EXISTENCE AND UNIQUENESS RESULTS FOR THE BENDING ELASTIC BEAM EQUATIONS

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ABSTRACT. We examine the existence and uniqueness of solutions to a particular class of two-point boundary value problems involving fourth-order ordinary differential equations. These problems have fascinating applications in the modeling of deflections of beams. Applying the fixed point theorems by Banach and Rus is one of the main instruments used in this investigation. Our theoretical findings are applied to deflections of an elastic beam when subjected to a loading force with both ends hinged. We also ensure the existence and uniqueness of solutions to the model for certain classes of linear and nonlinear loading forces.

Keywords: Fourth-order, ordinary differential equation, boundary value problem, Green's function, fixed point, existence, uniqueness, deflections of beams.

AMS Subject Classification: 34B60.

1. INTRODUCTION

We consider the nonlinear fourth-order differential equation

$$y'''' + \beta^2 y'' = f(x, y), \quad x \in [0, L], \tag{1}$$

together with the boundary conditions

$$y(0) = 0, \quad y(L) = 0, \quad y''(0) = 0, \quad y''(L) = 0. \tag{2}$$

Here L, $\beta \in \mathbb{R}^+$, $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $f(\cdot, 0) \neq 0$. The assumption $f(\cdot, 0) \neq 0$ excludes the possibility of the trivial solution. By a solution to (1) - (2), we mean a function $y : [0, L] \to \mathbb{R}$ such that y is four times differentiable, with a continuous fourth-order derivative on $[0, L]$, which we denote by $y \in C^4([0, L])$, and satisfies both (1) and (2).

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Using fixed point theorems, this work aims to establish and compare results on the existence of a unique solution to (1) - (2). Our significant findings indicate that there exists a unique nontrivial solution to the problem if and only if L is small and the function f meets the Lipschitz condition. To get these results, we first construct the related Green's function and rephrase our problem (1) - (2) as an equivalent integral equation. Next, we utilize an infinite strip to apply the Banach fixed point theorem. The Banach fixed point theorem is then used inside a closed and bounded set to extend the conclusion to a larger class of functions. Ultimately, we utilize Rus's fixed point theorem to extend the length of the valid interval for the result. In order to compare the established results, we examine a few examples.

The analysis of deflections in elastic beams naturally motivates the study of fourth-order boundary value problems. Consider a thin beam whose ends are hinged at $x = 0$ and $x = L$ on the x-axis. A transverse load $h(x)$ varies throughout the beam's length, and a compressive force P are two of the forces acting on it. If $y = y(x)$ represents the resultant deflection of the beam at position x, the differential equation

$$y'''' + \beta^2 y'' = h(x), \quad x \in [0, L], \tag{3}$$

depicts the displacement of the beam in the transverse direction caused by buckling with $\beta = \sqrt{\frac{P}{EI}}$, where E is the slender member's Young's modulus and I is the beam's moment of inertia along the direction of its length. Assume that the compressive load P and $E \cdot I$ are constants for the sake of simplicity. It is obvious that β needs to be bigger than zero; otherwise, $P = 0$ would result from $\beta = 0$. The beam has hinged ends at $x = 0$ and $x = L$. Hence, the problem is susceptible to the boundary conditions (2). The fourth-order differential equation (1) is obtained if we consider the beam's transverse load, which is provided by $f(x, y)$. This load may not be linear.

The study of solutions to boundary value problems often involves examining the construction of Green functions specific to those problems. Consequently, Green functions hold significance in the theory of boundary value problems. Fourth-order boundary value problems and their relevance to elastic beam deflections have been extensively researched. Numerous well-known studies focus on determining the solvability of fourth-order boundary value problems and verifying the existence and uniqueness of solutions. One powerful and efficient method for proving the existence or uniqueness of solutions to nonlinear boundary value problems is to use fixed point theorems. Numerous authors have investigated the existence of solutions to fourth-order boundary value problems using different fixed-point theorems. We refer to [16, 17, 1, 10, 5, 2, 3, 6, 7, 18, 9, 19, 4] and the references therein for a few recent publications in this area among an enormous number of papers dealing with the solvability of fourth-order nonlinear differential equations subject to a variety of boundary conditions using fixed point theory.

