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INJECTIVE COLORING OF CENTRAL GRAPHS

SHAHRZAD SADAT MIRDAMAD[1], DOOST ALI MOJDEH[1,*]

Abstract. For a given graph $G = (V(G), E(G))$, researchers have introduced different colorings based on the distances of the vertices. An injective coloring of a graph G is an assignment of colors to the vertices of G such that no two vertices with a common neighbor receive the same color. The injective chromatic number of G, denoted by $\chi_i(G)$, is the minimum number of colors required for an injective coloring of G. The concept of a central graph of any graph has been a widely studied topic among mathematical researchers and graph theorists nowadays. The central graph of a given graph G, denoted by $C(G)$, is the graph obtained by subdividing each edge of G exactly once and also adding an edge between each pair of non-adjacent vertices of G.

In this work, we present some results on injective coloring of central graph $C(G)$ of G. We show that for a graph G of order n and maximum degree $\Delta(G)$,

$$n - 1 \leq \chi_i(C(G)) \leq n^2 - 3n - (n - 3)\Delta(G) + 3.$$

Next, we will closely examine the injective chromatic number of the central graph of some special graphs and trees. Finally, for any graph H, and the corona product $(H \circ K_1)$, $(H \circ K_2)$, we will have a precise determination of the injective chromatic number of $C(H \circ K_1)$ and $C(H \circ K_2)$ in terms of $\chi_i(C(H))$ and order of H.

Keywords: Graph coloring, injective coloring, central graphs, corona product.

AMS Subject Classification: 05C15.

1. Introduction and preliminaries

All graphs $G = (V, E)$ in this paper are finite simple graphs, with vertex set $V = V(G)$ and edge set $E = E(G)$. For graph theoretic terminology and standard notation in this paper, we refer the reader to [3, 12] as references. The maximum degree, the minimum degree and the clique number of a graph G are denoted, respectively, by $\Delta(G)$, $\delta(G)$ and $\omega(G)$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u|uv \in E\}$, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. We call a vertex v of degree 0 an isolated vertex, while we refer to a vertex of degree $n-1$ as a universal vertex. We call a vertex v of degree 1 a pendant vertex, and its neighbor a

[1] Department of Mathematics - Faculty of Mathematical Sciences - University of Mazandaran- Babolsar - Iran.
e-mail: s.mirdamad03@umail.umz.ac.ir; ORCID: 0009-0006-2129-6658.
e-mail: damojdeh@umz.ac.ir; ORCID: 0000-0001-9373-3390.
[*] Corresponding author
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support vertex. For any two vertices u and v of G, the distance $d_G(u,v)$ is the length of a shortest path between them. The notation $[n]$ denotes the set $\{1,2,\ldots,n\}$.

We determine the injective chromatic number for the central graph of the certain family of standard graphs, which we introduce below. The path, cycle and complete graph with n vertices are denoted by P_n, C_n and K_n, respectively. The wheel and star graphs with $n+1$ vertices are denoted by W_n and S_n, respectively. A double star graph is a graph consisting of the union of two star graphs S_n and S_m, with one edge joining their support vertices. The double star graph with $n+m+2$ vertices is denoted by $S_{n,m}$.

The complete bipartite graph with partite sets of sizes n and m is denoted by $K_{n,m}$. A quartic graph is a graph where all vertices have degree 4. The octahedral graph is the platonic graph with 6 vertices, 12 edges and 8 faces. The octahedral graph is a 3 partite symmetric quartic graph.

1.1. **Pertinent concepts.**

The injective coloring was first introduced in 2002 by Hahn et al. [6] and has since been further studied in [2, 4, 7, 8, 9, 10]. An injective k-coloring of a graph G is a function $f: V \to \{1,2,3,\ldots,k\}$ such that if two vertices share a common neighbor, then they receive different colors. In other words, if we observe a path $P_3 = xyz$ in the graph, then $f(x) \neq f(z)$. The injective chromatic number of G, denoted by $\chi_i(G)$, is the minimum positive integer k for which G has an injective k-coloring. Note that the injective coloring of G is not necessarily a proper coloring. The injective coloring of graphs originates from complexity theory and finds applications in various fields such as random access machines, error-correcting codes, and the design of computer networks [1]. Additionally, the injective chromatic number of the hypercube has applications in the theory of error-correcting codes [6].

A 2-*distance coloring* of a graph G is a function $f: V \to \{1,2,3,\ldots,k\}$ such that no pair of vertices at distance at most 2 receive the same color. Every 2-distance coloring is an injective coloring. Thus the inequality $\chi_i(G) \leq \chi_2(G)$ obviously holds. Since all the neighbors of a common vertex receive distinct colors, it is clear that $\Delta \leq \chi_i(G)$. On the other hand, it is clear that for any graph G, $\chi_2(G) \leq \Delta^2(G)+1$.

Hahn et al. [6] introduced the concept of the common neighbor graph. The common neighbor graph $N_c(G)$ of a given graph G is defined as $V(N_c(G)) = V(G)$, and two distinct vertices u, v are adjacent in $N_c(G)$ if and only if they have a common neighbor in G. Thus, $\chi_i(G) = \chi(N_c(G))$.

One of the concepts that we need here is the central graph $C(G)$ of a given graph G, introduced by Vernold in 2007 [11].

A *central graph* $C(G)$ of a graph G, is obtained by:

(i). Subdividing each edge of G exactly once.

(ii). Adding an edge between each pair of non-adjacent vertices of G.

In [5], Frucht and Harary introduced the concept of the *corona product* in 1970. Let G and H be simple graphs. The corona product $G \circ H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)| = n$ copies of H and putting them around G, and joining each vertex of the k-th copy of H to the k-th vertex of G where $1 \leq k \leq n$.

Our purpose in this work is as follows. In Section 2, we show that

$$n-1 \leq \chi_i(C(G)) \leq n^2 - 3n - (n-3)\Delta(G) + 3.$$

In Section 3, we derive the injective chromatic number of $C(P_n)$, $C(C_n)$, $C(K_n)$, $C(W_n)$, $C(S_n)$, $C(S_{n,m})$ and $C(K_{n,m})$. For any tree T of order n, we show that $\chi_i(C(T)) = n$ in Section 4, and finally in Section 5, we show that

$$\chi_i(C(H \circ K_2)) = \begin{cases} \chi_i(C(H)) + 2n & \text{if } \chi_i(C(H)) \geq n \\ 3n & \text{if } \chi_i(C(H)) = n-1 \end{cases}$$

$$\chi_i(C(H \circ K_1)) = \begin{cases} \chi_i(C(H)) + n & \text{if } \chi_i(C(H)) \geq n \\ 2n & \text{if } \chi_i(C(H)) = n-1 \end{cases}.$$

2. General results on injective chromatic number

Let G be an arbitrary graph of order $n = |V(G)|$, size $m = |E(G)|$, and maximum degree $\Delta(G)$. According to the structure of central graph $C(G)$, $\Delta(C(G)) = n - 1$. On the other hand, we have the following lemma from [6].

Lemma 2.1. ([6] Lemma 5) *Let G have maximum degree Δ. Then, $\chi_i(G) \leq \Delta(\Delta-1)+1$.*

Therefore, we have the following:

Proposition 2.1. *For any graph G of order n, and size m. If $C(G)$ is the central graph of G, then $n-1 \leq \chi_i(C(G)) \leq n^2 - 3n + 3$. The lower bound is sharp for $C(W_5)$ and the upper bound is sharp for $C(C_3)$.*

Proof. Since $\Delta(G) \leq n-1$, Lemma 2.1 shows that $\chi_i(G) \leq (n-1)(n-2)+1 = n^2-3n+3$. On the other hand, $\Delta(C(G)) = n-1$ and $\chi_i(C(G)) \geq \Delta(C(G)) = n-1$. □

We also improve the Proposition 2.1 as follows.

Theorem 2.1. *For any graph G of order n, and maximum degree $\Delta(G)$. Then we have*

$$\chi_i(C(G)) \leq n^2 - 3n - (n-3)\Delta(G) + 3.$$

This bound is sharp.

