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ON SEPARATION PROPERTIES IN SOFT TOPOLOGICAL SPACES

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Abstract. In this paper, we explore separation axioms T_i, for $i = 0, 1, 2$, within the framework of soft topological spaces, utilizing the concept of soft points as defined in [16]. We define T_0 and T_1 in terms of the mapping $\tau_{\mathcal{F}}$ and establish that a soft space is T_1 iff the soft point is soft closed, assuming the soft topology is enriched. Furthermore, we provide a new characterization of T_2 soft spaces, contributing to the understanding of separation properties in soft topology.

Keywords: Soft sets, Soft topology, Soft product, Soft separation axioms.

AMS Subject Classification: 54A40

1. Introduction

Soft set theory, first introduced by Molodtsov [11] in 1999, is a mathematical framework aimed at addressing uncertainty, imprecision, and vagueness in decision-making scenarios. Unlike conventional set theories that struggle with ambiguity, soft set theory offers a more adaptable approach by linking parameters to elements, which facilitates a more detailed representation of information. This theory has been extensively researched and applied across various disciplines, including algebra, topology, and decision analysis. Soft topological spaces [14] expand on classical topology by incorporating open and closed sets, as well as continuity within a soft set context. In topology, separation axioms are criteria applied to a topological space to distinguish and separate points or sets, which aid in categorizing different types of topological spaces based on their separation characteristics. Owing to the various definitions of soft points and the concepts of soft continuity, soft separation axioms also admit multiple interpretations. This diversity has prompted several studies to explore soft separation axioms from different perspectives. For example, T_i- soft spaces have been defined using distinct crisp points ([9],[14], [16]) as well as distinct soft points ([4],[8],[15]) respectively. The authors in [5] introduced two new soft belonging relations and used these relations to define novel soft separation axioms. El-Shafei and Al-Shami utilized these relations to establish e-soft T_i spaces [1] and w-soft T_i spaces [6]. In [7], each soft set $\mathcal{F} : \mathfrak{A} \longrightarrow P(\mathfrak{X})$ is linked to a mapping $\tau_{\mathcal{F}} : \mathfrak{X} \longrightarrow P(\mathfrak{A})$. It was also

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demonstrated that $\mathcal{F}$ is completely characterized by the mapping $\tau_{\mathcal{F}}$. In this paper, we study separation axioms $T_i, (i = 0, 1, 2)$ in the context of soft topological spaces, based on the definition of soft points from [16]. We define T_0 and T_1 in terms of the mapping $\tau_{\mathcal{F}}$. We demonstrate that a soft space is T_1 iff the soft point is soft closed, provided that the soft topology is enriched. Finally, we give a new characterization of T_2 soft spaces.

2. Preliminaries

2.1. **Soft sets.** Let $\mathfrak{X}$ represent a universe set, $P(\mathfrak{X})$ be the power set of $\mathfrak{X}$, and $\mathfrak{A}$ denotes the set of all possible parameters related to $\mathfrak{X}$. A Soft set (**S-set**) is defined as follows:

Definition 2.1. [11] A pair $(\mathcal{F}, \mathfrak{A})$ is referred to a **S-set** over $\mathfrak{X}$, where $\mathcal{F} : \mathfrak{A} \longrightarrow P(\mathfrak{X})$ is a set-valued mapping.

The set $\mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ represents the collection of all **S-set**s over $\mathfrak{X}$. For simplicity, we write $\mathcal{F}$ instead of the pair $(\mathcal{F}, \mathfrak{A})$.

Definition 2.2. [8, 13] Let $\mathcal{F}, \mathcal{G} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$, then

(1) $\mathcal{F} \sqsubseteq \mathcal{G}$, if $F(\alpha) \subseteq G(\alpha)$, for all $\alpha \in \mathfrak{A}$.
(2) $\mathcal{F} = \mathcal{G}$, if $\mathcal{F}(\alpha) = \mathcal{G}(\alpha)$, for all $\alpha \in \mathfrak{A}$.
(3) $(\mathcal{F} \sqcap \mathcal{G})(\alpha) = \mathcal{F}(\alpha) \cap \mathcal{G}(\alpha)$, for all $\alpha \in \mathfrak{A}$.
(4) $(\mathcal{F} \sqcup \mathcal{G})(\alpha) = \mathcal{F}(\alpha) \cup \mathcal{G}(\alpha)$, for all $\alpha \in \mathfrak{A}$.
(5) $\mathcal{F}^c(\alpha) = \mathfrak{X} - F(\alpha)$, for all $\alpha \in \mathfrak{A}$.
(6) $\mathcal{F} \times \mathcal{G} \in \mathcal{SS}(\mathfrak{X} \times \mathfrak{X}, \mathfrak{A})$, defined as $(\mathcal{F} \times \mathcal{G})(\alpha) = \mathcal{F}(\alpha) \times \mathcal{G}(\alpha)$.