The problem under consideration is distinct from the aforementioned works. Furthermore, we highlight that our method of utilizing Rus's fixed point theorem seems to hold a distinct place in the literature to guarantee the existence and uniqueness of solutions to boundary value problems of the fourth order. The findings presented here represent an improvement over conventional methods, like using Banach's fixed point theorem. This is achieved by applying Rus's fixed point theorem along with two metrics. As we will see, this allows for a clearer understanding of the existence and uniqueness of solutions to a broader class of problems. This involves fine-tuning the Lipschitz constants in closed and bounded domains as well as in a global (unbounded) setting.

Since our primary tools in this paper are fixed point theorems, let us state Banach and Rus's fixed point theorems for the reader's convenience.

Theorem 1.1. [8] *Let X be a nonempty set, and d be a metric on X such that (X, d) forms a complete metric space. If the mapping $T : X \to X$ satisfies*

$$d(Ty, Tz) \leq \alpha d(y, z) \text{ for some } \alpha \in (0, 1) \text{ and all } y, z \in X;$$

then there is a unique $y_0 \in X$ such that $Ty_0 = y_0$.

Theorem 1.2. [11] *Let X be a nonempty set, and d and ρ be two metrics on X such that (X, d) forms a complete metric space. If the mapping $T : X \to X$ is continuous with respect to d on X and*

(1) *there exists $c > 0$ such that*

$$d(Ty, Tz) \leq c\rho(y, z) \text{ for all } y, z \in X;$$

(2) *there exists $\alpha \in (0, 1)$ such that*

$$\rho(Ty, Tz) \leq \alpha\rho(y, z) \text{ for all } y, z \in X;$$

then there is a unique $y_0 \in X$ such that $Ty_0 = y_0$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function corresponding to the boundary value problem (1) - (2) by employing the variation of parameters formula and some additional assumptions. Section 3 is devoted to the estimation of an integral that involves the Green's function. In Section 4, we prove our main theorems on the existence of a unique solution to to the boundary value problem (1) - (2). Also, we provide a few examples to illustrate the applicability of established results.

2. CONSTRUCTION OF THE GREEN'S FUNCTION

The goal of this section is to rewrite the boundary value problem (1) - (2) as an equivalent integral equation. So, let us consider the linear equation (3) together with the boundary conditions (2).

Proposition 2.1. *Assume $\sin \beta L \neq 0$. If $h : [0, L] \to \mathbb{R}$ is a continuous function, then the boundary value problem (3) - (2) has a unique solution which we can write as*

$$y(x) = \int_0^L G(x, \xi)h(\xi)d\xi, \quad 0 \leq x \leq L, \tag{4}$$

where the Green's function is given by

$$G(x, \xi) = \begin{cases} G_1(x, \xi), & 0 \leq \xi \leq x \leq L, \\ G_2(x, \xi), & 0 \leq x \leq \xi \leq L. \end{cases} \tag{5}$$

Here

$$\mathcal{K}(x, \xi) = \frac{1}{\beta^3} \left[\beta(x - \xi) - \sin \beta(x - \xi) \right],$$

$$G_1(x, \xi) = -\frac{x(L - \xi)}{L\beta^2} + \frac{\sin \beta x \sin \beta(L - \xi)}{\beta^3 \sin \beta L} + \mathcal{K}(x, \xi), \tag{6}$$

and

$$G_2(x, \xi) = -\frac{x(L - \xi)}{L\beta^2} + \frac{\sin \beta x \sin \beta(L - \xi)}{\beta^3 \sin \beta L}. \tag{7}$$

Proof. The general solution of (3) is given by

$$y(x) = c_1 + c_2 x + c_3 \cos \beta x + c_4 \sin \beta x + \int_0^x \mathcal{K}(x, \xi)h(\xi)d\xi, \quad 0 \leq x \leq L, \tag{8}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. From (8), we have

$$y'(x) = c_2 - \beta c_3 \sin \beta x + \beta c_4 \cos \beta x + \int_0^x \mathcal{K}_x(x, \xi)h(\xi)d\xi, \quad 0 \leq x \leq L, \tag{9}$$

$$y''(x) = -\beta^2 c_3 \cos \beta x - \beta^2 c_4 \sin \beta x + \int_0^x \mathcal{K}_{xx}(x,\xi)h(\xi)d\xi, \quad 0 \leq x \leq L, \qquad (10)$$

where

$$\mathcal{K}_x(x,\xi) = \frac{1}{\beta^2}\left[1 - \cos \beta(x-\xi)\right], \quad \mathcal{K}_{xx}(x,\xi) = \frac{\sin \beta(x-\xi)}{\beta}.$$

