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NEWLY DEVELOPED SINGLE-STEP BLOCK METHOD FOR NUMERICAL SOLUTION OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This paper focuses on the development of a new block method for solving fourth order initial value problems of ordinary differential equations. Applying Chebyshev polynomial as a basis function, the method was developed using interpolation and collocation approaches. The convergence property of the method was established with zero-stability and consistency. Comparison was made with existing method, and the newly developed method compares favourably well.

Numerical solution, Chebyshev polynomial, Error analysis, Zero stability, Convergence

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1. Introduction

Mathematical formulations arise to aid the understanding of physical phenomena. These formulations generate differential equations that involve different derivatives of an unknown function of one or more variables. These equations appear in different fields such as physical and biological sciences, engineering, agricultural sciences, medicine, etc. Kuboye and Omar [10] provided solution to cubic initial value problem of ordinary aspect of the differential equations by applying a multi-step sequencing approach involving a new block method satisfying zero-stability, consistency and convergence. Adeyeye and Omar [3] developed a hybrid block method involving equidistant hybrid points for solving third order ODEs. Moreover in [2], they discussed maximal order block approach for solving second order ODEs. Furthermore, Adeyeye and Omar [4] suggested an algorithm for developing k-step block methods to solve m-order ODEs. The algorithm applies the concept of Taylor series approach to develop multi-step linear methods. Omar and Abdelrahim [11] applied a single step involving three hybrid points block method in solving some third order ODEs.

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Ishak et al [15] discussed the application of two-point block method for solving functional differential equations. Sagir [14] applied discrete linear multistep block method with uniform order in solving first order initial value problems of ODEs. A method with higher hidden volumes for direct numerical solving of quaternary ODEs was discussed in Atabo and Adee [6]. The construction of the new formula consists of 15 steps performed using interpolation and sequencing techniques. Moreover, Adeyefa et al [1] discussed orthogonal dependent numerical method for second order initial value problems in ODEs.

In this paper, we develop a new numerical method for solving fourth order initial value problem of ODEs directly using Chebyshev polynomial as the basis functions. This is achieved by developing a continuous implicit scheme with step number $k = 1$, deriving discrete schemes and its derivatives from the continuous implicit scheme, and developing a new block method. We further examine properties of the newly developed block method which include consistency, zero-stability and convergence; implement the method using maple software, and compare the result with some existing methods. Other part of the work is given as follows: Section 2 discusses the methodology; the new single-step block method for solving fourth order ODEs was developed in Section 3, while Section 4 discusses the implementation of the new method and necessary comparison with some existing methods. Then, conclusion follows.

2. Methodology

Consider the general fourth-order initial value problem of the form:

$$y^{iv}(x) = f(x, y, y', y'', y'''), \tag{1}$$

with initial conditions;

$$y(a) = \eta_0, y'(a) = \eta_1, y''(a) = \eta_2, y'''(a) = \eta_3, \quad x \in [a, b].$$

The starting point plays no special role in a one-step method, that is, the value of y_{n+1} can be obtained if y_n is known; while in a multistep method, the value of y_{n+1} requires the exact knowledge of both y_n and a certain number of preceding values $y_{n-1}, y_{n-2}, \ldots$.

Block Method is a numerical technique used for solving ODEs. It involves dividing the interval of integration into smaller subintervals or blocks and solving the differential equation iteratively over these blocks. This method is particularly useful for solving stiff ODEs and systems of ODEs. In the block method, the interval $[a, b]$ is divided into n subintervals, and the solution is computed at discrete points within each subinterval. The method can use implicit or explicit formulas to estimate the values of the solution at multiple points simultaneously.

Step-size h refers to the distance between consecutive points in the interval over which the solution is being approximated. It determines the precision and stability of the numerical solution. A smaller h gives a better accurate solution but involves more computational steps. Thus, the step size gives increment in an independent variable between successive points where solution is computed.