Definition 2.3. [10] Let $\mathfrak{X}, \mathfrak{Y}$ be two nonempty sets and $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a mapping, then

(1) f maps $\mathcal{F} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ into $\overrightarrow{f} F \in \mathcal{SS}(\mathfrak{Y}, \mathfrak{A})$ such that $\overrightarrow{f} \mathcal{F}(\alpha) = f(\mathcal{F}(\alpha)), \forall \alpha \in \mathfrak{A}$.
(2) The inverse image of $G \in \mathcal{SS}(\mathfrak{Y}, \mathfrak{A})$ is $\overleftarrow{f} G \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ where $\overleftarrow{f} \mathcal{G}(\alpha) = f^{-1}(G(\alpha))$, for all $\alpha \in \mathfrak{A}$.

In [7], every **S-set** $\mathcal{F} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ is linked to a mapping $\tau_{\mathcal{F}} : \mathfrak{X} \longrightarrow P(\mathfrak{A})$ where

$$\tau_{\mathcal{F}}(x) = \{\alpha \in \mathfrak{A} : x \in \mathcal{F}(\alpha)\}$$

Also, the author showed that $\mathcal{F}$ is fully defined through the mapping $\tau_{\mathcal{F}}$. Below, the relationship between $\tau_{\mathcal{F}}$ and soft operations is examined.

Lemma 2.1. [7] Let $\mathcal{F}, \mathcal{G} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$, then for all $x, y \in \mathfrak{X}$

(1) $\tau_{\widehat{\phi}}(x) = \phi$ and $\alpha_{\widehat{\mathfrak{X}}}(x) = \mathfrak{A}$,
(2) $\mathcal{F} \sqsubseteq \mathcal{G} \Rightarrow \tau_{\mathcal{F}}(x) \subseteq \tau_{\mathcal{G}}(x)$,
(3) $\tau_{\mathcal{F} \sqcap \mathcal{G}}(x) = \tau_{\mathcal{F}}(x) \cap \tau_{\mathcal{G}}(x)$,
(4) $\tau_{\mathcal{F} \sqcup \mathcal{G}}(x) = \tau_{\mathcal{F}}(x) \cup \tau_{\mathcal{G}}(x)$,
(5) $\tau_{\mathcal{F}^c}(x) = \mathfrak{A} - \tau_{\mathcal{F}}(x)$,
(6) $\tau_{\mathcal{F} \times \mathcal{G}}(x, y) = \tau_{\mathcal{F}}(x) \cap \tau_{\mathcal{G}}(y)$

2.2. **Soft topology.** By $\widehat{\mathcal{U}} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$, we mean a constant **S-set** (or, Stable soft set [5]) over $\mathfrak{X}$ taking value $\mathcal{U} \in P(\mathfrak{X})$, that is $\widehat{\mathcal{U}}(\alpha) = \mathcal{U}$, for all $\alpha \in \mathfrak{A}$.

Definition 2.4. [14] A soft topology (**S-top**) over $\mathfrak{X}$ is a collection $T \subseteq \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ satisfies the following:

(ST1) $\widehat{\phi}, \widehat{\mathfrak{X}} \in T$,
(ST2) $\mathcal{F}, \mathcal{G} \in T \Rightarrow \mathcal{F} \sqcap \mathcal{G} \in T$,
(ST3) If $\{\mathcal{F}_i, i \in I\} \subseteq T$, then $\bigsqcup_i \mathcal{F}_i \in T$.

The triplet $(\mathfrak{X}, T, \mathfrak{A})$ is referred to as a **S-top** space over $\mathfrak{X}$, which the elements of T are called soft open sets (**S-opS**) in $\mathfrak{X}$.

Definition 2.5. *$\mathcal{F} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ is called a soft closed set (**S-clS**) in $\mathfrak{X}$ if $\mathcal{F}^c$ is a **S-opS** in $\mathfrak{X}$.*

Definition 2.6. *$T = \{\widehat{\phi}, \widehat{\mathfrak{X}}\}$ and $\Delta = \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ are called indiscrete **S-top** and discrete **S-top** on $\mathfrak{X}$, respectively.*

In **S-top** spaces, the concept of soft points can take on multiple meanings. Below, we illustrate a few of these variations.

Definition 2.7.
1. *For $\alpha \in \mathfrak{A}, x \in \mathfrak{X}$, a soft point over $\mathfrak{X}$ is a **S-set** over $\mathfrak{X}$ such that*
$$\alpha_x(\beta) = \begin{cases} \{x\} & \text{if } \alpha = \beta \\ \phi & \text{otherwise} \end{cases}$$
*for all $\beta \in \mathfrak{A}$. Two soft points α_x, α_y are distinct if $x \neq y$. The soft point α_y belongs to the **S-set** $\mathcal{G}$, denoted by $\alpha_y \in \mathcal{G}$, if $\{y\} \subseteq \mathcal{G}(\alpha)$*[17].
2. *The **S-set** $\mathcal{F}$ is called a soft point, denoted by $\alpha_{\mathcal{F}}$, if $\mathcal{F}(\alpha) \neq \phi$ and $\mathcal{F}(\beta) = \phi, \forall \beta \in \mathfrak{A} - \{\alpha\}$. We write $\alpha_{\mathcal{F}} \neq \alpha_{\mathcal{G}} \iff \mathcal{F} \neq \mathcal{G}$ and $\alpha_{\mathcal{F}} \in \mathcal{H}$ if $\mathcal{F}(\alpha) \subseteq \mathcal{H}(\alpha)$* [10].
3. *We say $x \in \mathcal{F}$,a soft element of $\mathcal{F}$, whenever $x \in \mathcal{F}(\alpha), \forall \alpha \in \mathfrak{A}$. Note that, $x \notin \mathcal{F}$, if $x \notin \mathcal{F}(\alpha)$ for some $\alpha \in \mathcal{A}$* [14].