Using boundary conditions (2) in (8) - (10) and rearranging the terms, we get

$$c_1 = 0,$$

$$c_2 = -\frac{1}{L\beta^2}\int_0^L (L-\xi)h(\xi)d\xi,$$

$$c_3 = 0,$$

$$c_4 = \frac{1}{\beta^3 \sin \beta L}\int_0^L \sin \beta(L-\xi)h(\xi)d\xi.$$

Substituting the constants c_1, c_2, c_3 and c_4 in (8) and rearranging the terms, we obtain (4). Hence, the boundary value problem (3) - (2) has a unique solution (4). To verify that $y \in C^4[0,L]$, one can differentiate (4) four times and verify its continuity. $\square$

3. Estimation of the Green's Function

In this section, we prove a useful inequality for an integral that involves the Green's function.

Proposition 3.1. *Assume $\sin \beta L \neq 0$. The Green's function in (5) satisfies*

$$\int_0^L |G(x,\xi)|\, d\xi \leq \frac{L^2}{2}k_1 + Lk_2 + \frac{L^4}{24}, \qquad (11)$$

where

$$k_1 = \sup_{x \in [0,L]} \left|\frac{x}{L\beta^2}\right| = \frac{1}{\beta^2}, \qquad (12)$$

and

$$k_2 = \sup_{x \in [0,L]} \left|\frac{\sin \beta x}{\beta^3 \sin \beta L}\right|. \qquad (13)$$

Proof. For all $x \in [0,L]$, we have

$$\int_0^L |G(x,\xi)|\, d\xi = \int_0^x |G(x,\xi)|\, d\xi + \int_x^L |G(x,\xi)|\, d\xi$$

$$\leq \int_0^x \left|-\frac{x(L-\xi)}{L\beta^2}\right| d\xi + \int_0^x \left|\frac{\sin \beta x \sin \beta(L-\xi)}{\beta^3 \sin \beta L}\right| d\xi + \int_0^x |\mathcal{K}(x,\xi)|\, d\xi$$

$$+ \int_x^L \left|-\frac{x(L-\xi)}{L\beta^2}\right| d\xi + \int_x^L \left|\frac{\sin \beta x \sin \beta(L-\xi)}{\beta^3 \sin \beta L}\right| d\xi$$

$$\leq k_1\left[\int_0^x (L-\xi)d\xi + \int_x^L (L-\xi)d\xi\right]$$

$$+ k_2\left[\int_0^x |\sin \beta(L-\xi)|\, d\xi + \int_x^L |\sin \beta(L-\xi)|\, d\xi\right]$$

$$+ \int_0^x \mathcal{K}(x,\xi)d\xi$$

$$= k_1 \int_0^L (L - \xi) + k_2 \int_0^L |\sin \beta(L - \xi)| \, d\xi + \int_0^x \mathcal{K}(x, \xi) d\xi$$

$$\leq \frac{L^2}{2} k_1 + L k_2 + \frac{x^4}{24}$$

$$\leq \frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24}.$$

The proof is complete. $\qquad \square$

4. Existence of a Unique Solution

In this section, we will apply fixed point theorems to prove our results on the existence of a unique solution to the boundary value problem (1) - (2) and compare them. For this, let us define two metrics on the set X of continuous functions defined on $[0, L]$ such that

$$d(y, z) = \sup_{x \in [0,L]} |y(x) - z(x)|,$$

and

$$\rho(y, z) = \left(\int_0^L |y(x) - z(x)|^2 \, dx \right)^{\frac{1}{2}},$$

for all $y, z \in X$. It is easy to show that (X, ρ) is a metric space and (X, d) forms a complete metric space.

4.1. Application of Theorem 1.1 on an Infinite Strip.

Theorem 4.1. *Assume* $\sin \beta L \neq 0$ *and* f *satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant* K. *If*

$$\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} < \frac{1}{K}, \tag{14}$$

then there exists a unique non-trivial solution to the boundary value problem (1) - (2).