Proof. Let G be a graph with maximum degree $\Delta = \Delta(G)$ and v be a vertex of degree $\deg(v) = \Delta$. The vertex v in $C(G)$ will be adjacent to Δ new vertices of degree 2 in $C(G)$ and also will be adjacent with $n-1-\Delta$ vertices of G in $C(G)$ which was not adjacent to them in G yet. On the other hand, each of the former vertices (the vertices of G except v) will be adjacent to $n-2$ other vertices (except v and Δ new adjacent vertices of v of degree 2) in $C(G)$. Now if we assign distinct colors to Δ new neighbor vertices of v with degree 2, assign different colors to all $n-2$ neighbors of each vertex u in $N_{C(G)}(v)$ of degree $n-1$ and one of these $n-2$ colors to u and assign a color to v that has not been used yet, then this coloring is an injective coloring of $C(G)$. Therefore

$$\chi_i(C(G)) \leq \Delta + ((n-1)-\Delta)(n-2) + 1 = n^2 - 3n - (n-3)\Delta(G) + 3.$$

This bound is sharp for complete graph K_n, wheel graph W_n ($n \neq 5$) and star graph S_n, see Propositions 3.4, 3.5 and 3.6. □

3. Central of some standard graphs

There are several special graphs whose injective coloring has been studied so far, for example $P_n, C_n, K_n, K_{n,m}$ and so on. In this section, we want to study the injective chromatic number of central of some standard graphs, such as $C(P_n), C(C_n), C(K_n), C(W_n), C(S_n)$, $C(S_{n,m})$ and $C(K_{n,m})$.

3.1. $C(P_n)$ **and** $C(C_n)$**.** In this subsection, we investigate the injective chromatic numbers of central of paths and cycles.

Proposition 3.1. ([6]) *For the path P_n and the cycle C_n, we have*

(1) $\chi_i(P_n) = \begin{cases} 1 & n = 1, 2 \\ 2 & otherwise \end{cases}$

(2) $\chi_i(C_n) = \begin{cases} 2 & n \equiv 0 \pmod 4 \\ 3 & otherwise \end{cases}$.

Proposition 3.2. *For path P_n, we have $\chi_i(C(P_n)) = n$.*

Proof. Let $V(P_n) = \{v_i | \ 1 \leq i \leq n\}$ be the vertex set of P_n, and $C(P_n)$ be central of P_n with vertex set $V(P_n) \bigcup \{u_{1,2}, u_{2,3}, \ldots, u_{n-1,n}\}$, where $u_{i,i+1}$ is the new vertex corresponding to once subdividing of edge $e = v_iv_{i+1}$, $(1 \leq i \leq n-1)$.

Let $n = 2, 3$. It is clear that $C(P_2) = P_3$ and $C(P_3) = C_5$. Thus $\chi_i(C(P_2)) = \chi_i(P_3) = 2$ and $\chi_i(C(P_3)) = \chi_i(C_5) = 3$.

Let $4 \leq n \leq 6$. It is easy to see that in any three vertices in $C(P_n)$, at least two of them have a common neighbor. Thus no three vertices can receive the same color and then, since for $4 \leq n \leq 6$, we have $2n - 1$ vertices in $C(P_n)$, we infer that $\chi_i(C(P_n)) \geq n$ for $4 \leq n \leq 6$.

Let $n \geq 7$. Then both distinct vertices v_i and v_j are shared by a vertex and so they receive different colors under any injective coloring P_n for $n \geq 7$. Therefore, $\chi_i(C(P_n)) \geq n$ for $n \geq 7$.

Now we show an injective coloring with n colors of $C(P_n)$. We assign color i to the vertices $v_i, u_{i,i+1}$, $1 \leq i \leq n-1$ and we assign color n to the vertex v_n. This coloring is an injective coloring, because there is no path of length 2 between any two vertices of the same color. Therefore $\chi_i(C(P_n)) = n$ for $n \geq 1$, (see Figure 1).

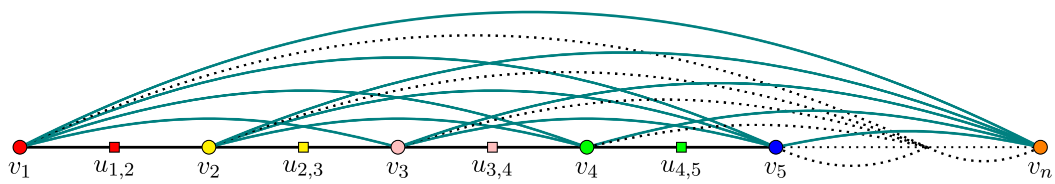


FIGURE 1. Injective coloring of $C(P_n)$

□

Proposition 3.3. *For cycle C_n, we have $\chi_i(C(C_n)) = n$.*

Proof. Let $V(C_n) = \{v_i | \ 1 \leq i \leq n\}$ be the vertex set of C_n. Let $C(C_n)$ be central of C_n with vertex set $V(C_n) \bigcup \{u_{1,2}, u_{2,3}, \ldots, u_{n-1,n}, u_{n,n+1 \ (mod \ n)}\}$, where $u_{i,i+1}$ is the vertex corresponding to once subdividing of edge $e = v_iv_{i+1}$ $(mod\ n)$ where $1 \leq i \leq n$.

If $n = 3$, then $C(C_3) = C_6$. Thus $\chi_i(C(C_3)) = \chi_i(C_6) = 3$.

Let $4 \leq n \leq 6$. It is easy to see that in any three vertices in $C(C_n)$, at least two of them have common neighbor. Thus no three vertices can receive the same color and then, since for $4 \leq n \leq 6$, we have $2n$ vertices in $C(C_n)$, we infer that $\chi_i(C(C_n)) \geq n$ for $4 \leq n \leq 6$.

Let $n \geq 7$. Then both distinct vertices v_i and v_j are shared by a vertex and so they receive different colors under any injective coloring of C_n for $n \geq 7$. Therefore, $\chi_i(C(C_n)) \geq n$ for $n \geq 7$.

Now by assign color i to the vertices $v_i, u_{i,i+1\ (mod\ n)}$, $1 \leq i \leq n$ demonstrate an injective coloring with n colors of $C(C_n)$. Since there is no path of length 2 between any two vertices of the same color. Therefore $\chi_i(C(C_n)) = n$ for $n \geq 3$, (see Figure 2).

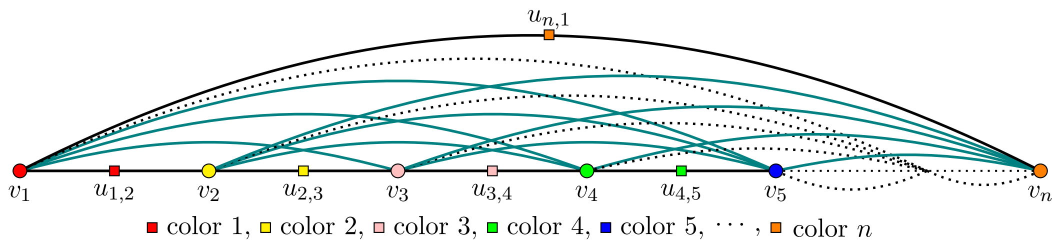


FIGURE 2. Injective coloring of $C(C_n)$

□

3.2. $C(K_n)$**.** Now we want to see the injective chromatic number of central of a complete graph $C(K_n)$. Before we prove the injective chromatic number of central of a complete graph, we give examples of some specific complete graphs and examine their injective chromatic number. Let $u_{i,j}$ be the vertex that divides the edge $e_{i,j} = v_iv_j$ where, $i < j$.

Example 3.1. The Figure 3 (1), demonstrates that, $\chi_i(C(K_6)) \geq 6$. Now we assign colors $1, 2, 3, \cdots, 6$ to the vertices $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 6$ and $j = n - (i + t)\ (mod\ 6)$, $-1 \leq t \leq 4$, respectively as follows:

We assign color 1 to the vertices v_1, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 6$ and $j = 6 - (i - 1)\ (mod\ 6)$; which are vertices $u_{1,6}$, $u_{2,5}$ and $u_{3,4}$.

We assign color 2 to the vertices v_2, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 6$ and $j = 6-(i)\ (mod\ 6)$; which are vertices $u_{1,5}$, and $u_{2,4}$.

We assign color 3 to the vertices v_3, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 6$ and $j = 6 - (i + 1)\ (mod\ 6)$; which are vertices $u_{1,4}$, $u_{2,3}$ and $u_{5,6}$.

We assign color 4 to the vertices v_4, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 6$ and $j = 6 - (i + 2)\ (mod\ 6)$; which are vertices $u_{1,3}$, $u_{4,6}$.

We assign color 5 to the vertices v_5, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 6$ and $j = 6 - (i + 3)\ (mod\ 6)$; which are vertices $u_{1,2}$, $u_{3,6}$ and $u_{4,5}$.

Finally, we assign color 6 to the vertices v_6, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 6$ and $j = 6 - (i + 4)\ (mod\ 6)$; which are vertices $u_{2,6}, u_{3,5}$.

With a simple inspection of Figure 3 (1), one can see that, there is no path of length 2 between any two vertices of the same color. Therefore $\chi_i(C(K_6)) = 6$.