Remark: We observe the following implications:
- $x \in \mathcal{G} \Rightarrow \alpha_x \in \mathcal{G}$.
- $\alpha_{\mathcal{F}} \in \mathcal{G} \Rightarrow \alpha_x \in \mathcal{G}, if x \in \mathcal{F}(\alpha)$.

Throughout this paper, $\mathcal{SP}(\mathfrak{X}, \mathfrak{A})$ represents the collection of soft points, as outlined in item (i) of the definition provided above.

Definition 2.8. [3] $\mathcal{F} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ is called pseudo-constant (**PC-**) set if $\mathcal{F}(\alpha) = \phi$ or $\mathfrak{X}$, $\forall \mathcal{F} \in \mathfrak{A}$. We say $(\mathfrak{X}, T, \mathfrak{A})$ is an enriched **S-top** space if T contains all **PC-** sets.

Definition 2.9. [13] Let $(\mathfrak{X}, T, \mathfrak{A})$ be a **S-top** space over $\mathfrak{X}$. A soft set $\mathcal{F}$ is called a soft neighborhood (**S-nghd**) of the soft point α_x if $\exists\, \mathcal{G} \in T$ such that $\alpha_x \in \mathcal{G} \sqsubseteq \mathcal{F}$. We denote the set of all **S-nghd**s of α_x by $\mathbf{SN}(\alpha_x)$.

Definition 2.10. [13] A function $f : (\mathfrak{X}, T, \mathfrak{A}) \longrightarrow (\mathfrak{Y}, D, \mathfrak{A})$ is called a soft continuous (**S-con**) function at α_x if for each $\mathcal{H} \in \mathbf{SN}(\overrightarrow{f}\alpha_x)$ there exists $\mathcal{F} \in \mathbf{SN}(\alpha_x)$ such that $\overrightarrow{f}\mathcal{F} \sqsubseteq \mathcal{H}$. We say f is a **S-con** function if it is a **S-con** mapping for all α_x.

Theorem 2.1. [15] Let $f : (\mathfrak{X}, T, \mathfrak{A}) \longrightarrow (\mathfrak{Y}, D, \mathfrak{A})$ be a (**S-con**) function. The following statements are therefore equivalent:
1. f is a (**S-con**) function
2. for each $\mathcal{G} \in D$, $\overleftarrow{f}\mathcal{G} \in T$.
3. for each **S-clS**, $\mathcal{H} \in D$, $\overleftarrow{f}\mathcal{H}$ is a **S-clS** $\in T$.

3. Main Results

Definition 3.1. *Let $\mathfrak{Z} \subseteq \mathfrak{X}$, then $\mathfrak{Z} \in \mathcal{S}(\mathfrak{X}, \mathfrak{A})$ is a soft set defined by:*
$$\tau_{\mathfrak{Z}}(x) = \begin{cases} \mathfrak{A} & \text{if } x \in \mathfrak{Z} \\ \phi & \text{otherwise} \end{cases}$$
for all $x \in \mathfrak{X}$

Definition 3.2. [15] Let $(\mathfrak{X}, T, \mathfrak{A})$ be a **S-top** space over $\mathfrak{X}$ and $\alpha_x, \alpha_y \in \mathcal{SP}(\mathfrak{X}, \mathfrak{A})$ where $\alpha_x \neq \alpha_y$. If there exists $\mathcal{F} \in T$ such that: **case I:**$\alpha_x \in \mathcal{F}, \alpha_y \notin \mathcal{F}$ or **case II:**$\alpha_x \notin \mathcal{F}, \alpha_y \in \mathcal{F}$, then $(\mathfrak{X}, T, \mathfrak{A})$ is called a T_0-**S-top** space.

Proposition 3.1. *$(\mathfrak{X}, T, \mathfrak{A})$ is a T_0-**S-top** space $\iff$ for each $x \neq y \in \mathfrak{X}$, there exists $\mathcal{F} \in T$ such that $\tau_{\mathcal{F}}(x) \neq \tau_{\mathcal{F}}(y)$.*