Proof. It follows from Proposition 2.1 that the boundary value problem (1) - (2) is equivalent to the integral equation

$$y(x) = \int_0^L G(x, \xi) f(\xi, y(\xi)) d\xi, \quad 0 \leq x \leq L.$$

Define the mapping $T : X \to X$ by

$$(Ty)(x) = \int_0^L G(x, \xi) f(\xi, y(\xi)) d\xi, \quad 0 \leq x \leq L.$$

Clearly, y is a solution of (1) - (2) iff y is a fixed point of T. To establish the existence of a unique fixed point of T, we show that the conditions of Theorem 1.1 hold. To see this, let $y, z \in X$, $x \in [0, L]$ and consider

$$|(Ty)(x) - (Tz)(x)| = \left| \int_0^L G(x, \xi) f(\xi, y(\xi)) d\xi - \int_0^L G(x, \xi) f(\xi, z(\xi)) d\xi \right|$$

$$\leq \int_0^L |G(x, \xi)| \, |f(\xi, y(\xi)) - f(\xi, z(\xi))| \, d\xi$$

$$\leq K \int_0^L |G(x, \xi)| \, |y(\xi) - z(\xi)| \, d\xi$$

$$\leq K d(y,z) \int_0^L |G(x,\xi)| \, d\xi$$

$$\leq K \left(\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \right) d(y,z),$$

implying that

$$d(Ty, Tz) \leq K \left(\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \right) d(y,z),$$

for all $y, z \in X$. Since

$$K \left(\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \right) < 1,$$

the mapping T is a contraction. Hence, by Theorem 1.1, T has a unique fixed point in X. Therefore, the boundary value problem (1) - (2) has a unique non-trivial solution $y \in X$. The proof is complete. $\qquad\square$

4.2. **Application of Theorem 1.1 within a Closed and Bounded Set.** Consider a closed ball B_N with radius N in X as follows:

$$B_N = \{ y \in X : d(y,0) \leq N \}.$$

Since B_N is a closed subspace of X, the pair (B_N, d) forms a complete metric space. Clearly, $T : B_N \to X$.

Theorem 4.2. *Let $f : [0,L] \times [-N,N] \to \mathbb{R}$ be a continuous function and $f(\cdot, 0) \neq 0$. Assume $\sin \beta L \neq 0$ and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If L satisfies the inequalities (14) and*

$$\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \leq \frac{N}{M}, \tag{15}$$

where

$$M = \sup_{(x,y) \in [0,L] \times [-N,N]} |f(x,y)|,$$

then there exists a unique non-trivial solution y to the boundary value problem (1) - (2) such that

$$|y(x)| \leq N, \quad x \in [0,L].$$

Proof. First, we show that $T : B_N \to B_N$. To see this, let $y \in B_N$, $x \in [0,L]$ and consider

$$|(Ty)(x)| \leq \int_0^L |G(x,\xi)| \, |f(\xi, y(\xi))| \, d\xi$$

$$\leq M \int_0^L |G(x,\xi)| \, d\xi$$

$$\leq M \left(\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \right)$$

implying that

$$d(Ty, 0) \leq M \left(\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \right) \leq N.$$

Thus, $Ty \in B_N$. Therefore, $T : B_N \to B_N$. It follows from the proof of Theorem 4.1 that $T : B_N \to B_N$ is a contraction. Hence, by Theorem 1.1, T has a unique fixed point in B_N. Therefore, the boundary value problem (1) - (2) has a unique non-trivial solution $y \in B_N$. The proof is complete. $\qquad\square$

4.3. **Application of Theorem 1.2 on an Infinite Strip.**

Theorem 4.3. *Assume* $\sin \beta L \neq 0$ *and* f *satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant* K. *If*

$$\left(\frac{k_1^2 L^4}{3} + k_2^2 L^2 + \frac{L^8}{252} + k_1 k_2 L^3 + \frac{k_2 L^5}{12} + \frac{k_1 L^6}{12} \right)^{\frac{1}{2}} < \frac{1}{K}, \tag{16}$$

then there exists a unique non-trivial solution to the boundary value problem (1) - (2).

