|p'(z)| \geq |\beta| \frac{n}{1+s_t} |p(z)| + n s_t^{-n+1} \left| \frac{\beta}{1+s_t} \right| \min_{|z|=k} |p(z)|, \tag{14}$$

for $|z| = 1$.

By letting $|\beta| \longrightarrow 1$, we get the following result which is a refinement of Inequality (3).

Corollary 2.4. *If $p(z) = a_n z^n + \sum\limits_{v=t}^{n} a_{n-v} z^{n-v}$ $1 \le t \le n$, be a polynomial of degree n, having all zeros in $|z| \le k$, $k \le 1$ then,*

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+s_t}\Big(\max_{|z|=1} |p(z)| + \frac{1}{s_t^{n-1}} \min_{|z|=k} |p(z)|\Big). \tag{15}$$

The result is optimal, and equality holds for $p(z) = (z + k)^n$.

Remark 2.2. *If $t = 1$, then $s_1 \le k$, and hence Inequality (15) improves upon the result of Mezerji [9].*

By applying Theorem 2.1 to the polynomials $p(z)$ and $M\left(\frac{z}{t}\right)^n$, where $M = \max\limits_{|z|=k}|p(z)|$, we obtain the following result.

Corollary 2.5. *If $p(z) = a_n z^n + \sum\limits_{v=t}^{n} a_{n-v} z^{n-v}$, $1 \le t \le n$, be a polynomial of degree at the most n, then*

$$\max_{|z|=1} \Big|zp'(z) + \frac{n\beta}{1+s_t} p(z)\Big| \le \frac{n}{s_t^n}\Big|1 + \frac{\beta}{1+s_t}\Big| \max_{|z|=k} |p(z)|. \tag{16}$$

This is the best possible result, with equality holding for $p(z) = \left(\frac{z}{k}\right)^n$.

Remark 2.3. *By selecting a suitable argument for β and letting $|\beta| \longrightarrow 1$ in Corollary 2.5, we obtain,*

$$\max_{|z|=k} |p'(z)| \le \frac{n}{1+s_t}\Big\{\max_{|z|=1} |p(z)| + \frac{1}{s_t^{n-1}} \max_{|z|=k} |p(z)|\Big\}. \tag{17}$$

For $t = 1$, Inequality (17) is an improvment upon the result by Mezerji [9]. If we consider $q(z) = z^n \overline{p\big(\frac{1}{\bar z}\big)}$ and $M = \max\limits_{|z|=k}|p(z)|$, then by applying Theorem 2.1 for the polynomials q(z) and Mz^n, we obtain the following result.

Corollary 2.6. *If $p(z) = a_n z^n + \sum\limits_{v=t}^{n} a_{n-v} z^{n-v}$, $1 \le t \le n$, be a polynomial of degree at most n, and $M = \max\limits_{|z|=k}|p(z)|$, $k \ge 1$, then for β with $|\beta| \le 1$,*

$$\Big|zq'(z) + \frac{n\beta k}{1+s_t} q(z)\Big| \le Mn\Big|1 + \frac{\beta k}{1+s_t}\Big| \ \ for \ \ |z| = 1, \tag{18}$$

where $q(z) = z^n \overline{p\big(\frac{1}{\bar z}\big)}$.

Let $p(z)$ be a polynomial of degree n and $M = \max\limits_{|z|=k}|p(z)|$, then the polynomial $s(z) = p(z) - \alpha M$ has no zeros in $|z| < k$ for $|\alpha| > 1$. Therefor the polynomial $T(z) = z^n \overline{s\Big(\frac{1}{\bar z}\Big)} = q(z) - \bar\alpha M z^n$ has all it's s zeros in$|z| \le \frac{1}{k}$ and $|T(z)| = \frac{1}{k^n}|s(k^2 z)|$ for $|z| = \frac{1}{k}$. By applying Theorem 2.1 to polynomials $k^n T(Z)$ and $s(k^2 z)$, then for $|z| = 1$ and $|\beta| \le 1$, and by choosing a suitabel argument of α, applying Corollary 2.10 and letting $|\alpha| \longrightarrow 1$, we get the following result,

$$\Big|zk^2 p'(k^2 z) + \frac{n\beta k}{1+s_t} p(k^2 z)\Big| + k^n\Big|zq'(z) + \frac{n\beta k}{1+s_t} q(z)\Big| \le Mn\Big\{k^n\Big|1 + \frac{\beta k}{1+s_t}\Big| + \Big|\frac{\beta k}{1+s_t}\Big|\Big\}. \tag{19}$$

Next, we prove the following theorem which is a generalization of Theorem 1.14 due to [9].