Proof. $(\Longrightarrow)$ Let $\alpha_x \neq \alpha_y$ be soft points and $\mathcal{F} \in T$ such that: **case I:** $\alpha_x \in \mathcal{F}, \alpha_y \notin \mathcal{F}$, then $x \in \mathcal{F}(\alpha), y \notin \mathcal{F}(\alpha)$. That is, $\alpha \in \tau_{\mathcal{F}}(x), \alpha \notin \tau_{\mathcal{F}}(y)$ which means that $\forall x \neq y, \tau_{\mathcal{F}}(x) \neq \tau_{\mathcal{F}}(y)$. Similarly, we arrive at the same result when examining **case II.**
$(\Longleftarrow)$ For $x \in \mathfrak{X}, y \in \mathfrak{X} - \{x\}$, the soft points α_x, α_y are distinct, and then there is $\mathcal{F} \in T$ such that $\tau_{\mathcal{F}}(x) \neq \tau_{\mathcal{F}}(y)$. Since $\alpha \in \mathfrak{A}$ is arbitrary, then $\alpha \in \tau_{\mathcal{F}}(x), \alpha \notin \tau_{\mathcal{F}}(y) \Longrightarrow \alpha_x \in \mathcal{F}, \alpha_y \notin \mathcal{F}$ which is **case I**. A similar argument verifies **case II**. □

Example 3.1. *Let $\mathfrak{A} = \{\alpha, \beta\}, \mathfrak{X} = \{x, y\}$ and $(\mathfrak{X}, T, \mathfrak{A})$ be a **S-top** space over $\mathfrak{X}$, where $T = \{\widehat{\phi}, \widehat{\mathfrak{X}}, \mathcal{F}, \mathcal{G}, \mathcal{H}\}$ such that*

$$\mathcal{F}(\alpha) = \mathfrak{X},\ \mathcal{F}(\beta) = \{y\},\ \mathcal{G}(\alpha) = \{x\},\ \mathcal{G}(\beta) = \mathfrak{X},\ \mathcal{H}(\alpha) = \{x\},\ \mathcal{H}(\beta) = \{y\}.$$

Then, according to Definition 3.2,$(\mathfrak{X}, T, \mathfrak{A})$ is a T_0-**S-top** space [15]. Now, to verify Proposition 3.1, consider the corresponding mappings:

$$\tau_{\mathcal{F}} = \{(x, \{\alpha\}), (y, \mathfrak{A})\},\ \tau_{\mathcal{G}} = \{(x, \mathfrak{A}), (y, \{\beta\})\},\ \tau_{\mathcal{H}} = \{(x, \{\alpha\}), (y, , \{\beta\})\},$$

obviously, for each $x \neq y \in \mathfrak{X}$ there exists $\mathcal{H} \in T$ such that $\tau_{\mathcal{H}}(x) = \{\alpha\} \neq \{\beta\} = \tau_{\mathcal{H}}(y)$.

Definition 3.3. [15] Let $(\mathfrak{X}, T, \mathfrak{A})$ be a **S-top** space over $\mathfrak{X}$ and $\alpha_x, \beta_y \in \mathcal{SP}(\mathfrak{X}, \mathfrak{A})$ where $\alpha_x \neq \alpha_y$. If there exist $\mathcal{F}, \mathcal{G} \in T$ such that: **case I:** $\alpha_x \in \mathcal{F}, \alpha_y \notin \mathcal{F}$ and **case II:** $\alpha_x \notin \mathcal{G}, \alpha_y \in \mathcal{G}$, then $(\mathfrak{X}, T, \mathfrak{A})$ is called a T_1-**S-top** space.

The authors in [15] demonstrated, in the following lemma, that if every soft point of a **S-top** space is a closed set, then the space is a T_1-**S-top** space. Additionally, they provided a counterexample showing that the converse does not hold in general. In Theorem 3.2, we introduce an additional condition on the soft space under which the converse of Lemma 3.1 is valid.

Lemma 3.1. [15] If every $\alpha_x \in \mathcal{SP}(\mathfrak{X}, \mathfrak{A})$ is a **S-clS**, then $(\mathfrak{X}, T, \mathfrak{A})$ is called a T_1-**S-top** space.

Theorem 3.1 ([17], Theorem 5.16.)**.** *A soft set in $(\mathfrak{X}, T, \mathfrak{A})$ is **S-opS** iff it is a **S-nghd** of each of its soft points.*

Theorem 3.2. *Let $(\mathfrak{X}, T, \mathfrak{A})$ be an enriched **S-top** space, then*

$$(\mathfrak{X}, T, \mathfrak{A}) \textit{ is called a } T_1 - \textbf{\textit{S-top}} \textit{ space} \iff \textit{each } \alpha_x \textit{ is a } \textbf{\textit{S-clS.}}$$

Proof. $(\Longleftarrow)$ follows directly from Lemma 3.1.
$(\Rightarrow)$ Let $x \in \mathfrak{X}, \alpha \in \mathfrak{A}$. To prove that α_x is a **S-clS.**, we show that α_x^c defined by

$$\alpha_x^c(\beta) = \begin{cases} \mathfrak{X} - \{x\} & \text{if } \beta = \alpha \\ \mathfrak{X} & \text{otherwise} \end{cases}$$

is a **S-opS.** Let $y \in\in \mathfrak{X} - \{x\}$. Then $\alpha_x \neq \alpha_y \Rightarrow$ there exists $\mathcal{G}_y \in T$ such that $\alpha_y \in \mathcal{G}_y$ and $\alpha_x \notin \mathcal{G}_y$. Thus, we have $\alpha_y \in \mathcal{G} \sqsubseteq \alpha_x^c$. Assume that $\mathcal{H} \in T$ is a **PC**-set defined by $\mathcal{H}(\alpha) = \phi$ and $\mathcal{H}(\beta) = \mathfrak{X}$, $\forall \beta \in \mathfrak{A} - \{\alpha\}$. Now, we get

$$(\sqcup_{y \in \{x\}^c} \mathcal{G}_y)(\alpha) \cup \mathcal{H}(\alpha) = \{x\}^c, \text{ and } (\sqcup_{y \in \{x\}^c} \mathcal{G}_y)(\beta) \cup \mathcal{H}(\beta) = \mathfrak{X}.$$