Proof. To establish the existence of a unique fixed point of T using Theorem 1.2, we have to show that the conditions of Theorem 1.2 hold. For this purpose, let $y, z \in X$, $x \in [0, L]$ and consider

$$
\begin{aligned}
|(Ty)(x) - (Tz)(x)| &= \left| \int_0^L G(x, \xi) f(\xi, y(\xi)) d\xi - \int_0^L G(x, \xi) f(\xi, z(\xi)) d\xi \right| \\
&\leq \int_0^L |G(x, \xi)| \, |f(\xi, y(\xi)) - f(\xi, z(\xi))| \, d\xi \\
&\leq K \int_0^L |G(x, \xi)| \, |y(\xi) - z(\xi)| \, d\xi \\
&\leq K \left(\int_0^L |G(x, \xi)|^2 \, d\xi \right)^{\frac{1}{2}} \left(\int_0^L |y(\xi) - z(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \\
&\leq K \sup_{0 \leq x \leq L} \left(\int_0^L |G(x, \xi)|^2 \, d\xi \right)^{\frac{1}{2}} \rho(y, z) \\
&\leq c \rho(y, z),
\end{aligned}
$$

implying that

$$d(Ty, Tz) \leq c \rho(y, z),$$

for all $y, z \in X$. Here

$$c = K \sup_{0 \leq x \leq L} \left(\int_0^L |G(x, \xi)|^2 \, d\xi \right)^{\frac{1}{2}} > 0.$$

Also,

$$
\begin{aligned}
\rho(y, z) &= \left(\int_0^L |y(x) - z(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^L \sup_{0 \leq x \leq L} |y(x) - z(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \sup_{0 \leq x \leq L} |y(x) - z(x)| \left(\int_0^L dx \right)^{\frac{1}{2}} \\
&= L^{\frac{1}{2}} d(y, z).
\end{aligned}
$$

Thus, we obtain that

$$d(Ty, Tz) \leq c \rho(y, z) \leq c L^{\frac{1}{2}} d(y, z),$$

for all $y, z \in X$. Then, for any $\epsilon > 0$, choose $\delta = \dfrac{\epsilon}{cL^{\frac{1}{2}}}$ such that $d(Ty, Tz) < \epsilon$ whenever $d(y, z) < \delta$. Therefore, T is continuous with respect to d on X. Consider

$$\left(\int_0^L |(Ty)(x) - (Tz)(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \left(\int_0^L \left[K \left(\int_0^L |G(x, \xi)|^2 \, d\xi \right)^{\frac{1}{2}} \rho(y, z) \right]^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq K\rho(y, z) \left(\int_0^L \left(\int_0^L |G(x, \xi)|^2 \, d\xi \right) dx \right)^{\frac{1}{2}}.$$

Now, consider

$$\int_0^L |G(x, \xi)|^2 \, d\xi = \int_0^x |G(x, \xi)|^2 \, d\xi + \int_x^L |G(x, \xi)|^2 \, d\xi$$

$$\leq \int_0^x \left(k_1^2(L - \xi)^2 + k_2^2 + \frac{(x - \xi)^6}{36} + 2k_1 k_2(L - \xi) \right.$$

$$\left. + \frac{k_1(x - \xi)^3(L - \xi)}{3} + \frac{k_2(x - \xi)^3}{3} \right) d\xi$$

$$+ \int_x^L \left(k_1^2(L - \xi)^2 + k_2^2 + 2k_1 k_2(L - \xi) \right) d\xi$$

$$\leq \frac{k_1^2 L^3}{3} + k_2^2 L + \frac{L^7}{252} + k_1 k_2 L^2 + \frac{k_2 L^4}{12}$$

$$+ \int_0^x \frac{k_1(x - \xi)^3(L - \xi)}{3} d\xi$$

$$\leq \frac{k_1^2 L^3}{3} + k_2^2 L + \frac{L^7}{252} + k_1 k_2 L^2 + \frac{k_2 L^4}{12}$$

$$+ \frac{k_1}{3} \left(-\left[\frac{(x - \xi)^4(L - \xi)}{4} \right]_0^x - \int_0^x \frac{(x - \xi)^4}{4} d\xi \right)$$

$$\leq \frac{k_1^2 L^3}{3} + k_2^2 L + \frac{L^7}{252} + k_1 k_2 L^2 + \frac{k_2 L^4}{12}$$

$$+ \frac{k_1}{3} \left(\frac{L^5}{4} + \left[\frac{(x - \xi)^5}{20} \right]_0^x \right)$$

$$\leq \frac{k_1^2 L^3}{3} + k_2^2 L + \frac{L^7}{252} + k_1 k_2 L^2 + \frac{k_2 L^4}{12} + \frac{k_1 L^5}{12}.$$