Collocation is a method used in numerical analysis to approximate solutions of differential equations. In this method, the solution is assumed to be a linear combination of a set of basis functions, and the coefficients of these basis functions are determined by ensuring that the differential equation is satisfied at specific points referred to as collocation points within the given domain.

Interpolation is a method of constructing new data points within the range of a discrete set of known data points. It involves finding a function that exactly passes through given data points.

Chebyshev method is a numerical technique used to solve ODEs and approximate functions. It leverages Chebyshev polynomials, which are a sequence of orthogonal polynomials that are useful in approximation theory.

The major concern of this work is the solution of ODEs by numerical methods, the focus is on the initial value problems (IVPs) of the fourth order differential equations.

2.1. **Formulation of the New Method.** We propose a block method of the form:

$$\sum_{i=0}^{k} \alpha_i(t) y_{n+i} = h^4 \left(\sum_{i=0}^{k} \beta_i(t) f_{n+i} \right)$$

for the solution of

$$\begin{cases} y^{(iv)}(x) & = f(x, y(x)), y'(x), y''(x), y'''(x) \\ y(x_0) & = y_0,\ y'(x_0) = y'_0,\ y''(x_0) = y''_0,\ y'''(x_0) = y'''_0 \end{cases} \tag{2}$$

where either of $\alpha_0(t)$ and $\beta_0(t)$ do not vanish, $\alpha_k(t) = 1$, $\beta_k(t) = 0$ and $k = 1$.

Chebyshev polynomials of the first kind serve as our basis function which was interpolated at x_n while the fourth derivatives of the basis function were collocated at selected grid and off-grid points. The resulting equations were reduced to system of equation in matrix form $\mathbf{AX} = \mathbf{B}$ and solved using Gaussian elimination method for the values $a_j s$. These were substituted into the basis function and simplified to produce continuous implicit schemes which were evaluated at non-interpolation points and were applied in a block-by-block manner as numerical integrators for fourth-order IVPs of ODEs.

3. Results

With initial conditions

$$y(x_0) = y_0,\ y'(x_0) = y'_0,\ y''(x_0) = y''_0,\ y'''(x_0) = y'''_0; \tag{3}$$

a new one-step hybrid method capable of solving fourth order ODEs is formulated by employing Chebyshev polynomials as our basis function. Thus, we introduce

$$\sum_{j=0}^{k+5} a_j T_n^j = y_{n+r} \tag{4}$$

using a single step length ($k = 1$) where a_j's are parameters to be determined and $t_j(x)$ represent parameters of the Chebyshev polynomials.
Equation(4) is interpolated at x_{n+r} where $r = 0, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}$; while fourth derivative of equation (4) which is;

$$\sum_{j=2}^{k+8} j(j-3) a_j T_n^{j-4} = f_{n+m} \tag{5}$$

is collocated at x_{n+m} where $m = 0, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}$ and 1.

The interpolating equation is as follows:
At $x = x_n$,

$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - a_9 = y_n. \tag{6}$$

At $x = x_{n+\frac{h}{24}}$,

$$a_0 - \frac{11}{12}a_1 + \frac{49}{72}a_2 - \frac{143}{432}a_3 - \frac{191}{2592}a_4 + \frac{7249}{15552}a_5 - \frac{72863}{93312}a_6 + \frac{540529}{559872}a_7 - \frac{3322751}{3359232}a_8 + \frac{17091217}{20155392}a_9 = y_{n+\frac{h}{24}}. \tag{7}$$

At $x = x_{n+\frac{h}{12}}$,

$$a_0 - \frac{5}{6}a_1 + \frac{7}{18}a_2 + \frac{5}{27}a_3 - \frac{113}{162}a_4 + \frac{475}{486}a_5 - \frac{679}{729}a_6 + \frac{2515}{4374}a_7 - \frac{353}{13122}a_8 - \frac{10435}{19683}a_9 = y_{n+\frac{h}{12}}. \tag{8}$$