Theorem 2.2. *If* $p(z) = a_n z^n + \sum\limits_{v=t}^{n} a_{n-v} z^{n-v}$ *,* $1 \leq t \leq n$*, is a polynomial of degree* n*, having no zeros in* $|z| < k$*,* $k \geq 1$*, then for* β *with* $|\beta| \leq 1$*,*

$$\max_{|z|=k^2} \left| zp'(z) + \frac{n\beta k}{1+s_t} p(z) \right| \leq$$
$$\frac{n}{2}\left[\left\{ \left| \frac{\beta k}{1+s_t} \right| + k^n \left| 1 + \frac{\beta k}{1+s_t} \right| \right\} \max_{|z|=k} |p(z)| - \left\{ k^n \left| 1 + \frac{\beta k}{1+s_t} \right| - \left| \frac{\beta k}{1+s_t} \right| \right\} \min_{|z|=k} |p(z)| \right]. \tag{20}$$

The result is best possible and equality holds for $p(z) = \lambda_1 k^n + \lambda_2 z^n$, where $|\lambda_1| = |\lambda_2| = \frac{1}{2}$. For $t = k = 1$, Theorem 2.2 reduces to Theorem 1.3 and for $\beta = 0$ and $k = 1$, Inequality (20) reduces to the following inequality proved by Aziz and Dawood [2],

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2}\left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}.$$

3. Lemmas

We will need the following lemmas to prove our theorems.

Lemma 3.1. *If all zeros of an* n^{th} *degree polynomial,* $p(z)$ *lie in a circular region* C *and* w *is any zero of* $D_\alpha p(z)$*, then at most one of the points* w *and* α *may lie outside* C *[9].*

Lemma 3.2. *If* $p(z) = a_n z^n + \sum\limits_{v=t}^{n} a_{n-v} z^{n-v}$*,* $1 \leq t \leq n$ *, is a polynomial of degree* n *having all it's s zeros in* $|z| \leq k \leq 1$*, then on* $|z| = 1$ *for* $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$*,*

$$|q'(z)| \leq s_t |p'(z)|, \tag{21}$$

and $\frac{t}{n}\left|\frac{a_{n-t}}{n}\right| \leq k^t$*, where*

$$s_t = \frac{n|a_n|k^{2t} + t|a_{n-t}|k^{t-1}}{n|a_n|k^{t-1} + t|a_{n-k}|}. \tag{22}$$

The above lemma is due to Aziz and Rather [3].

Lemma 3.3. *If* $p(z) = a_n z^n + \sum\limits_{v=t}^{n} a_{n-v} z^{n-v}$*;* $1 \leq t \leq n$ *is a polynomial of degree* n *having all it's s zeros in* $|z| \leq k \leq 1$*, then on* $|z| = 1$*,*

$$|zp'(z)| \geq \frac{n}{1+s_t} |p(z)|. \tag{23}$$

Proof . Since

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)},$$

we have

$$q'(z) = nz^n \overline{p\left(\frac{1}{\bar{z}}\right)} - z^{n-z} \overline{p'\left(\frac{1}{\bar{z}}\right)},$$

equivalently

$$zq'(z) = nz^n \overline{p\left(\frac{1}{\bar{z}}\right)} - z^{n-1} \overline{p'\left(\frac{1}{\bar{z}}\right)}.$$

Which implies for $|z| = 1$,

$$|q'(z)| = |np(z) - zp'(z)|. \tag{24}$$

Now using the Inequalities (21) and (12) for $|z| = 1$ we get

$$\begin{aligned}|np(z)| = |np(z) - zp'(z) + zp'(z)| &\leq |np(z) - zp'(z)| + |zp'(z)| \\ &= |q'(z)| + |zp'(z)| \leq (s_t + 1)|p'(z)|.\end{aligned}$$

This proves Lemma 3.3.

4. Proofs of the Theorems

Proof of Theorem 2.1.

Since $|\alpha f(z)| < |f(z) \leq |F(z)|$ (1), for $|\alpha| < 1$, $|z| = k$, by using Rouche's s theorem we can say that for $|\alpha| < 1$, $F(z) - \alpha f(z)$ and $F(z)$ have the same number of zeros in $|z| < k$. By Inequality (1), for $|z| = k$, any zero of $F(z)$ that lying on $|z| = k$, is also a zero of $f(z)$. Therefore, $F(z) - \alpha f(z)$ has all it's s zeros in $|z| \leq k$, and using the Lemma 3.3, we have for $|z| = 1$,

$$|zF'(z) - \alpha zf'(z)| \geq \frac{n}{1 + s_t}|F(z) - \alpha f(z)|. \tag{25}$$

For any β with $|\beta| < 1$, we have for $|z| = 1$,

$$\begin{aligned}|zF'(z) - \alpha zf'(z)| &\geq \frac{n}{1 + s_t}|F(z) - \alpha f(z)| \\ &> \frac{n|\beta|}{1 + s_t}|F(z) - \alpha f(z)|.\end{aligned} \tag{26}$$

This implies

$$T(z) = \big(zF'(z) - \alpha zf'(z)\big) + \beta\frac{n}{1 + s_t}\big(F(z) - \alpha f(z)\big), \tag{27}$$

for $|z| = 1$, which is equivalent to

$$T(z) = zF'(z) + \frac{n\beta}{1 + s_t}F(z) - \alpha\Big(zf'(z) + \frac{n\beta}{1 + s_t}f(z)\Big) \neq 0, \tag{28}$$

for $|z| = 1$.