Clearly $\alpha_x^c = \sqcup_{y \in \{x\}^c} \mathcal{G}_y \sqcup \mathcal{H}$, showing that α_x is a **S-clS** for every $x \in \mathfrak{X}$. □

Theorem 3.3. *Let $(\mathfrak{X}, T, \mathfrak{A})$ be a **S-top** space over $\mathfrak{X}$. Then $\widehat{\{x\}}$ is a **S-clS** $\iff$ for each $x \neq y \in \mathfrak{X}$ there exist $\mathcal{F}, \mathcal{G} \in T$ such that*

$$\tau_{\mathcal{F}}(x) = \mathfrak{A}, \tau_{\mathcal{F}}(y) = \phi \text{ and } \tau_{\mathcal{G}}(y) = \mathfrak{A}, \tau_{\mathcal{G}}(x) = \phi. \tag{1}$$

Proof. ($\Longrightarrow$) For $x \in \mathfrak{X}, y \in \mathfrak{X} - \{x\}$, the soft points α_x, α_y are distinct. The soft sets $\widehat{\{x\}}^c, \widehat{\{y\}}^c$ are **S-opS**s. By Definition 3.1, we have, $\tau_{\widehat{\{x\}}^c}(x) = (\tau_{\widehat{\{x\}}}(x))^c = \phi$, $\tau_{\widehat{\{x\}}^c}(y) = (\tau_{\widehat{\{x\}}}(y))^c = \mathfrak{A}$ and $\tau_{\widehat{\{y\}}^c}(x) = (\tau_{\widehat{\{y\}}}(x))^c = \mathfrak{A}$, $\tau_{\widehat{\{y\}}^c}(y) = (\tau_{\widehat{\{y\}}}(y))^c = \phi$.

($\Longleftarrow$) For $x \in \mathfrak{X}, y \in \mathfrak{X} - \{x\}$, the soft points α_x, α_y are distinct, and then $\exists\, \mathcal{F}, \mathcal{G} \in T$ satisfy condition (1). Since $\alpha \in \mathfrak{A}$ is arbitrary, then $\alpha \in \tau_{\mathcal{F}}(x), \alpha \notin \tau_{\mathcal{F}}(y) \Longrightarrow \alpha_x \in \mathcal{F}, \alpha_y \notin \mathcal{F}$ which is **case I** in Definition 3.3. A similar argument verifies **case II**. □

The following example demonstrates that an enriched **S-top** space may not qualify as a T_1-**S-top** space as defined in Definitions 3.3 and 4.1.

Example 3.2. *Let $\mathfrak{A} = \{\alpha, \beta\}, \mathfrak{X} = \{x, y\}$ and $(\mathfrak{X}, T, \mathfrak{A})$ be an enriched **S-top** space over $\mathfrak{X}[2]$, where $T = \{\widehat{\phi}, \widehat{\mathfrak{X}}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}\}$ such that*

$$\tau_{\mathcal{F}} = \{(x, \{\alpha\}), (y, \mathfrak{A})\},\ \tau_{\mathcal{G}} = \{(x, \phi), (y, \{\beta\}),$$

$$\tau_{\mathcal{H}} = \{(x, \{\beta\}), (y, \{\beta\}),\ \tau_{\mathcal{K}} = \{(x, \{\alpha\}), (y, \{\alpha\}\}.$$

Then, according to Definition 3.3, Theorem 3.2 implies that $(\mathfrak{X}, T, \mathfrak{A})$ is not T_1-**S-top** space . Also, Theorem 3.3 implies that condition (1) is not hold and cosequently $(\mathfrak{X}, T, \mathfrak{A})$ is not T_1-**S-top** space according to Definition 4.1.

Theorem 3.4. *Let $(\mathfrak{X}, T, \mathfrak{A})$ be an enriched **S-top** space over $\mathfrak{X}$. Then each $\alpha_x \in \mathcal{SP}(\mathfrak{X}, \mathfrak{A})$ and each $\widehat{\{x\}}, \forall x \in \mathfrak{X}$, is a **S-clS** $\iff$ for each $x \neq y \in \mathfrak{X}$ there exists $\mathcal{F}, \mathcal{G} \in T$ satisfy condition(1).*

Proof. ($\Longleftarrow$) By observing, that T includes all the **PC**-sets, and as a result, each **PC**-set is also a **S-clS**. Let α_x be a soft point, and $\mathcal{H} \in T$ be a **PC**-set defined as: $\mathcal{H}(\alpha) = \mathfrak{X}$, $\mathcal{H}(\beta) = \phi$ for all $\beta \in \mathfrak{A} - \{\alpha\}$. Then clearly, $\alpha_x = \widehat{\{x\}} \sqcap \mathcal{H}$. By Theorem 3.3, since α_x is an intersection of **S-clS**s, it must also be **S-clS**.