At $x = x_{n+\frac{h}{8}}$,

$$a_0 - \frac{3}{4}a_1 + \frac{1}{8}a_2 + \frac{9}{16}a_3 - \frac{31}{32}a_4 + \frac{57}{64}a_5 - \frac{47}{128}a_6 + \frac{87}{256}a_7 + \frac{449}{512}a_8 - \frac{999}{1024}a_9 = y_{n+\frac{h}{8}}. \tag{9}$$

The collocating equation:
At $x = x_n$,

$$\frac{3072_{a_4}}{h^4} - \frac{30720_{a_5}}{h^4} + \frac{165888_{a_6}}{h^4} - \frac{645120_{a_7}}{h^4} + \frac{2027520_{a_8}}{h^4} - \frac{5474304_{a_9}}{h^4} = f_n. \tag{10}$$

At $x = x_{n+\frac{h}{24}}$,

$$\frac{3072_{a_4}}{h^4} - \frac{28160_{a_5}}{h^4} + \frac{136448_{a_6}}{h^4} - \frac{4188800}{9}\frac{a_7}{h^4} + \frac{33796480}{27}\frac{a_8}{h^4} - \frac{25327808}{9}\frac{a_9}{h^4} = f_{n+\frac{h}{24}}. \tag{11}$$

At $x = x_{n+\frac{h}{12}}$,

$$\frac{3072_{a_4}}{h^4} - \frac{25600_{a_5}}{h^4} + \frac{109568_{a_6}}{h^4} - \frac{2867200}{9}\frac{a_7}{h^4} + \frac{18810880}{27}\frac{a_8}{h^4} - \frac{10700800}{9}\frac{a_9}{h^4} = f_{n+\frac{h}{12}}. \tag{12}$$

At $x = x_{n+\frac{1}{8}}$,

$$\frac{3072_{a_4}}{h^4} - \frac{23040_{a_5}}{h^4} + \frac{85248_{a_6}}{h^4} - \frac{201600_{a_7}}{h^4} + \frac{320640a_8}{h^4} - \frac{295488_{a_9}}{h^4} = f_{n+\frac{h}{8}}. \tag{13}$$

At $x = x_{n+\frac{h}{6}}$,

$$\frac{3072_{a_4}}{h^4} - \frac{20480_{a_5}}{h^4} + \frac{63488_{a_6}}{h^4} - \frac{1003520}{9}\frac{a_7}{h^4} + \frac{2314240}{27}\frac{a_8}{h^4} + \frac{987136}{9}\frac{a_9}{h^4} = f_{n+\frac{h}{6}}. \tag{14}$$

At $x = x_{n+1}$,

$$\frac{3072_{a_4}}{h^4} + \frac{30720_{a_5}}{h^4} + \frac{165888_{a_6}}{h^4} - \frac{645120_{a_7}}{h^4} + \frac{2027520_{a_8}}{h^4} - \frac{5474304_{a_9}}{h^4} = f_{n+\frac{n}{+h}}. \tag{15}$$

Consequently, this yields an nxn matrix of the form

$$\mathbf{AX} = \mathbf{B} \tag{16}$$

where

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}^T,$$

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{11}{12} & \frac{49}{72} & -\frac{143}{432} & -\frac{191}{2592} & \frac{7249}{15552} & -\frac{72863}{93312} & \frac{540529}{559872} & -\frac{3322751}{3359232} & \frac{17091217}{20155392} \\ 1 & -\frac{5}{6} & \frac{7}{18} & \frac{5}{27} & -\frac{113}{162} & \frac{475}{486} & -\frac{679}{729} & \frac{2515}{4374} & -\frac{353}{13122} & -\frac{10435}{19683} \\ 1 & -\frac{3}{4} & \frac{1}{8} & \frac{9}{16} & -\frac{31}{32} & \frac{57}{64} & -\frac{47}{128} & -\frac{87}{256} & \frac{449}{512} & -\frac{999}{1024} \\ 0 & 0 & 0 & 0 & 3072 & -30720 & 165888 & -645120 & 2027520 & -5474304 \\ 0 & 0 & 0 & 0 & 3072 & -28160 & 136448 & -\frac{4188800}{9} & \frac{33796480}{27} & -\frac{25327808}{9} \\ 0 & 0 & 0 & 0 & 3072 & -25600 & 109568 & -\frac{2867200}{9} & \frac{18810880}{27} & -\frac{10700800}{9} \\ 0 & 0 & 0 & 0 & 3072 & -23040 & 85248 & -201600 & 320640 & -295488 \\ 0 & 0 & 0 & 0 & 3072 & -20480 & 63488 & -\frac{1003520}{9} & \frac{2314240}{27} & \frac{987136}{9} \\ 0 & 0 & 0 & 0 & 3072 & 30720 & 165888 & 645120 & 2027520 & 5474304 \end{bmatrix},$$