Hence,

$$\left(\int_0^L |(Ty)(x) - (Tz)(x)|^2 dx \right)^{1/2}$$

$$\leq K\rho(y, z) \left(\int_0^L \left(\frac{k_1^2 L^3}{3} + k_2^2 L + \frac{L^7}{252} + k_1 k_2 L^2 + \frac{k_2 L^4}{12} + \frac{k_1 L^5}{12} \right) dx \right)^{1/2}$$

$$= K \left(\frac{k_1^2 L^4}{3} + k_2^2 L^2 + \frac{L^8}{252} + k_1 k_2 L^3 + \frac{k_2 L^5}{12} + \frac{k_1 L^6}{12} \right)^{\frac{1}{2}} \rho(y, z),$$

implying that

$$\rho(Ty, Tz) \leq \alpha \rho(y, z),$$

for all y, $z \in X$. Here

$$\alpha = K \left(\frac{k_1^2 L^4}{3} + k_2^2 L^2 + \frac{L^8}{252} + k_1 k_2 L^3 + \frac{k_2 L^5}{12} + \frac{k_1 L^6}{12} \right)^{\frac{1}{2}} < 1.$$

Hence, by Theorem 1.2, T has a unique fixed point in X. Therefore, the boundary value problem (1) - (2) has a unique non-trivial solution $y \in X$. The proof is complete. $\square$

4.4. Application of Theorem 1.2 within a Closed and Bounded Set.

Theorem 4.4. *Let $f : [0, L] \times [-N, N] \to \mathbb{R}$ be a continuous function and $f(\cdot, 0) \neq 0$. Assume $\sin \beta L \neq 0$ and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant K. If L satisfies the inequalities (15) and (16), then there exists a unique non-trivial solution y to the boundary value problem (1) - (2) such that*

$$|y(x)| \leq N, \quad x \in [0, L].$$

Proof. To establish the existence of a unique fixed point of T using Theorem 1.2, we have to show that the conditions of Theorem 1.2 hold. First, we show that $T : B_N \to B_N$. To see this, let $y \in B_N$, $x \in [0, L]$ and consider

$$|(Ty)(x)| \leq \int_0^L |G(x, \xi)| \, |f(\xi, y(\xi))| \, d\xi$$

$$\leq M \int_0^L |G(x, \xi)| \, d\xi$$

$$\leq M \left(\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \right)$$

implying that

$$d(Ty, 0) \leq M \left(\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \right) \leq N.$$

Thus, $Ty \in B_N$. Therefore, $T : B_N \to B_N$. It follows from the proof of Theorem 4.3 that

(1) T is continuous with respect to d on B_N;
(2) T is a contraction with respect to ρ on B_N.

Hence, by Theorem 1.2, T has a unique fixed point in B_N. Therefore, the boundary value problem (1) - (2) has a unique non-trivial solution $y \in B_N$. The proof is complete. $\square$

Remark 4.1. *From (14) and (16), we observe that*

$$\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} < \left(\frac{k_1^2 L^4}{3} + k_2^2 L^2 + \frac{L^8}{252} + k_1 k_2 L^3 + \frac{k_2 L^5}{12} + \frac{k_1 L^6}{12} \right)^{\frac{1}{2}}. \qquad (17)$$

Consequently, Theorem 4.3 $\Rightarrow$ Theorem 4.1, and the converse need not be true. Similarly, Theorem 4.4 $\Rightarrow$ Theorem 4.2, and the converse need not be true.

5. Examples

In this section, we provide a few examples to illustrate the applicability of results established in the previous section.