($\Longrightarrow$) The proof used for Theorem 3.3 can be applied here to demonstrate that condition (1) is satisfied.

□

Notation: For a nonempty set $\mathfrak{X}$, define $\mathcal{E} = \{(x, x) \in \mathfrak{X} \times \mathfrak{X}\}$.

Theorem 3.5. *Let $(\mathfrak{X}, T, \mathfrak{A})$ be a **S-top** space. Then $\widehat{\mathcal{E}}$ is a **S-clS** in $(\mathfrak{X} \times \mathfrak{X}, T \times \Delta, \mathfrak{A}) \iff \widehat{\{x\}}$ is a **S-clS** in $(\mathfrak{X}, T, \mathfrak{A})$, $\forall x \in \mathfrak{X}$.*

Proof. ($\Longrightarrow$) For all $x \in \mathfrak{X}$, it is sufficient to demonstrate that $\{x\}^c$ is a soft open in $\mathfrak{X}$. Let $y \in \mathfrak{X} - \{x\}, \alpha_y \in \{x\}^c$. Then $x \neq y$, that is $(x, y) \in \mathfrak{X} \times \mathfrak{X} - \{x\}$. Definition 3.1 implies that $\alpha_{(x,y)} \in \mathfrak{Z} = \mathfrak{X} \times \mathfrak{X} - \{x\}$, and consequently, there exist $\mathcal{F} \in T, \mathcal{G} \in \Delta$ such that $\alpha_{(x,y)} \in \mathcal{F} \times \mathcal{G} \sqsubseteq \mathfrak{Z}$. Hence

$$\alpha \in \tau_{\mathcal{F} \times \mathcal{G}}(x, y) = \tau_{\mathcal{F}}(x) \cap \tau_{\mathcal{G}}(y) \subseteq \tau_{\mathcal{G}}(y).$$

Then $\alpha \in \tau_{\mathcal{G}}(y)$ implies $\alpha_y \in \mathcal{G}$. Furthermore, we have

$$\tau_{\mathcal{F} \times \mathcal{G}}(x, x) = \tau_{\mathcal{F}}(x) \cap \tau_{\mathcal{G}}(x) \subseteq \mathfrak{Z}(x, x) = \phi. \quad [\text{Since } (x, x) \in \mathcal{E}].$$

Since $\tau_{\mathcal{G}}(x) \neq \phi$, then $\tau_{\mathcal{F}}(x) = \phi \Longrightarrow \mathcal{F} \sqsubseteq \{x\}^c$. Thus $\alpha_y \in \mathcal{F} \sqsubseteq \{x\}^c$. In view of Theorem 3.1, the soft set $\{x\}^c$ is soft open.

($\Longleftarrow$) Let $\alpha_{(x,y)} \in \mathfrak{Z} = \mathfrak{X} \times \mathfrak{X} - \mathcal{E}$, then $x \neq y$. Since $\{y\}^c$ is soft open, then $\alpha_y \in \mathcal{F} \sqsubseteq \mathfrak{Z}$, where $\mathcal{F} \in T$. By hypotheses, consider $\mathcal{F} \times \{y\} \in T \times \Delta$. We obtain

$$\tau_{\mathcal{F}\times\{y\}}(x,y) = \tau_{\mathcal{F}}(x) \cap \tau_{\{y\}}(y) = \tau_{\mathcal{F}}(x).$$

This implies that $\alpha_{(x,y)} \in \mathcal{F} \times \{y\}$. Now observe that $\mathcal{F} \sqsubseteq \{y\}^c \Longrightarrow \tau_{\mathcal{F}}(y) = \phi$. This, along with the fact that $\tau_y(x) = \phi$ when $x \neq y$, verifying that

$$\tau_{\mathcal{F}\times\{y\}}(z,z) = \tau_{\mathcal{F}}(z) \cap \tau_{\{y\}}(z) = \phi.$$

therefore, $\alpha_{(x,y)} \in \mathcal{F} \times \{y\} \sqsubseteq \mathfrak{Z}$, showing that $\mathfrak{Z} = \mathfrak{X} \times \mathfrak{X} - \mathcal{E}$ is soft open. □

Proposition 3.2. *$(\mathfrak{X}, T, \mathfrak{A})$ is a T_1- **S-top** space if $\widehat{\{x\}}$ is a **S-clS** ,$\forall x \in \mathfrak{X}$.*

Proof. Suppose that $\widehat{\{x\}}$ is a **S-clS** ,$\forall x \in \mathfrak{X}$ then $\widehat{\{x\}}^c$ is a **S-opS**. Let $y \in \{x\}^c$, then $\alpha_x \neq \alpha_y$ and $\alpha_y \in \widehat{\{x\}}^c, \alpha_x \notin \widehat{\{x\}}^c$. Similarly, $\widehat{\{y\}}^c \in T$ such that $\alpha_x \in \widehat{\{y\}}^c, \alpha_y \notin \widehat{\{y\}}^c$. Thus $(\mathfrak{X}, T, \mathfrak{A})$ is a T_1- **S-top** space . □

In general, the converse of Proposition 3.2 is not true.