and

$$\mathbf{B} = \left[[y_n \quad y_{n+\frac{1}{24}} \quad y_{n+\frac{1}{12}} \quad y_{n+\frac{1}{8}} \quad h^4 f_n \quad h^4 f_{n+\frac{1}{24}} \quad h^4 f_{n+\frac{1}{12}} \quad h^4 f_{n+\frac{1}{8}} \quad h^4 f_{n+\frac{1}{6}} \quad h^4 f_{n+1}]^T \right].$$

Solving equation (16) with Gaussian elimination approach, the values of the unknown variables a_j's are derived:
a_j's where $(j = 0(1)9)$ are substituted into equation (4) which yield the implicit schemes;

$$y_{n+\frac{1}{6}} = \frac{31}{59719680} h^4 f_{n+\frac{1}{24}} + \frac{79}{39813120} h^4 f_{n+\frac{1}{12}} + \frac{31}{59719680} h^4 f_{n+\frac{1}{8}} - \frac{1}{238878720} h^4 f_{n+\frac{1}{6}} - \frac{1}{238878720} h^4 f_n + 4y_{n+\frac{1}{8}} - 6y_{n+\frac{1}{12}} + 4y_{n+\frac{1}{24}} - y_n \tag{17}$$

$$y_{n+1} = \frac{19807733}{3317760} h^4 f_{n+\frac{1}{24}} + \frac{7583951}{737280} h^4 f_{n+\frac{1}{12}} - \frac{1668520579}{209018880} h^4 f_{n+\frac{1}{8}} + \frac{379993}{1672151040} h^4 f_{n+1} - \frac{314183749}{238878720} h^4 f_n + 2024y_{n+\frac{1}{8}} - 5796y_{n+\frac{1}{12}} + 5544y_{n+\frac{1}{24}} - 1771y_n \tag{18}$$

and the derivatives of the discrete scheme at starting point to cater for the initial conditions are;

$$y'_n = \frac{1151}{1672151040} h^3 f_n - \frac{1}{2115271065600} h^3 f_{n+1} + \frac{1}{696729600} h^3 f_{n+\frac{1}{6}} + \frac{17}{209018880} h^3 f_{n+\frac{1}{8}} - \frac{457}{102187008} h^3 f_{n+\frac{1}{12}} - \frac{20843}{1602478080} h^3 f_{n+\frac{1}{24}} - \frac{44}{h} y_n + \frac{8}{h} y_{n+\frac{1}{8}} - \frac{36}{h} y_{n+\frac{1}{12}} + \frac{72}{h} y_{n\frac{1}{24}} \tag{19}$$

$$y''_n = \frac{52981}{418037760} h^2 f_n + \frac{1}{6437781504} h^2 f_{n+1} - \frac{199}{23224320} h^2 f_{n+\frac{1}{6}} + \frac{7093}{182891520} h^2 f_{n+\frac{1}{8}} + \frac{779}{3548160} h^2 f_{n+\frac{1}{12}} + \frac{3527}{2903040} h^2 f_{n+\frac{1}{24}} - \frac{1152}{h^2} y_n - \frac{576}{h^2} y_{n+\frac{1}{8}} + \frac{2304}{h^2} y_{n+\frac{1}{12}} - \frac{2880}{h^2} y_{n\frac{1}{24}} \tag{20}$$