Example 3.2. The Figure 3 (2), demonstrates that, $\chi_i(C(K_7)) \geq 7$. Now we assign colors $1, 2, 3, \ldots, 7$ to the vertices $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = n - (i + t)\ (mod\ 7)$, $-1 \leq t \leq 5$, respectively as follows:

we assign color 1 to the vertices v_1, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = 7 - (i - 1)\ (mod\ 7)$; which are vertices $u_{1,7}$, $u_{2,6}$ and $u_{3,5}$.

We assign color 2 to the vertices v_2, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = 7-(i)\ (mod\ 7)$; which are vertices $u_{1,6}$, $u_{2,5}$ and $u_{3,4}$.

We assign color 3 to the vertices v_3, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = 7 - (i + 1)\ (mod\ 7)$; which are vertices $u_{1,5}$, $u_{2,4}$ and $u_{6,7}$.

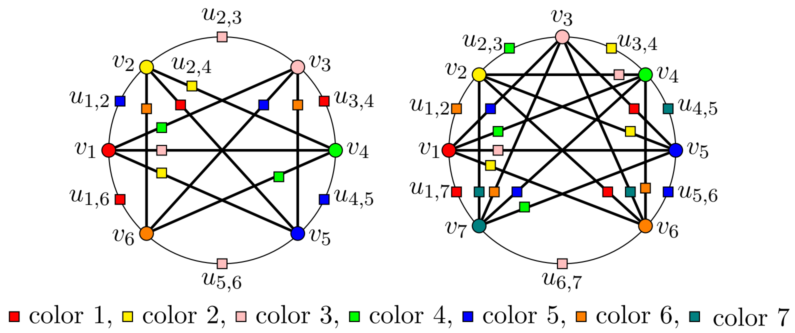
We assign color 4 to the vertices v_4, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = 7 - (i + 2)\ (mod\ 7)$; which are vertices $u_{1,4}$, $u_{2,3}$ and $u_{5,7}$.

We assign color 5 to the vertices v_5, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = 7 - (i + 3)\ (mod\ 7)$; which are vertices $u_{1,3}$, $u_{4,7}$ and $u_{5,6}$.

We assign color 6 to the vertices v_6, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = 7 - (i + 4)\ (mod\ 7)$; which are vertices $u_{1,2}$, $u_{3,7}$ and $u_{4,6}$.

Finally, we assign color 7 to the vertices v_7, $u_{i,j}$; $(i < j)$ where $1 \leq i \leq 7$ and $j = 7 - (i + 5)\ (mod\ 7)$; which are vertices $u_{2,7}, u_{4,5}, u_{3,6}$, (see Figure 3) (2).

With a simple inspection of Figure 3 (2), one can see that, there is no path of length 2 between any two vertices of the same color. Therefore $\chi_i(C(K_n)) = n$.



(1) $C(K_6)$ (2) $C(K_7)$

FIGURE 3. Injective coloring of $C(K_6)$ and $C(K_7)$

We prove now the result in general.

Proposition 3.4. *For complete K_n, we have $\chi_i(C(K_n)) = n$.*

Proof. Let $V(K_n) = \{v_i|\ 1 \leq i \leq n\}$ be the vertex set of K_n. Let $C(K_n)$ be central of K_n with vertex set $V(K_n) \bigcup \{u_{i,j} : 1 \leq i < j \leq n\}$, where $u_{i,j}$ is the new vertex corresponding to once subdividing of edge $e = v_i v_j$ where $1 \leq i < j \leq n$. According to the central graph structure, we know that there is a path of length 2 between any two vertices of $V(K_n)$. Therefore $\chi_i(C(K_n)) \geq n$. We now give an injective coloring with n colors for $C(K_n)$. For this coloring, there are two situations that must be considered.

Case 1. If n is even, following the injective coloring of Example 3.1, we assign color t for $1 \leq t \leq n$ to vertices v_i and $u_{i,j}$; $(i < j)$ where $i \in [n]$, $j = n - (i + t - 2)\ (mod\ n)$. As we see in the Figure 3 (1), similarly, one can see that, there is no path of length 2 between any two vertices with the same color. Therefore $\chi_i(C(K_n)) = n$.

Case 2. If n is odd, following the injective coloring of Example 3.2, we assign color t for $1 \leq t \leq n$ to vertices v_i and $u_{i,j}$; $(i < j)$ where $i \in [n]$, $j = n - (i + t - 2)\ (mod\ n)$ respectively. Same as in Case 1, as we see in the Figure 3 (2), similarly, one can see that, there is no path of length 2 between any two vertices with the same color. Therefore $\chi_i(C(K_n)) = n$. □

3.3. $C(W_n)$**,** $C(S_n)$ **and** $C(S_{m,n})$**.** In this subsection, we investigate the injective chromatic numbers of central of wheels, stars, and double stars.

Proposition 3.5. *For wheel W_n, we have*

$$\chi_i(C(W_n)) = \begin{cases} 5 & n = 5 \\ n+1 & otherwise \end{cases}.$$

Proof. Let $V(W_n) = \{v_i| \ 1 \leq i \leq n+1\}$ be the vertex set of W_n and v_{n+1} be a universal vertex. Let $C(W_n)$ be central of W_n with vertex set $V(W_n) \bigcup \{u_{i,j} : 1 \leq i < j \leq n+1\}$, where $u_{i,j}$ is the new vertex corresponding to once subdividing of edge $e = v_iv_j$ where $1 \leq i < j \leq n+1$.

Let $n \in \{4, 5, 6\}$. According to the Figure 4, the proof is obvious.

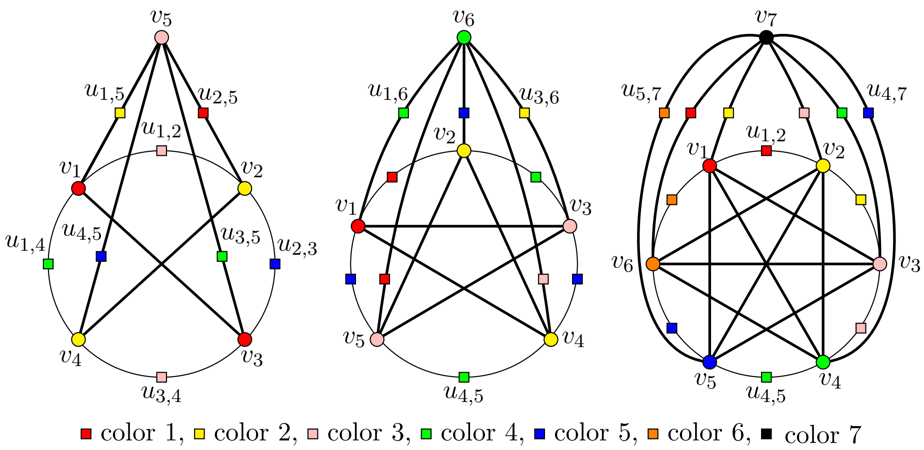


FIGURE 4. Injective coloring of $C(W_n)$ for $n \in \{4, 5, 6\}$

Let $n \geq 7$. According to the structure of graph $C(W_n)$, any two vertices $V(W_n)$ lie either on paths $v_iu_{i,i+1}v_{i+1}$, $v_iu_{i,i-1}v_{i-1}(mod\ n)$, $v_iu_{i,n+1}v_{n+1}$ or on a triangle. Therefore there is a path of length 2 between any two different vertices v_i, v_j in $C(W_n)$. Thus $\chi_i(C(W_n)) \geq n+1$.

Now we construct an injective coloring with $n+1$ colors of $C(W_n)$. We assign color i to the vertices v_i, $u_{i,i+1\ (mod\ n)}, u_{i+2\ (mod\ n),n+1}$, $(1 \leq i \leq n)$ and assign color $n+1$ to the universal vertex v_{n+1} in W_n. This coloring is an injective coloring, because there is no path of length 2 between any two vertices with the same color. Therefore $\chi_i(C(W_n)) = n+1$. □

Proposition 3.6. *If S_n is a star, then $\chi_i(C(S_n)) = n+1$.*

Proof. Let $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ and v_0 be a universal vertex in S_n. Let $C(S_n)$ be the central of S_n with vertex set $V(C(S_n)) = V(S_n) \bigcup \{u_{0,1}, u_{0,2}, \ldots, u_{0,n}\}$ where $u_{0,i}$ is the corresponding vertex to once subdividing of edge $e_i = v_0v_i$ $(1 \leq i \leq n)$ of S_n. There exists a path $v_0u_{0,i}v_i$ of length 2 between every pair of vertices v_0, v_i $(1 \leq i \leq n)$ and the vertices $\{v_1, v_2, \ldots, v_n\}$ form a clique of size n in $C(S_n)$. Thus $\chi_i(C(S_n)) \geq n+1$. Now we construct an injective coloring with $n+1$ colors of $C(S_n)$. We assign color i to the vertices $v_i, u_{0,i}$, $(1 \leq i \leq n)$ and assign color $n+1$ to the vertex v_0. This coloring is an injective coloring. Therefore $\chi_i(C(S_n)) = n+1$. (See Figure 5).