Example 3.3. *Let $\mathfrak{X} = \{x, y\}, \mathfrak{A} = \{\alpha, \beta\}$ and $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathcal{SS}(\mathfrak{X}, \mathfrak{A})$ where,*

$$\tau_{\mathcal{F}} = \{(x, \{\alpha\}), (y, \{\beta\})\},\ \tau_{\mathcal{G}} = \{(x, \{\beta\}), (y, \{\alpha\})\}$$

*be a T_1-**S-top** space over $\mathfrak{X}$ where $T = \{\widehat{\phi}, \widehat{\mathfrak{X}}, \mathcal{F}, \mathcal{G}\}$* [16].
*Now, since $\tau_{\widehat{\{x\}}^c}(x) = \phi$, $\tau_{\widehat{\{x\}}^c}(y) = \mathfrak{A}$, then $\widehat{\{x\}}^c \in T$. That is, $\widehat{\{x\}}$ is not **S-clS**.*

Definition 3.4. [4] Let $(\mathfrak{X}, T, \mathfrak{A})$ be a **S-top** space over $\mathfrak{X}$ and $\alpha_x, \beta_y \in \mathcal{SP}(\mathfrak{X}, \mathfrak{A})$ where $\alpha_x \neq \alpha_y$. If there exist $\mathcal{F}, \mathcal{G} \in T$ such that $\alpha_x \in \mathcal{F}$, $\alpha_y \in \mathcal{G}$ and $\mathcal{F} \sqcap \mathcal{G} = \widehat{\phi}$, then $(\mathfrak{X}, T, \mathfrak{A})$ is called a T_2-**S-top** space.

For any **S-con** functions $f, g : (\mathfrak{X}, T, \mathfrak{A}) \longrightarrow (\mathfrak{Y}, D, \mathfrak{A})$, consider the two sets $\mathbf{A} = \{x \in \mathfrak{X} : f(x) = g(x)\}$ and $\mathbf{G} = \{(x, f(x)) \in \mathfrak{X} \times \mathfrak{Y} : x \in \mathfrak{X}.\}$

Theorem 3.6. *Let $(\mathfrak{Y}, D, \mathfrak{A})$ be an enriched **S-top** space. Then*

(1) *$(\mathfrak{Y}, D, \mathfrak{A})$ is a T_2-**S-top** space.*
(2) *$\widehat{\mathcal{E}}$ is a **S-clS** in $\mathfrak{Y} \times \mathfrak{Y}$.*
(3) *$\widehat{\mathbf{A}}$ is a **S-clS** in $\mathfrak{X}$.*
(4) *$\widehat{\mathbf{G}}$ is a **S-clS** in $\mathfrak{X} \times \mathfrak{Y}$.*

Proof. (1) $\Rightarrow$ (2) : Let $(y, z) \in \mathfrak{Z} = \mathfrak{Y} \times \mathfrak{Y} - \mathcal{E}$, then $\alpha_y \neq \alpha_z$. By hypotheses, $\exists\, \mathcal{F}, \mathcal{G} \in D$ containing α_y, α_z respectively. Now, we have

$$\alpha \in \tau_{\mathcal{F}\times\mathcal{G}}(y, z) = \tau_{\mathcal{F}}(y) \cap \tau_{\mathcal{G}}(z).$$

This implies that $\alpha_{(y,z)} \in \mathcal{F} \times \mathcal{G}$. Further, $\forall (y_1, y_1) \in \mathfrak{Y} \times \mathfrak{Y}$,

$$\tau_{\mathcal{F}\times\mathcal{G}}(y_1, y_1) = \tau_{\mathcal{F}}(y_1) \cap \tau_{\mathcal{G}}(y_1) = \tau_{\mathcal{F}\sqcap\mathcal{G}}(y_1) = \phi. \text{ (being } \mathfrak{Y}\text{ , } T_2 \text{ space)}$$

This, along with the fact that $\tau_{\mathcal{F}\times\mathcal{G}}(y, z) \sqsubseteq \mathfrak{A}$ when $y \neq z$, verifying that

$$\alpha_{(y,z)} \in \tau_{\mathcal{F}\times\mathcal{G}} \sqsubseteq \mathfrak{Z}.$$