$$y'''_n = \frac{476401}{34836480} h f_n + \frac{883}{61695406080} h f_{n+1} + \frac{5419}{5806080} h f_{n+\frac{1}{6}} - \frac{36167}{7620480} h f_{n+\frac{1}{8}} + \frac{3655}{1064448} h f_{n+\frac{1}{12}} - \frac{404339}{8346240} h^2 f_{n+\frac{1}{24}} - \frac{13824}{h^3} y_n + \frac{13824}{h^3} y_{n+\frac{1}{8}} - \frac{41472}{h^3} y_{n+\frac{1}{12}} + \frac{41472}{h^3} y_{n\frac{1}{24}} \tag{21}$$

These schemes and derivatives are combined together in matrix form and by using the matrix inversion, a block method of the following form is produced

$$U\begin{bmatrix} y_{n+\frac{1}{24}} \\ y_{n+\frac{1}{12}} \\ y_{n+\frac{1}{8}} \\ y_{n+\frac{1}{6}} \\ y_{n+1} \end{bmatrix} = B\begin{bmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + C\begin{bmatrix} g_{n-4} \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{bmatrix} + E\begin{bmatrix} y''_{n-4} \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{bmatrix} + F\begin{bmatrix} y'''_{n-4} \\ y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_n \end{bmatrix} + G\begin{bmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + H\begin{bmatrix} f_{n+\frac{1}{24}} \\ f_{n+\frac{1}{12}} \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{6}} \\ f_{n+1} \end{bmatrix}$$

$$U=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, C=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{24}h \\ 0 & 0 & 0 & 0 & \frac{1}{12}h \\ 0 & 0 & 0 & 0 & \frac{1}{8}h \\ 0 & 0 & 0 & 0 & \frac{1}{6}h \\ 0 & 0 & 0 & 0 & h \end{bmatrix}$$

$$E=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{1152}h^2 \\ 0 & 0 & 0 & 0 & \frac{1}{288}h^2 \\ 0 & 0 & 0 & 0 & \frac{1}{128} \\ 0 & 0 & 0 & 0 & \frac{1}{72}h^2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}h^2 \end{bmatrix}, F=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{82944}h^3 \\ 0 & 0 & 0 & 0 & \frac{1}{10368}h^3 \\ 0 & 0 & 0 & 0 & \frac{1}{3072}h^3 \\ 0 & 0 & 0 & 0 & \frac{1}{1296}h^3 \\ 0 & 0 & 0 & 0 & \frac{1}{6}h^3 \end{bmatrix},$$

$$G=\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{241387}{2889476997120} \\ 0 & 0 & 0 & 0 & \frac{1321}{1410877440} \\ 0 & 0 & 0 & 0 & \frac{4687}{1321205760} \\ 0 & 0 & 0 & 0 & \frac{157}{17635968} \\ 0 & 0 & 0 & 0 & \frac{83}{63} \end{bmatrix},$$

$$H=\begin{bmatrix} \frac{3089}{43266908160} & -\frac{20147}{441447874560} & \frac{6373}{316036546560} & -\frac{9329}{2407897497600} & \frac{1469}{25586318809497600} \\ \frac{4159}{2704181760} & -\frac{113}{156764160} & \frac{781}{2469035520} & -\frac{71}{1175731200} & \frac{1}{1135756339200} \\ \frac{417}{52756480} & -\frac{459}{201850880} & \frac{89}{72253440} & -\frac{87}{367001600} & \frac{41}{11699277004800} \\ \frac{7}{301806} & -\frac{1}{336798} & \frac{1}{275562} & -\frac{89}{146966400} & \frac{1}{111547497600} \\ -\frac{960}{161} & \frac{72}{7} & -\frac{3520}{441} & \frac{417}{175} & \frac{461}{2028600} \end{bmatrix}$$