□

Proposition 3.7. *If $S_{n,m}$ is a double star, then $\chi_i(C(S_{n,m})) = n+m+2$.*

Proof. Let $V(S_{n,m}) = \{v_i| \ 1 \leq i \leq n+m+2\}$ be the vertex set of $S_{n,m}$. Two vertices v_{n+1}, v_{m+1} are support vertices in double stars $S_{n,m}$. Let $C(S_{n,m})$ be the central of $S_{n,m}$

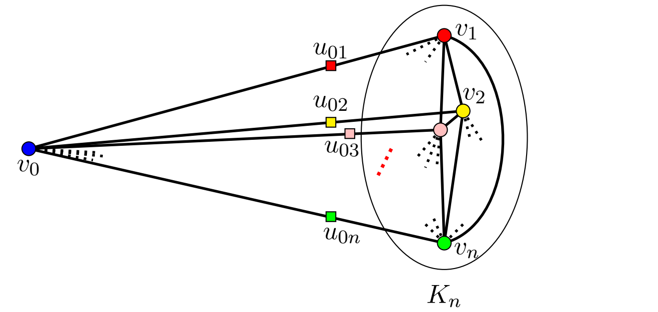


FIGURE 5. Injective coloring of $C(S_n)$

with vertex set $V(C(S_{n,m})) = V(S_{n,m}) \bigcup \{u_{i,n+1}, u_{j,m+1}, u_{n+1,m+1};\ (1 \leq i \leq n, 1 \leq j \leq m)\}$ where u is the corresponding vertex to once subdividing of edge e of $S_{n,m}$.

According to the structure of graph $C(S_{n,m})$, any two different vertices v_i, v_j lie either on paths $v_i u_{i,n+1} v_{n+1}$, $v_j u_{j,m+1} v_{m+1}$, $v_{n+1} u_{n+1,m+1} v_{m+1}$, $(1 \leq i \leq n, 1 \leq j \leq m)$ or on a triangle. So we need at least $n+m+2$ colors. On the other hand, there is no path of length 2 between any two different vertices $v_i, u_{i,n+1}$ and $v_j, u_{j,m+1}$ and $v_{n+1}, u_{n+1,m+1}$. This shows that $v_i, u_{i,n+1}$, $v_j, u_{j,m+1}$, and $v_{n+1}, u_{n+1,m+1}$ can be assigned same color. Therefore $\chi_i(C(S_{n,m})) = n+m+2$. (See Figure 6).

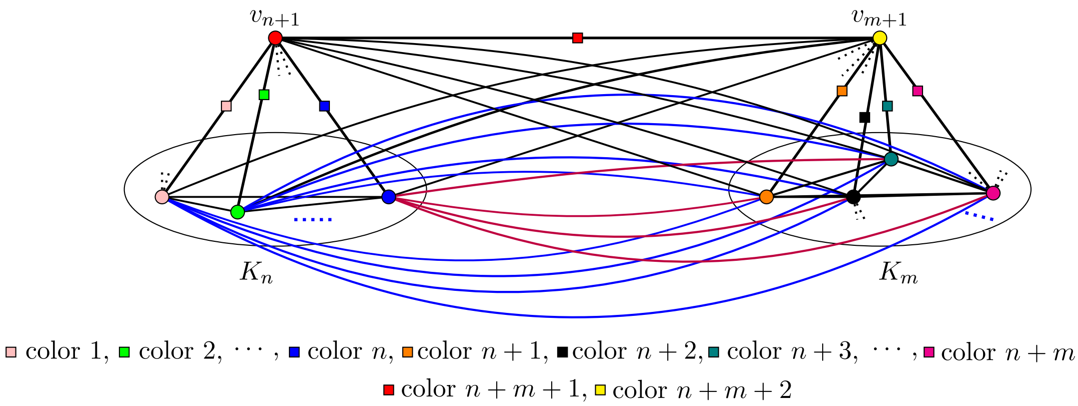


FIGURE 6. Injective coloring of $C(S_{n,m})$

□

3.4. $C(K_{n,m})$**.** To obtain the injective chromatic number of central of complete bipartite graphs, we first present two examples $\chi_i(C(K_{2,5}))$ and $\chi_i(C(K_{3,5}))$, so that we can simplify the proof of $\chi_i(C(K_{n,m}))$.

Example 3.3. $\chi_i(C(K_{2,5})) = 9$. Suppose vertices v_1, v_2 and w_1, w_2, w_3, w_4, w_5 be the vertices of the first and second partite sets of $K_{2,5}$ respectively. Assume that, $u_{1,j}, u_{2,j}$ for $j \in [5]$ be the vertices subdivide the edges $v_1 w_j$ and $v_2 w_j$ in $C(K_{2,5})$ respectively, see Figure 7. Since w_js in $C(K_{2,5})$ form a clique K_5, they get 5 distinct colors. On the other hand, if c is an injective color function of $C(K_{2,5})$, then $c(u_{1,j}) \neq c(u_{2,j})$ $(j \in [5])$, $c(u_{1,i}) \neq c(u_{1,j})$ and $c(u_{2,i}) \neq c(u_{2,j})$ for $1 \leq i \neq j \leq 5$. Hence we infer at most one of $u_{1,j}$ or $u_{2,j}$ can take the color $c(w_j)$. It is obvious that $c(v_k) \neq c(w_j)$ for $k \in [2]$ and $j \in [5]$. The above statements denote that the vertices v_1, v_2 and five vertices in $\{u_{k,j} : k \in [2] \text{ and } j \in [5]\}$ must accept new colors other than $c(w_j)$. Now, if $c(v_1) = c(v_2)$, since non of the five vertices $u_{k,j}$s cannot take $c(v_i)$s, then at least three new colors for these five vertices. if $c(v_1) \neq c(v_2)$, since at most two of these five vertices can take $c(v_i)$s, then at least two new colors for three remaining vertices $u_{k,j}$s. Therefore $\chi_i(C(K_{2,5})) \geq 9$.

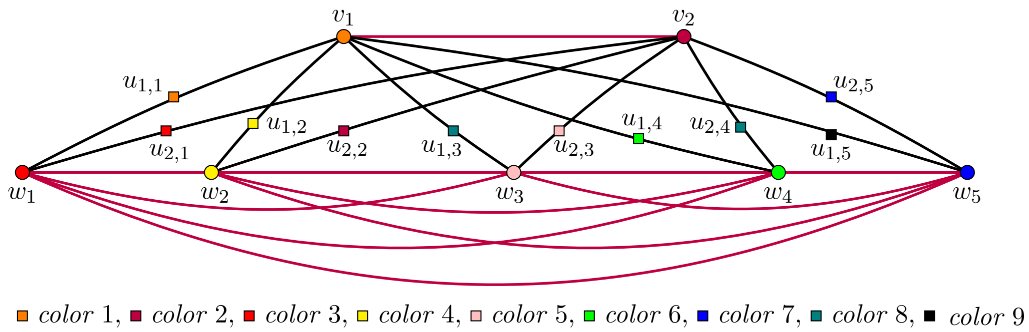


FIGURE 7. Injective coloring of $C(K_{2,5})$

Now if $c(u_{1,1}) = 1 = c(v_1)$, $c(u_{2,2}) = 2 = c(v_2)$, $c(u_{2,1}) = 3 = c(w_1)$, $c(u_{1,2}) = 4 = c(w_2)$, $c(u_{2,3}) = 5 = c(w_3)$, $c(u_{1,4}) = 6 = c(w_4)$, $c(u_{2,5}) = 7 = c(w_5)$, $c(u_{2,4}) = c(u_{1,3}) = 8$ and $c(u_{1,5}) = 9$. Therefore $\chi_i(C(K_{2,5})) = 9$.

Example 3.4. $\chi_i(C(K_{3,5})) = 11$. By examining the Example 3.3, in this example the vertices in each partite set take different colors. If c is an injective color function of $C(K_{3,5})$, then $c(v_k) \neq c(w_j)$ and also the relationships between $c(w_j)$s, $c(v_k)$s and $c(u_{i,j})$s are the same relationships as in the previous example, see Figures 7 and 8. Hence we infer that at most three of $u_{i,j}$s can accept the colors of v_ks and at most five of $u_{i,j}$s can accept the colors of w_js. Also according to the Figure 8 at most three of $u_{i,j}$s which their indices are pairwise distinct, can be assigned with same new color at the same time. Since we have $15 - 8 = 7$, $u_{i,j}$s which has not been colored so far, thus $\chi_i(C(K_{3,5})) \geq 11$.