Example 5.1. *Consider (1) - (2) with $\beta = L = 1$ and $f(x,y) = \dfrac{y^2}{2y^2+2} + 10x + 1$. Clearly, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(\cdot, 0) \neq 0$. Also, $\sin 1 = 0.0175 \neq 0$ and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 0.5$. Further, we obtain*

$$k_1 = \frac{1}{\beta^2} = 1,$$

and

$$k_2 = \sup_{x \in [0,L]} \left| \frac{\sin \beta x}{\beta^3 \sin \beta L} \right| = \sup_{x \in [0,1]} \left| \frac{\sin x}{\sin 1} \right| = 1.$$

We have

$$\left(\frac{k_1^2 L^4}{3} + k_2^2 L^2 + \frac{L^8}{252} + k_1 k_2 L^3 + \frac{k_2 L^5}{12} + \frac{k_1 L^6}{12} \right)^{\frac{1}{2}} \approx 1.5824 < 2,$$

where $\frac{1}{K} = 2$, implying the inequality (16) holds. Hence, Theorem 4.3 is applicable for (1) - (2). Consequently, by Remark 4.1, Theorem 4.1 is also applicable for (1) - (2).

Example 5.2. *Consider (1) - (2) with $\beta = 1$, $L = 2$ and $f(x,y) = \dfrac{y^2}{2y^2+2} + 10x + 1$. Clearly, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(\cdot, 0) \neq 0$. Also, $\sin 2 = 0.0349 \neq 0$ and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 0.5$. Further, we obtain*

$$k_1 = \frac{1}{\beta^2} = 1,$$

and

$$k_2 = \sup_{x \in [0,L]} \left| \frac{\sin \beta x}{\beta^3 \sin \beta L} \right| = \sup_{x \in [0,2]} \left| \frac{\sin x}{\sin 2} \right| \approx 1.0998.$$

Since

$$\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \approx 4.8663 > 2,$$

where $\frac{1}{K} = 2$, the inequality (14) does not hold. Hence, Theorem 4.1 is not applicable in this case. Consequently, by Remark 4.1, Theorem 4.3 is also not applicable in this case.

Example 5.3. *Consider (1) - (2) with $\beta = L = 1$ and $f(x,y) = x^2 y^2 + 1$. Clearly, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(\cdot, 0) \neq 0$. Also, $\sin 1 = 0.0175 \neq 0$. But, f doesn't satisfies the Lipschitz condition with respect to its second argument. Hence, Theorem 4.1 is not applicable in this case. Consequently, by Remark 4.1, Theorem 4.3 is also not applicable in this case.*

Example 5.4. *Consider (1) - (2) with $\beta = 1$, $L = 0.5$ and $f(x,y) = x^2 y^2 + 0.5$. Choose $N = 1$. Clearly, $f : [0, 0.5] \times [-1, 1] \to \mathbb{R}$ is a continuous function and $f(\cdot, 0) \neq 0$. Also, $\sin(0.5) = 0.0087 \neq 0$ and f satisfies the Lipschitz condition with respect to its second argument with a Lipschitz constant $K = 0.5$. Further, we obtain*

$$k_1 = \frac{1}{\beta^2} = 1,$$

and

$$k_2 = \sup_{x \in [0,L]} \left| \frac{\sin \beta x}{\beta^3 \sin \beta L} \right| = \sup_{x \in [0,0.5]} \left| \frac{\sin x}{\sin 0.5} \right| = 1,$$

and

$$M = \sup_{(x,y) \in [0,0.5] \times [-1,1]} |f(x,y)| = 0.75.$$

Since

$$\frac{L^2}{2} k_1 + L k_2 + \frac{L^4}{24} \approx 0.6276 < 1.3333,$$

and

$$\left(\frac{k_1^2 L^4}{3} + k_2^2 L^2 + \frac{L^8}{252} + k_1 k_2 L^3 + \frac{k_2 L^5}{12} + \frac{k_1 L^6}{12} \right)^{\frac{1}{2}} \approx 0.6323 < 2,$$

where $\frac{N}{M} \approx 1.3333$ and $\frac{1}{K} = 2$, the inequalities (15) and (16) hold. Hence, Theorem 4.4 is applicable for (1) - (2). Consequently, by Remark 4.1, Theorem 4.2 is also applicable for (1) - (2).