Equations (22, 23, 24, 25 and 26) listed below are the explicit presentation of the Block Method

$$\begin{aligned} y_{n+\frac{1}{24}} = y_n + \frac{1}{24}hy'_n + \frac{1}{1152}h^2y''_n + \frac{1}{82944}h^3y'''_n + \frac{241387}{2889476997120}f_n + \frac{3089}{43266908160}f_{n+\frac{1}{24}} \\ - \frac{20147}{441447874560}f_{n+\frac{1}{12}} + \frac{6373}{316036546560}f_{n+\frac{1}{8}} - \frac{9329}{2407897497600}f_{n+\frac{1}{6}} \\ + \frac{1469}{25586318809497600}f_{n+1}. \end{aligned} \tag{22}$$

$$y_{n+\frac{1}{12}} = y_n + \frac{1}{12}hy'_n + \frac{1}{288}h^2y''_n + \frac{1}{10368}h^3y'''_n + \frac{1321}{1410877440}f_n + \frac{4159}{2704181760}f_{n+\frac{1}{24}} \tag{23}$$

$$-\frac{113}{156764160}f_{n+\frac{1}{12}}+\frac{781}{2469035520}f_{n+\frac{1}{8}}-\frac{71}{1175731200}f_{n+\frac{1}{6}} +\frac{1}{1135756339200}f_{n+1}.$$

$$y_{n+\frac{1}{8}}=y_n+\frac{1}{8}hy'_n+\frac{1}{288}h^2y''_n+\frac{1}{3072}h^3y'''_n+\frac{4687}{1321205760}f_n+\frac{417}{52756480}f_{n+\frac{1}{24}} -\frac{459}{201850880}f_{n+\frac{1}{12}}+\frac{89}{72253440}f_{n+\frac{1}{8}}- -\frac{87}{367001600}f_{n+\frac{1}{6}}+\frac{41}{11699277004800}f_{n+1}. \tag{24}$$

$$y_{n+\frac{1}{6}}=y_n+\frac{1}{6}hy'_n+\frac{1}{72}h^2y''_n+\frac{1}{1296}h^3y'''_n+\frac{157}{17635968}f_n+\frac{7}{301806}f_{n+\frac{1}{24}}- \frac{1}{336798}f_{n+\frac{1}{12}}+\frac{1}{275562}f_{n+\frac{1}{8}}-\frac{89}{146966400}f_{n+\frac{1}{6}}+\frac{1}{111547497600}f_{n+1}. \tag{25}$$

$$y_{n+1}=y_n+hy'_n+\frac{1}{2}h^2y''_n+\frac{1}{6}h^3y'''_n+\frac{83}{63}f_n+\frac{960}{161}f_{n+\frac{1}{24}}-\frac{72}{7}f_{n+\frac{1}{12}}+\frac{3520}{441}f_{n+\frac{1}{8}} -\frac{417}{175}f_{n+\frac{1}{6}}+\frac{461}{2028600}f_{n+1}. \tag{26}$$

3.1. **Analysis of the Method.** In this section, the analysis of basic properties of this method such as order, error constant, zero stability and consistency are considered.

3.1.1. *Order and Error Constant.* Equations (22, 23, 24, 25 and 26) derived are discrete schemes belonging to the class of linear multistep methods of the form

$$\sum_{i=0}^{k}\alpha_i(t)y_{n+i}=h(\sum_{i=0}^{k}\beta_i(t)f_{n+i})+h^4(\sum_{i=0}^{k}\lambda i(t)g_{n+i}). \tag{27}$$

Following Fatunla [8] and Lambert [12], we define the local truncation error associated with equation (27) by the difference operator

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j y(x_n+jh)-h\beta(x_n+h)-h^4\alpha(x_n+jh)] \tag{28}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. Expanding (28) in Taylor series about the point x, we obtain the expression

$$L[y(x):h]=C_0y(x)+C_1hy'(x)+C_2hy''(x)+\cdots+C_{p+4}h^{p+4}y^{p+4}(x)$$

where the $C_0, C_1, C_2, ..., C_p, ..., C_{p+4}$ are obtained as:

$$C_0=\sum_{j=0}^{k}\alpha_j,$$

$$C_1=\sum_{j=1}^{k}j\alpha_j,$$

$$C_2=\frac{1}{2}\sum_{j=0}^{k}j^2\alpha_j,$$

$$C_q=\frac{1}{q!}[\sum_{j=1}^{k}j^q\alpha_j-q(q-1)\sum_{j=1}^{k}\beta_j j^{q-2}-q(q-1)(q-2)\sum_{j=1}^{k}\gamma_j j^{q-3}].$$