Now if $c(u_{i,i}) = i = c(v_i)$, $i \in [3]$, $c(u_{2,1}) = 4 = c(w_1)$, $c(u_{3,2}) = 5 = c(w_2)$, $c(u_{1,3}) = 6 = c(w_3)$, $c(u_{2,4}) = 7 = c(w_4)$, $c(u_{3,5}) = 8 = c(w_5)$, $c(u_{3,4}) = c(u_{2,3}) = c(u_{1,2}) = 9$, $c(u_{1,4}) = c(u_{2,5}) = c(u_{3,1}) = 10$ and $c(u_{1,5}) = 11$. Therefore $\chi_i(C(K_{3,5})) = 11$.

Now let's look at the injective chromatic number of an arbitrary complete bipartite graph.

Proposition 3.8. *For complete bipartite graph $K_{n,m}$ with $n, m \geq 2$, we have*

$$\chi_i(C(K_{n,m})) = n + m + \lceil \frac{nm - (n+m)}{min\{n, m\}} \rceil.$$

Proof. Let $N = \{v_1, v_2, \dots, v_n\}$, $M = \{w_1, w_2, \dots, w_m\}$ be two partite sets of graph $K_{n,m}$, $m \geq n \geq 2$. Let $C(K_{n,m})$ be the central of $K_{n,m}$ with vertex set $V(C(K_{n,m})) = V(K_{n,m}) \bigcup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ where $u_{i,j}$ is the corresponding vertex to once subdividing of edge $e_{ij} = v_i w_j$ of $K_{n,m}$. We now consider two situations.

1. Let $n = 2$, $m \geq 2$. If $n = 2$, $m = 2$, then $\chi_i(C(K_{2,2})) = \chi_i(C(C_4)) = 4$ and the result holds.

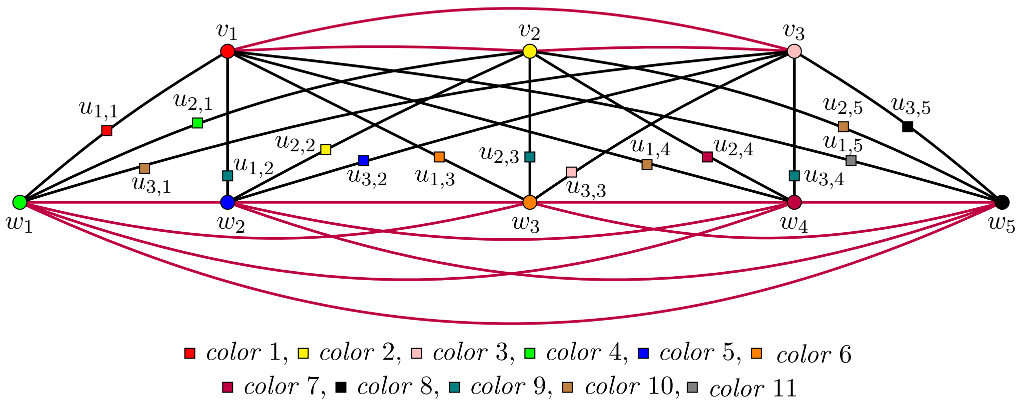


FIGURE 8. Injective coloring of $C(K_{3,5})$

Let $n = 2$, $m > 2$. According to the structure of graph $C(K_{2,m})$, the vertices w_1, $w_2, \ldots, w_m$ form a clique of size m, K_m in graph $C(K_{2,m})$, so we need to have m colors for the vertices $w_j s$, $(1 \leq j \leq m)$.

On the other hand, there exist paths $u_{1,k}w_kw_j$ and $u_{2,k}w_kw_j$, $(k \neq j)$, $u_{1,j}w_ju_{2,j}$, $v_tu_{t,j}w_j$ $(t \in \{1, 2\})$, $u_{1,j}v_1v_2$ and $u_{2,j}v_2v_1$, so the color that we assign to w_j can only be assigned to one of the vertices $u_{1,j}$ and $u_{2,j}$. Now we complete the injective coloring of $V(C(K_{2,m}))$ by coloring the remaining non colored m-vertices $u_{1,j}, u_{2,j}$ $(1 \leq j \leq m)$ and also two vertices v_1, v_2 with new colors. We show that, for injective coloring of remaining m-vertices $u_{1,j}, u_{2,j}$ we need to have at least $\lceil \frac{m}{2} \rceil$ new colors.

Since the vertices $u_{1,j}, u_{1,k}$ have a common neighbor v_1 and the vertices $u_{2,j}, u_{2,k}$, $(j \neq k)$ have a common neighbor v_2 and also the vertices $u_{1,j}, u_{2,j}$ have a common neighbor w_j, so they cannot be the same color. Also we infer that, the vertex $u_{1,j}$ $(u_{2,j})$ can receive the same color only with at most one of the vertices $u_{2,k}$ $(u_{1,k})$ for $1 \leq j \neq k \leq m$. Therefore for injective coloring of remaining m-vertices $u_{1,j}, u_{2,j}$ we need at least $\lceil \frac{m}{2} \rceil$ new colors.

Since the vertex v_1 (v_2) must be assigned with a new color(s) other than the colors of w_js and $u_{2,k}$ $(u_{1,k})$, depending on whether m is an even or odd integer, we deduce that $\chi_i(C(K_{2,m})) \geq 1 + m + \lceil \frac{m}{2} \rceil = 2 + m + \lceil \frac{2m-(2+m)}{2} \rceil$, see Example 3.3.

Now we show an injective coloring with $1 + m + \lceil \frac{m}{2} \rceil$ colors of $C(K_{2,m})$.

Let c be a color function on G. Consider $c(v_i) = i = c(u_{i,i})$ for $i \in [2]$, $c(w_j) = 2 + j$ for $j \in [m]$. Now we put $c(u_{2,r}) = c(w_r)$, for odd integers $r \in [m]$ and $c(u_{1,r}) = c(w_r)$, for even integers $r \in [m]$, and if m is odd, then we put $c(u_{1,2i+1}) = m + 2 + i = c(u_{2,2i+2})$ for $i \in [\frac{m-3}{2}]$ and $c(u_{1,m}) = m + 2 + \frac{m-1}{2}$, and if m is even, then we put $c(u_{1,2i+1}) = m + 2 + i = c(u_{2,2i+2})$ for $i \in [\frac{m-2}{2}]$. It is easy to see that the given color function is an injective color function with $1 + m + \lceil \frac{m}{2} \rceil = 2 + m + \lceil \frac{2m-(2+m)}{2} \rceil$ colors. Therefore,

$$\chi_i(C(K_{2,m})) = 2 + m + \lceil \frac{2m - (2 + m)}{2} \rceil.$$

2. Let $n, m > 2$ and without loss of generality $m \geq n \geq 3$. In two partite sets of $K_{n,m}$, the vertices $v_1, v_2, \ldots, v_n$ induces a clique of size n and the vertices $w_1, w_2, \ldots, w_m$ induce a clique of size m , $(n, m \geq 3)$. Also, there exists a path $v_iu_{i,j}w_j$, $(1 \leq i \leq n,\ 1 \leq j \leq m)$ of length 2 between any pair of vertices v_i, w_j in $C(K_{n,m})$. Thus we need to have $n + m$ distinct colors for the vertices $v_i s$, $w_j s$.

For an arbitrary vertex $u_{t,k}$, there exist paths $u_{t,k}w_kw_j$, $(k \neq j)$ and $u_{t,k}v_tv_i$, $(t \neq i)$, so we can only assign the color v_t or w_k to the vertex $u_{t,k}$. On the other hand, the vertices $u_{t,k}, u_{t,j}$, $(k \neq j)$ have a common neighbor v_t and also the vertices $u_{t,k}, u_{i,k}$, $(t \neq i)$ have a common neighbor w_k, thus the color that we have assigned to v_i so far, can only be used to at most one vertex $u_{i,j}$ and the color one we have assigned to w_j can only be used to at most one vertex $u_{i,j}$ too, for $i \in [n]$ and $j \in [m]$. We demonstrate, to the injective coloring for $mn - (m + n)$ the remaining vertices $u_{i,j}$s in $C(K_{n,m})$, we need at least $\lceil \frac{mn-(m+n)}{min\{n,m\}} \rceil$ new colors.