6. Conclusion

In this article, we studied the existence and uniqueness of solutions to a particular class of two-point boundary value problems involving fourth-order ordinary differential equations using Banach's and Rus's fixed point theorems. Such problems have exciting applications for modeling the deflections of beams. The future scope of this research involves studying the existence and uniqueness of solutions of a hinged variable cross-section elastic beam subjected to a loading force, or a hinged functionally graded elastic beam subjected to a loading force [12, 13, 14, 15].

References

[1] Cabada, A. and Saavedra, L., (2018), Existence of solutions for nth-order nonlinear differential boundary value problems by means of fixed point theorems, Nonlinear Anal. Real World Appl., 42, pp. 180-206.

[2] Granas, A., Guenther, R. and Lee, J., (1985), Nonlinear boundary value problems for ordinary differential equations, Dissertationes Math., 244, pp. 1-128.

[3] Gupta, C. P., (1988), Existence and uniqueness results for the bending of an elastic beam equation at resonance, J. Math. Anal. Appl., 135, pp. 208-225.

[4] Zhai, C. and Anderson, D. R., (2011), A sum operator equation and applications to nonlinear elastic beam equations and Lane–Emden–Fowler equations, J. Math. Anal. Appl., 375, pp. 388-400.

[5] Franco, D., O'Regan, D. and Peran, J., (2005), Fourth-order problems with nonlinear boundary conditions, J. Comput. Appl. Math., 174, pp. 315-327.

[6] Jiang, D., Liu, H. and Xu, X., (2005), Nonresonant singular fourth-order boundary value problems, Appl. Math. Lett., 18, pp. 69-75.

[7] Kaufmann, E. R. and Kosmatov, N., (2007), Elastic beam problem with higher order derivatives, Nonlinear Anal. Real World Appl., 8, pp. 811-821.

[8] Zeidler, E., (1986), Nonlinear functional analysis and its applications I: Fixed-point theorems, New York, Springer-Verlag.

[9] Minhos, F. and Coxe, I., (2017), Systems of coupled clamped beams equations with full nonlinear terms: existence and location results, Nonlinear Anal. Real World Appl., 35, pp. 45-60.

[10] Chen, H. and Cui, Y., (2023), Existence and uniqueness of solutions to the nonlinear boundary value problem for fourth-order differential equations with all derivatives, J. Inequal. Appl., 2023, pp. 1-13.

[11] Rus, I. A., (1977), On a fixed point theorem of Maia, Studia Univ. Babes-Bolyai Math., 22, pp. 40-42.

[12] Valdes, J. E. N., Guzmán, P. M., Lugo, L. M. and Kashuri, A., (2020), The local generalized derivative and Mittag–Leffler function, Sigma Journal of Engineering and Natural Sciences, 38, pp. 1007-1017.

[13] Cortez, M. V., Lugo, L. M., Valdés, J. E. N. and Samei, M. E., (2022), A Multi-Index generalized derivative; some introductory notes, Appl. Math. Inf. Sci, 16, pp. 883-890.

[14] Guzmán, P. M., Valdés, J. E. N. and Cortez, M. V., (2024), A new generalized derivative and related properties, Appl. Math., 18, pp. 923-932.
[15] Castillo, R. E., Valdés, J. E. N. and Chaparro, H., (2024), Omega derivative, Gulf Journal of Mathematics, 16, pp. 55-67.
[16] Agarwal, R. P., (1989), On fourth order boundary value problems arising in beam analysis, Differential Integral Equations, 2, pp. 91-110.
[17] Almuthaybiri, S. S. and Tisdell, C. C., (2020), Sharper existence and uniqueness results for solutions to fourth-order boundary value problems and elastic beam analysis, Open Math., 18, pp. 1006-1024.
[18] Li, Y. and Gao, Y., (2019), Existence and uniqueness results for the bending elastic beam equations, Appl. Math. Lett., 95, pp. 72-77.
[19] Yang, Y. S., (1988), Fourth-order two-point boundary value problems, Proc. Amer. Math. Soc., 104, pp. 175-180.
[20] Akbarov S.D., Bagirov E.T., Sardarova I.Z., (2025), The Response of an Infinite Elastic Medium with Inhomogeneous Initial Stresses to Moving Ring Load Acting in a Cylindrical Cavity Contained with This Medium, Appl. Comput. Math., Vol.24, No.1, pp.3-15

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