Hence, by Theorem 3.1 showing that $\mathfrak{Z} = \mathfrak{Y} \times \mathfrak{Y} - \mathcal{E}$ is a **S-opS**, and therefore, $\widehat{\mathcal{E}}$ is a **S-clS**.
(2) $\Rightarrow$ (3) : To demonstrate that $\mathbf{A}$ is a **S-clS**, we begin by confirming that the mapping

$$\pi : (\mathfrak{X}, T, \mathfrak{A}) \longrightarrow (\mathfrak{Y} \times \mathfrak{Y}, D \times D, \mathfrak{A}),\ x \to (f(x), g(x)),$$

is a **S-con** function. To establish this, it suffices to verify Theorem 2.1. Now, take $\mathcal{G} \times \mathcal{H} \in D \times D$, then for all $x \in \mathfrak{X}$

$$\tau_{\overleftarrow{\pi}(\mathcal{G}\times\mathcal{H})}(x) = \tau_{\mathcal{G}\times\mathcal{H}}(\pi(x)) = \tau_{\mathcal{G}\times\mathcal{H}}(f(x), g(x)) = \tau_{\mathcal{G}}(f(x)) \cap \tau_{\mathcal{H}}(g(x))$$
$$= \tau_{\overleftarrow{f}\mathcal{G}}(x) \cap \tau_{\overleftarrow{g}\mathcal{H}}(x) = \tau_{\overleftarrow{f}\mathcal{G}\sqcap\overleftarrow{g}\mathcal{H}}(x)$$

That is, $\overleftarrow{\pi}(\mathcal{G} \times \mathcal{H}) = \overleftarrow{f}\mathcal{G} \sqcap \overleftarrow{g}\mathcal{H}$. Since, $\overleftarrow{f}\mathcal{G}, \overleftarrow{g}\mathcal{H} \in T \Rightarrow \overleftarrow{\pi}(\mathcal{G} \times \mathcal{H}) \in T$. Hence π is a **S-con** function. Clearly, $\overleftarrow{\pi}\widehat{\mathcal{E}}(\alpha) = \pi^{-1}(\mathcal{E}) = \mathbf{A} = \widehat{\mathbf{A}}(\alpha)$, $\forall\alpha \in \mathfrak{A}$. Soft continuity of π and Theorem 2.1 imply that $\widehat{\mathbf{A}}$ is a **S-clS**.
(3) $\Rightarrow$ (4) : By applying (3), for the **S-con** functions $p_{\mathfrak{Y}}, h : \mathfrak{X} \times \mathfrak{Y} \longrightarrow \mathfrak{Y}$ defined as $p_{\mathfrak{Y}}(x, y) = y$[10], and $h(x, y) = f(x)$. Then, we have $\widehat{\mathbf{A}}(\alpha) = \{(x, y) : p_{\mathfrak{Y}}(x, y) = h(x, y)\} = \{(x, y) : y = f(x)\} = \widehat{\mathbf{G}}(\alpha), \forall\alpha \in \mathfrak{A}$. Therefore, $\widehat{\mathbf{G}}$ is a **S-clS** in $\mathfrak{X} \times \mathfrak{Y}$.
(4) $\Rightarrow$ (1) : By applying (4), for the **S-con** identity function $i : \mathfrak{Y} \longrightarrow \mathfrak{Y}$[10], the soft set $\widehat{\mathcal{E}}$ is a **S-clS** in $\mathfrak{Y} \times \mathfrak{Y}$. Now, let $\alpha_{(y,z)} \in \widehat{\mathcal{E}}^c \in D \times D$. Then $y \neq z \Rightarrow \alpha_y \neq \alpha_z$. From Theorem 3.1, there are $\mathcal{F}, \mathcal{H} \in D$ such that

$$\alpha_y \times \alpha_z = \alpha_{(y,z)} \in \mathcal{F} \times \mathcal{H} \sqsubseteq \widehat{\mathcal{E}}^c.$$

In addition, $\alpha_y \in \mathcal{F}$ and $\alpha_z \in \mathcal{H}$. Also, $\tau_{\mathcal{F}\times\mathcal{H}}(y, y) = \phi, \forall y \in \mathfrak{Y}$. In other words, $\tau_{\mathcal{F}\cap\mathcal{H}}(y) = \tau_{\mathcal{F}\times\mathcal{H}}(y, y) = \phi \;\forall y \in \mathfrak{Y}$. That is, for every $\alpha_y \neq \alpha_z$, $\exists\alpha_y \in \mathcal{F}$ and $\alpha_z \in \mathcal{H} \in D$ such that $\mathcal{F} \cap \mathcal{H} = \widehat{\phi}$. □

4. Conclusions

In classical topology, a space $(\mathfrak{X}, T)$ is considered a T_1 topological space iff the set $\{x\}$ is closed for every $x \in \mathfrak{X}$. However, this result does not hold in the context of **S-top** spaces (as shown in Proposition 3.2). We propose an alternative definition of a T_1-**S-top** spaces which reestablishes the statements of Proposition 3.2 as equivalent.

Definition 4.1. *An enriched **S-top** space* $(\mathfrak{Y}, D, \mathfrak{A})$ *is* T_1 *if for each* $x \neq y \in \mathfrak{X}$ *there exist* $\mathcal{F}, \mathcal{G} \in T$ *such that*

$$\tau_{\mathcal{F}}(x) = \mathfrak{A}, \tau_{\mathcal{F}}(y) = \phi \textit{ and } \tau_{\mathcal{G}}(y) = \mathfrak{A}, \tau_{\mathcal{G}}(x) = \phi.$$

In summary, we would like to emphasize that this definition preserves all the characteristics of the earlier definitions. Lastly, it offers a more concrete framework for work, as it is described through crisp points instead of soft points.

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