For this purpose, since the vertices $u_{i,j}, u_{i,k}$ where $j \neq k$ have a common neighbor v_i and the vertices $u_{i,j}, u_{t,j}$ where $i \neq t$ have a common neighbor w_j, so they cannot be the same color. Since there is no path of length 2 between the vertices $u_{i,j}, u_{t,k}$ where $i \neq t$ and $j \neq k$, they can receive the same new color. That is, n vertices $u_{t_1,l_1}, u_{t_2,l_2}, \ldots, u_{t_n,l_n}$ receive the same new color whenever the left indices t_is are pairwise distinct and the right indices l_is are pairwise distinct too. Since at most n vertices of $mn-(m+n)$ vertices can have same color, so we need at least $\lceil \frac{mn-(m+n)}{n} \rceil$ new colors for injective coloring of the remaining $mn-(m+n)$ vertices in $C(K_{n,m})$. Therefore we need at least $m+n+\lceil \frac{mn-(m+n)}{min\{n,m\}} \rceil$ different colors for injective coloring of $C(K_{n,m})$, and $\chi_i(C(K_{n,m})) \geq m+n+\lceil \frac{mn-(m+n)}{min\{n,m\}} \rceil$, see Example 3.4.

Now we give an injective coloring with $n+m+\lceil \frac{nm-(n+m)}{min\{n,m\}} \rceil$ colors for $C(K_{n,m})$. Assume that c is a color function on $C(K_{n,m})$. We consider $c(v_i) = i$ for $i \in [n]$ and $c(w_j) = n + j$ for $j \in [m]$. Since $n \leq m$, we may assume $m = kn + r$ $(0 \leq r \leq n - 1)$, then we let $c(u_{ii}) = i$ for $i \in [n]$, $c(u_{j+1,j}) = c(w_j)$ for $j \in [n-1]$ and $c(u_{j,j+sn-1}) = c(w_{j+sn-1})$ for $j \in [n]$ and $s \in [k-1]$ or $j \in [r+1]$ and $s = k$. It is clear, already we use $m + n$ colors for $2(m + n)$ vertices, and until now $mn - (m + n)$ vertices remain without colors.

Consider two vertices $u_{t,l}$ and $u_{s,p}$ where $u_{t,l}$ subdivides edge v_tw_l and $u_{s,p}$ subdivides edge v_sw_p. If $t \neq s$ and $l \neq p$, then $d(u_{t,l}, u_{s,p}) = 3$ and since there exist at most n vertices with the property such that the left indices are pairwise distinct and also the right indices, we put $c(u_{s,p}) = c(u_{t,l})$. On the other hand, there are exactly $\lceil \frac{nm-(n+m)}{n} \rceil$ sets of non colored $u_{i,j}$s that each set has at most n vertices with the mentioned property. Since $\lceil \frac{nm-(n+m)}{n} \rceil = m - k - 1$, let $S_1, S_2, \ldots, S_{m-k-1}$ be the all these sets. Then we put $c(S_h) = m + n + h$ for $h \in [m - k - 1]$. From the above process, the given color function c is an injective color function of $C(K_{n,m})$ with $n + m + \lceil \frac{nm-(n+m)}{min\{n,m\}} \rceil$ colors. Therefore,

$$\chi_i(C(K_{n,m})) = n + m + \lceil \frac{nm - (n + m)}{min\{n, m\}} \rceil.$$

□

4. Trees T

In Section 3 we studied the injective chromatic number of some special trees such as P_n, S_n, and $S_{n,m}$. In this section, we want to study the injective chromatic number of any tree.

Breadth First Search (BFS) is a fundamental algorithm in computer science for searching all vertices of a tree. The algorithm starts from the root vertex and examines all vertices at the current depth before moving to the vertices at the next depth level.

Proposition 4.1. *If $T_{p,q}$ is a tree of order $n = p+q+3$ which is obtained from double star $S_{p,q}$ with support vertices u and v, by subdividing the edge uv once, then $\chi_i(C(T_{p,q})) = n$.*

Proof. Let w be the vertex that subdivide edge uv. Then all leaves and the vertex w will be pairwise adjacent in $C(T_{p,q})$. On the other hand, from the property of central of $T_{p,q}$ the vertices u, v cannot be assigned by color of w. If the vertices u and v are same color, since we have $n-1$ vertices of degree two in $C(T_{p,q})$, then we cannot injective color to them with the $n-1$ colors which be already used for the vertices of $T_{p,q}$ in $C(T_{p,q})$. Therefore $\chi_i(C(T_{p,q})) \geq n$.

Now suppose that $w, v, v_1, \cdots, v_p$ and $u, u_1, \cdots, u_q$ are the vertices of $T_{p,q}$ such that w subdivides uv, v_i is a leaf adjacent to v and u_j is a leaf adjacent to u for $i \in [p]$ and $j \in [q]$ respectively. Let $x_{1,i}$, $y_{1,j}$, x and y subdivides vv_i, uu_j, wv and uw respectively in $C(T_{p,q})$. If c is a color function in which $c(v_i) = i = c(x_{1,i})$, $c(u_j) = p + j = c(y_{1,j})$, $c(w) = p + q + 1 = c(x)$, $c(v) = p + q + 2 = c(u)$ and $c(y) = p + q + 3 = n$ for $i \in [p]$ and $j \in [q]$ respectively. Then c is an injective color function for $C(T_{p,q})$. Therefore $\chi_i(C(T_{p,q})) = n$. □

Theorem 4.1. *For any tree T of order at least $n \geq 3$,*

$$\chi_i(C(T)) = n.$$

Proof. If T is one of S_n, $S_{p,q}$, or $T_{p,q}$, it has been studied previously. Let $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ be the vertex set of T. Let v_0 be a vertex of maximum degree $\Delta = \Delta(T)$. We use v_0 as the root of a BFS of T, where the maximum distance from the root is d and define L_i as the set of vertices at distance i from the root. Let $C(T)$ be the central of T, with vertex set $V(T) \bigcup \{u_{i,j},\ 0 \leq i < j \leq n-1\}$, where $u_{i,j}$ is the vertex that subdivides the edge $e = v_i v_j$.

According to the structure of graph $C(T)$, we know that:

(1) All vertices $v_j s$ in each L_i for $i = 0, 1, 2, \ldots, d$ in T, induce a clique in $C(T)$, so we have $|L_i|$ cliques for each $0 \leq i \leq d$.

(2) If two vertices v_i, v_j are adjacent in T, then they have a common neighbor u_{ij} in $C(T)$.

(3) If two vertices v_i, v_j are non-adjacent in T, since $T \notin \{S_n, S_{p,q}, T_{p,q}\}$, there is a vertex u in T such that $u \notin N(v_i) \cup N(v_j)$, then three vertices u, v_i, v_j will be adjacent together in $C(T)$.

From the situations 1, 2, 3 we infer that any two vertices in T must be assigned with distinct colors in $C(T)$. Therefore $\chi_i(C(T)) \geq n$.

Now we show an injective coloring with n colors of $C(T)$. For this purpose, we assign color n to the vertex v_0 and assign color j, $1 \leq j \leq n-1$ to the vertices $u_{i,j}$ and v_j. It can be shown with a not too difficult examination, this coloring is an injective coloring of $C(T)$. Therefore, $\chi_i(C(T)) = n$, see Figure 9. □

5. Central of a graph with corona K_1 and K_2

In this section we argue on the injective coloring of central of the corona product of a graph with K_1 and K_2.

5.1. $H \circ K_1$**.** If $H = \overline{K_2}$ consists of two isolated vertices or K_2, then it is easy to verify that $\chi_i(C(H \circ \overline{K_1})) = 3$ and $\chi_i(C(H \circ K_1)) = 4$. Now in general we have.