We conclude that,

$$\Big|zf'(z) + \frac{n\beta}{1 + s_t}f(z)\Big| \leq \Big|zF'(z) + \frac{n\beta}{1 + s_t}F(z)\Big|, \quad for\ |z| = 1. \tag{29}$$

If Inequality (29) is violated, there exists z_0 with $|z_0| = 1$ such that

$$\Big|z_0 f'(z_0) + \frac{n\beta}{1 + s_t}f(z_0)\Big| \ > \Big|z_0 F'(z_0) + \frac{n\beta}{1 + s_t}F(z_0)\Big|, \quad for\ |z| = 1. \tag{30}$$

We take

$$\alpha = \frac{z_0 F'(z_0) + \frac{n\beta}{1+s_t}F(z_0)}{z_0 f'(z_0) + \frac{n\beta}{1+s_t}f(z_0)}, \tag{31}$$

with this choice of α, we find $T(z_0) = 0$ for $|z_0| = 1$, which contradicts (28). For β with $|\beta| = 1$, Inequality (29) follows by continuity.

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Proof of Theorem 2.2.

Let $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, and $m = \min\limits_{|z|=k}|p(z)|$. If $m = 0$, by applying Theorem 2.1 to the polynomials $p(k^2 z)$ and $k^n q(z)$, we obtain the following result for $|z| = 1$ and $|\beta| \leq 1$,

$$\Big|zk^2 p'(k^2 z) + \frac{n\beta k}{1 + s_t}p(k^2 z)\Big| \leq k^n\Big|zq'(z) + \frac{n\beta k}{1 + s_t}q(z)\Big|. \tag{32}$$

Combining (32) and (19),confirms the theorem. Next, assume $m \neq 0$. For α with $|\alpha| < 1$, consider the polynomial $G(z) = p(z) - m\alpha$, which has no zeros in $|z| < k$. Define $H(z) = z^n \overline{G(\frac{1}{\bar{z}})} = q(z) - m\bar{\alpha}z^n$. All zeros of $H(z)$ are contained whithin $|z| \leq \frac{1}{k}$ and $k^n|H(z)| = |G(k^2 z)|$ for $|z| = \frac{1}{k}$,

$$\begin{aligned}&\Big|zk^2p'(k^2z) + \frac{n\beta k}{1+s_t}p(k^2z) - \frac{n\beta k}{1+s_t}m\alpha\Big| \\ &\leq k^n\Big|zq'(z) + \frac{n\beta k}{1+s_t}q(z) - mn\bar{\alpha}z^n\Big(1 + \frac{\beta k}{1+k}\Big)\Big|. \end{aligned} \tag{33}$$

By selecting an appropriate argument for α on the right-hand side of (33) and applying Corollary 2.2, we derive

$$\begin{aligned}&\Big|zk^2p'(k^2z) + \frac{n\beta k}{1+s_t}p(k^2z)\Big| - \frac{nk}{1+s_t}m|\alpha\beta| \\ &\leq k^n\Big\{\Big|zq'(z) + \frac{n\beta k}{1+s_t}q(z)\Big| - mn|\bar{\alpha}|\Big|1 + \frac{\beta k}{1+s_t}\Big|\Big\}. \end{aligned} \tag{34}$$

As $|\alpha| \longrightarrow 1$, the following holds,

$$\begin{aligned}&\Big|zk^2p'(k^2z) + \frac{n\beta k}{1+s_t}p(k^2z)\Big| \leq k^n\Big|zq'(z) + \frac{n\beta k}{1+s_t}q(z)\Big| \\ &- mn\Big\{k^n\Big|1 + \frac{\beta k}{1+s_t}\Big| - \Big|\frac{\beta k}{1+s_t}\Big|\Big\}, \end{aligned} \tag{35}$$

which implies for $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}2&\Big|zk^2p'(k^2z) + \frac{n\beta k}{1+s_t}p(k^2z)\Big| \leq \Big|zk^2p'(k^2z) + \frac{n\beta k}{1+s_t}p(k^2z)\Big| \\ &+ k^n\Big|zq'(z) + \frac{n\beta k}{1+s_t}q(z)\Big| - mn\Big\{k^n|1 + \frac{\beta k}{1+s_t}\Big| - \Big|\frac{\beta k}{1+s_t}\Big|\Big\}, \end{aligned} \tag{36}$$

this in conjanction with Inequality (19) gives for $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}&2\Big|zk^2p'(k^2z) + \frac{n\beta k}{1+s_t}p(k^2z)\Big| \leq \\ &n\Big[\Big\{\Big|\frac{\beta k}{1+s_t}\Big| + k^n\Big|1 + \frac{\beta k}{1+s_t}\Big|\Big\}M - \Big\{k^n\Big|1 + \frac{\beta k}{1+k}\Big| - \Big|\frac{\beta k}{1+k}\Big|\Big\}m\Big], \end{aligned} \tag{37}$$

and therefore the theorem follows.

Acknowledgement The authors wish to sincerely thank the referees, for the careful reading of the paper and for the helpful suggestions and comments.

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