Theorem 5.1. *Let H be a graph of order $n \geq 3$ and $G = H \circ K_1$. Then*

$$\chi_i(C(H \circ K_1)) = \begin{cases} \chi_i(C(H)) + n & \text{if } \chi_i(C(H)) \geq n \\ 2n & \text{if } \chi_i(C(H)) = n-1 \end{cases}$$

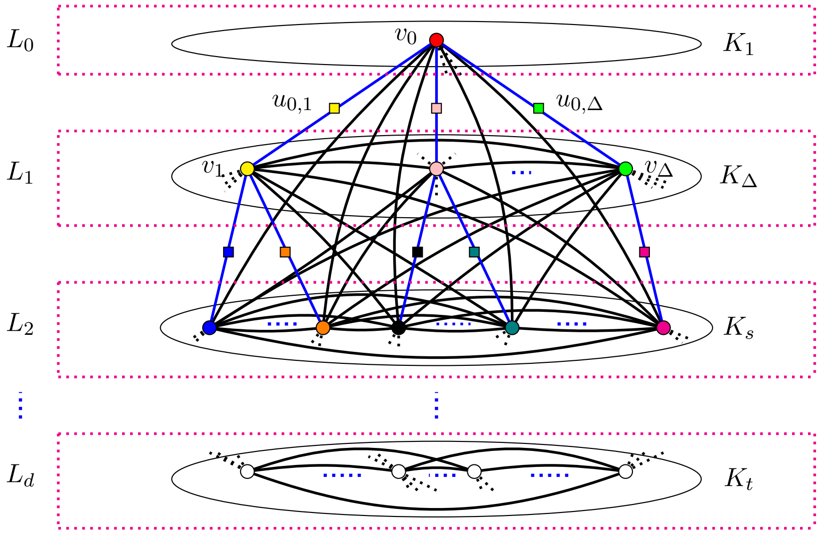


FIGURE 9. Injective coloring of $C(T)$ with maximum degree Δ

Proof. Let $V(H) = \{v_1, \dots, v_n\}$ and $V(G) = V(H) \bigcup \{w_1, w_2, \dots, w_n\}$. Let $V(C(H)) = V(H) \bigcup \{u_{i,j} : 1 \le i < j \le n\}$ and $V(C(G)) = V(G) \bigcup \{u_{i,j} : 1 \le i < j \le n\} \bigcup \{u_{k,k} : 1 \le k \le n\}$, where $u_{i,j}$ and $u_{k,k}$ are the new vertices subdivide the edges $e = v_iv_j$ and $e = v_kw_k$, respectively, see Figure 10.

Let $\chi_i(C(H)) = k \ge n$ and c be an injective coloring function of $C(G)$. Since $C(H)$ is an induced subgraph of $C(G)$, then $\chi_i(C(G)) \ge k$. According to the structure of graph $C(G)$, the set of vertices $\{w_1, w_2, \cdots, w_n\}$ forms a clique of order n (K_n) in the graph $C(G)$, thus we $c(w_i) \ne c(wj)$ for $i \ne j$. We explore that for any vertex w_i, $c(w_i) \ne c(v_j)$ and $c(w_i) \ne c(u_{r,s})$ for any v_j and any $u_{r,s}$. For this aim, consider the following cases:

(1) We observe the path $w_iu_{i,i}v_i$, $(1 \le i \le n)$ in graph $C(G)$, so it is clearly $c(w_i) \ne c(v_i)$.

(2) We observe the path $w_iw_jv_k$, $(1 \le i \ne j \le n)$ and $(1 \le j \ne k \le n)$ in graph $C(G)$, so we have $c(w_i) \ne c(v_k)$.

(3) We observe the path $w_iv_ju_{k,j}$ or $w_iv_ju_{j,l}$ $(1 \le i \ne j \le n)$, $(1 \le k < j < l \le n)$ in graph $C(G)$, so we explicitly have $c(w_i) \ne c(u_{k,j})$, $c(w_i) \ne c(u_{j,l})$. Therefore $\chi_i(C(G)) \ge \chi_i(C(H)) + n$.

Now we give an injective coloring of $C(G)$ using $\chi_i(C(H)) + n$ distinct colors as follows.

According to the suppositions, let us have an injective coloring of $C(H)$ using m colors, such that all vertices $V(H)$ receive distinct colors, then we assign new color $m + i$ to the vertices $w_i, u_{i,i}$ where $1 \le i \le n$. This coloring is an injective coloring of $C(G)$, see Figure 10. Therefore $\chi_i(C(G)) = \chi_i(C(H \circ K_1)) = \chi_i(C(H)) + n$.

Let $\chi_i(C(H)) = n - 1$ and c be an injective coloring function of $C(G)$. Similar to the part (3) of the above, $c(w_i) \ne c(u_{k,j})$, $c(w_i) \ne c(u_{j,l})$. On the other hand, since we have the path $v_rw_jv_s$ $(r \ne s$ and $j \notin \{r, s\}$ in $C(G))$, then $c(v_r) \ne c(v_s)$. Thus in graph $C(G)$ the vertices of $C(H)$ should be injective colored with at least n different colors. Therefore $\chi_i(C(G)) \ge 2n$. Now we give an injective coloring of $C(G)$ with $2n$ distinct colors as follows.

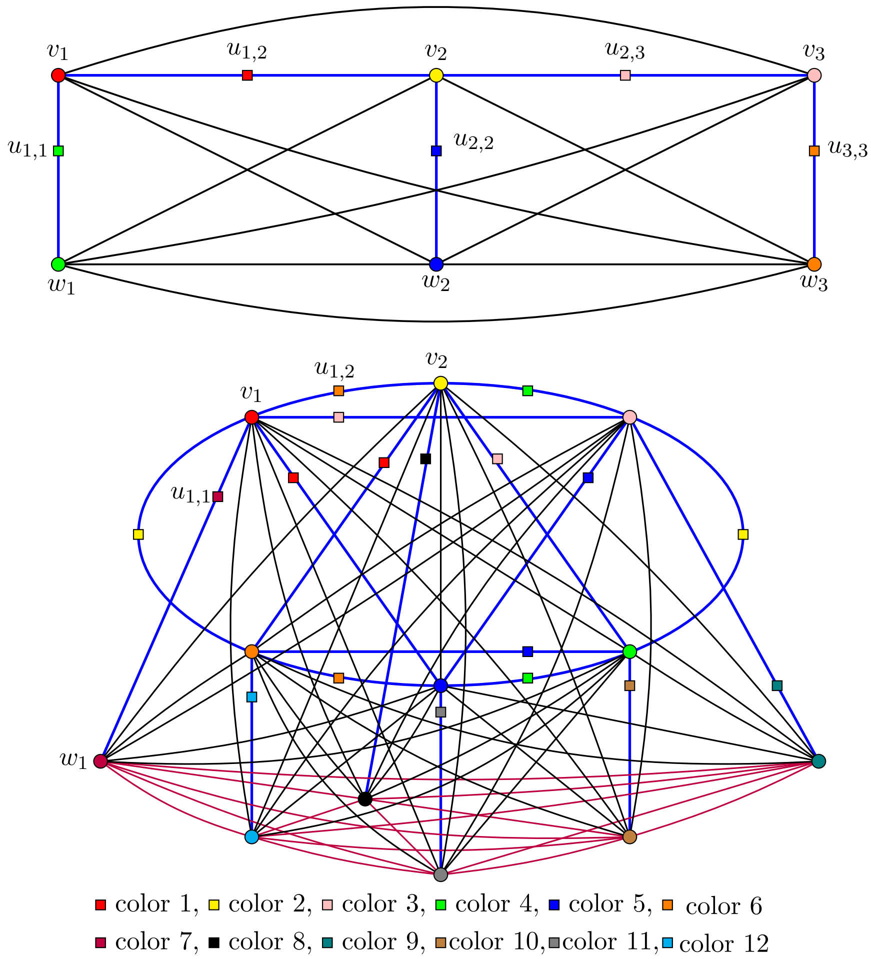


FIGURE 10. Injective coloring of central of the corona product of P_3 and the octahedral with K_1

Let we have an injective coloring of $C(H)$ using $n-1$ colors, without loss of generality, we change the colors v_i in $(C(H)$ such that $c(v_r) \neq c(v_s)$ for $r \neq s$. Then we assign new color $n+i$ to the vertices $w_i, u_{i,i}$ where $1 \leq i \leq n$. This coloring is an injective coloring of $C(G)$. Therefore $\chi_i(C(G)) = \chi_i(C(H \circ K_1)) = 2n$, see Figure 10. □

5.2. $H \circ K_2$**.** In this subsection, we investigate the injective chromatic of $G \circ K_2$ for any graph G. If $H = \overline{K_2}$ consists of two isolated vertices, then it is easy to verify that $\chi_i(C(H \circ K_2)) = 6$ and $\chi_i(C(H \circ \overline{K_2})) = 5$. Now in general we have.

Theorem 5.2. *Let $H \neq \overline{K_2}$ be a graph of order n and $G = H \circ K_2$. Then*

$$\chi_i(C(H \circ K_2)) = \begin{cases} \chi_i(C(H)) + 2n & \text{if } \chi_i(C(H)) \geq n \\ 3n & \text{if } \chi_i(C(H)) = n-1 \end{cases}$$

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$ and $V(G) = V(H) \bigcup \{w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_n, \}$. Let $V(C(H)) = V(H) \bigcup \{u_{i,j} : 1 \leq i < j \leq n\}$ and $V(C(G)) = V(G) \bigcup \{u_{i,j} : 1 \leq i < j \leq n\} \bigcup \{x_{k,k}, y_{k,k}, z_{k,k} : 1 \leq k \leq n\}$, where $u_{i,j}$, $x_{k,k}$, $y_{k,k}$ and $z_{k,k}$ are the new vertices that subdivide the edges $e = v_i v_j$, $e = u_k w_k$, $e = v_k w_k$ and $e = v_k u_k$ respectively, see Figure 11.

Let $\chi_i(C(H)) = k \geq n$ and c be an injective coloring function of $C(G)$. Since $C(H)$ is an induced subgraph of $C(G)$, then $\chi_i(C(G)) \geq k$. According to the structure of graph $C(G)$, the set of vertices $\{w_1, w_2, \ldots, w_n, u_1, u_2, \ldots, u_n\}$ induce a subgraph of order $2n$ which is isomorphic with $K_{2n} - M$ where M is the perfect matching $\{u_i w_i : 1 \leq i \leq n\}$ in $C(G)$.

Since in $C(G)$, $d(u_i, w_i) = 2$, under any injective coloring c of $C(G)$, we have $c(u_i) \neq c(w_i)$. On the other hand, it is obvious that any pair of vertices in $\{u_r, u_s w_i, w_j\}$ for distinct indices, share in a vertex, therefore all vertices u_is and w_js achieve distinct colors.

We now explore that the colors of vertices of u_is and w_is should be different from the colors of all vertices of $C(H)$. For this aim, consider the following cases:

(1) We observe the path $w_i y_{i,i} v_i$ and $u_i z_{i,i} v_i$, $(1 \leq i \leq n)$ in graph $C(G)$, so we explicitly have $c(w_i) \neq c(v_i)$ and $c(u_i) \neq c(v_i)$.

(2) We observe the path $w_i w_j v_k$ and $u_i u_j v_k$, $(1 \leq i \neq j \leq n)$ and $(1 \leq j \neq k \leq n)$ in graph $C(G)$, so we explicitly have $c(w_i) \neq c(v_k)$ and $c(u_i) \neq c(v_k)$

(3) We observe the paths $w_i v_j u_{k,j}$ or $w_i v_j u_{j,l}$, and the paths $u_i v_j u_{k,j}$ or $u_i v_j u_{j,l}$, $(1 \leq i \neq j \leq n)$, $(1 \leq k < j < l \leq n)$ in graph $C(G)$, so we explicitly have $c(w_i) \neq c(u_{k,j})$, $c(w_i) \neq c(u_{j,l})$, and $c(u_i) \neq c(u_{k,j})$, $c(u_i) \neq c(u_{j,l})$. Therefore $\chi_i(C(G)) \geq \chi_i(C(H)) + 2n$.

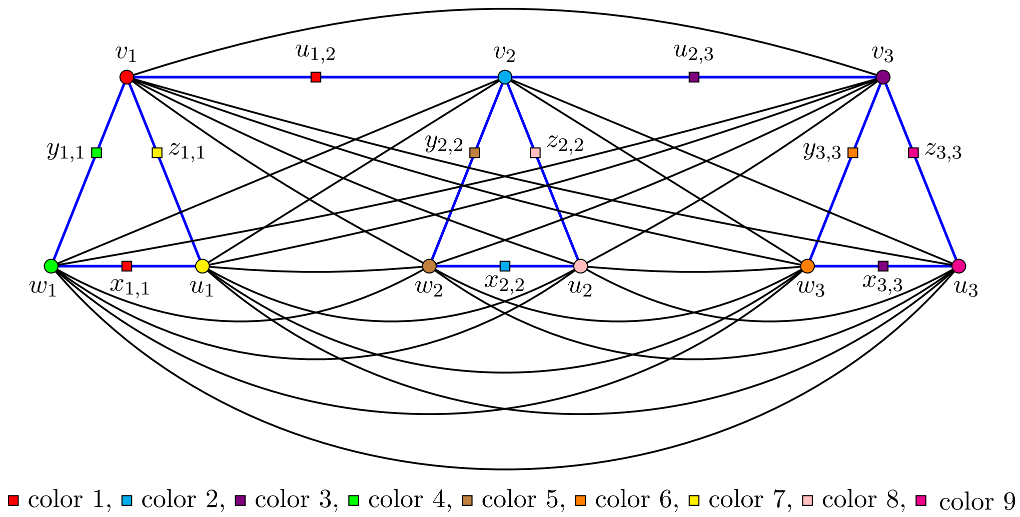


FIGURE 11. Injective coloring of $C(P_3 \circ K_2)$

Now, we give the injective coloring of $C(G)$ using $\chi_i(C(H)) + 2n$ distinct colors.

Let c be a chromatic injective function of $C(H)$ and $\chi_i(C(H)) = m$. Assign colors such that all vertices $V(H)$ receive distinct colors. If $c(v_i)$ be the color of v_i, then we consider $c(w_i) = m + i = c(y_{i,i})$, $c(u_i) = m + i + n = c(z_{i,i})$ and $c(x_{i,i}) = c(v_i)$ where $1 \leq i \leq n$. This coloring is an injective coloring of $C(G)$. Therefore $\chi_i(C(G)) = \chi_i(C(H \circ K_2)) = \chi_i(C(H)) + 2n$.

Let $\chi_i(C(H)) = n - 1$ and c be an injective coloring function of $C(G)$. Similar to the above proof we have $2n$ different colors for the injective coloring of u_is and w_js. Since we have the path $v_r w_j v_s$ ($r \neq s$ and $j \notin \{r, s\}$ in $C(G)$), then $c(v_r) \neq c(v_s)$ under injective coloring c of $C(G)$. Thus in graph $C(G)$ the vertices of $C(H)$ should be injective colored with at least n different colors. Therefore $\chi_i(C(G)) \geq 3n$.

Now, we give an injective coloring of $C(G)$ with $3n$ distinct colors as follows.

Let we have an injective coloring of $C(H)$ using $n-1$ colors, without loss of generality, we change the colors v_i in $C(H)$ such that $c(v_r) \neq c(v_s)$ for $r \neq s$. Then we consider $c(w_i) = n+i = c(y_{i,i})$, $c(u_i) = 2n+i = c(z_{i,i})$ and $c(x_{i,i}) = c(v_i)$ where $1 \leq i \leq n$. This coloring is an injective coloring of $C(G)$. Therefore $\chi_i(C(G)) = \chi_i(C(H \circ K_2)) = 3n$, see Figure 11. □

From Theorems 5.1, 5.2, and the results of Section 3 we obtain the following immediate results.

Corollary 5.1. *For some special graphs corona with K_1, we have:*

(1) $\chi_i(C(K_{n,m} \circ K_1)) = 2(n+m) + \lceil \frac{nm-(n+m)}{min\{n,m\}} \rceil$, $\chi_i(C(K_{n,m} \circ K_2)) = 3(n+m) + \lceil \frac{nm-(n+m)}{min\{n,m\}} \rceil$.

(2) $\chi_i(C(P_n \circ K_1)) = 2n$, $\chi_i(C(P_n \circ K_2)) = 3n$.

(3) $\chi_i(C(C_n \circ K_1)) = 2n$, $\chi_i(C(C_n \circ K_2)) = 3n$.

(4) $\chi_i(C(K_n \circ K_1)) = 2n$, $\chi_i(C(K_n \circ K_2)) = 3n$.

(5) $\chi_i(C(W_n \circ K_1)) = 2(n+1)$, $\chi_i(C(W_n \circ K_2)) = 3(n+1)$.

(6) $\chi_i(C(S_n \circ K_1)) = 2(n+1)$, $\chi_i(C(S_n \circ K_2)) = 3(n+1)$.

(7) $\chi_i(C(S_{n,m} \circ K_1)) = 2(n+m+2)$, $\chi_i(C(S_{n,m} \circ K_2)) = 3(n+m+2)$.

6. Concluding remarks

In this paper, we introduced and studied the concept of an injective coloring in the central graph of a graph. The concept of injective coloring in the middle graph of a graph, in the hypercube, in the generalization of Peterson graphs and in the family of Harary graphs has not been studied yet. The injective coloring of central, and middle of type of product of two graphs can be investigated.

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Shahrzad Sadat Mirdamad is pursuing a Ph.D. in Graph Theory and Combinatorics at the University of Mazandaran, Babolsar, Iran. She was an instructor of Statistics and Mathematics at the Faculty of Mathematics, Islamic Azad University, Tehran, Iran, from 2016 to 2023. Her main area of expertise is graph coloring. She has published several articles on various topics related to graph coloring in research journals.

Doost Ali Mojdeh received his Ph.D. degree in Mathematics from Sharif University of Technology, Tehran, Iran, in 1996. He is currently a full professor in the Department of Mathematics at the University of Mazandaran, Babolsar, Iran. His current research interests include Graph Theory (domination and coloring in graphs) and Combinatorics.