In view of Lambert [12], equation (3) is of order p if $C_0=C_1=C_2=\cdot\ \cdots=C_p=$

$C_{p+4\neq 0}$. The $C_{p+4\neq 0}$. is called the error constant and $C_{p+4}h^{p+4}y^{p+4}(x_n)$is the principal local truncation error at the point x_n.
Expanding (22, 23, 24, 25 and 26) in Taylor's series and comparing co-efficient, we have:
$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = C_9 = 0$.
Hence, the block (22- 26) are of order $p = [6, 6, 6, 6, 6,]^T$ and error constants
C_{p+4}

$$= \left[-\frac{109}{2156983020442091520}, -\frac{2083}{2696228775552614400}, -\frac{1}{324699527577600}, -\frac{83}{10532143654502400}, -\frac{307}{4180377600} \right]^T.$$

3.1.2. *Zero Stability of the Method.* The linear multistep method represented by (27) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation. To analyze the zero-stability of the method, we present (27) in vector notation form of column vectors
$e = (e_1, ..., e_\gamma)^T$,
$d = (d_1, ..., d_\gamma)^T$,
$y_m = (y_{n+1}, ..., y_{n+\gamma})^T$,
$F(y_m) = (f_{n+1}, ..., f_{n+\gamma})^T$,
$G(y_m) = (g_{n+1}, ..., g_{n+\gamma})^T$,
and matrices $A = (a_{ij}),\ B = (b_{ij})$.

The method (22- 26) forms the block formula

$$A^0_{ym} = hBF_{(ym)} + A^1_{yn} + hbf_n + h^4DG_{(ym)} + h^4dg_{(n)} \tag{29}$$

where h is a fixed mesh size within a block. In line with equation (29),

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first characteristic polynomial of (22- 26) are given by

$$\rho(R) = det(RA^0 - A^1) \tag{30}$$

Substituting A^0 and A^1 in equation (30), we have

$$\rho(R) = \begin{vmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{vmatrix}$$

$$= \lambda^5 - (5\,R - 1)\,\lambda^4 - \left(-10\,R^2 + 4\,R\right)\lambda^3 - \left(10\,R^3 - 6\,R^2\right)\lambda^2 - \left(-5\,R^4 + 4\,R^3\right)\lambda - (R-1)\,R^4$$
$$= -(R-1)\,R^4.$$

The values of R are obtained as 0, 0, 0, 0, and 1 for equations (22- 26).

According to Fatunla in ([8], [9]), the block formulae represented by equations (22- 26) is zero-stable since from equation (30), $\rho(R) = 0$ satisfy $|R_j| \leq 1,\ j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

3.1.3. *Consistency and Convergence of the Methods.* The linear multistep method represented by equation (27) is said to be consistent if it has order $p \geq 1$. Since (22- 26) are of order

$$p = [6, 6, 6, 6, 6]^T,$$

the method is consistent. According to the theorem of Dahlquist in Dahlquist [7], the necessary and sufficient condition for a linear multistep method to be convergent is to be consistent and zero stable. Since the method satisfy these two conditions, it is said to be convergent.

4. Application of the New Method

This section of the work deals with the implementation of the methods by solving initial value problem (IVPs) of fourth order differential equations (ODEs). The schemes are coded using Maple software packages, which were used to solve the problems.

4.1. **Presentation of Problem.** Test problems which include fourth order IVPs of ODEs are tested to ascertain the effectiveness of this new scheme.
Problem 1: We consider the fourth order IVP $y^{iv} - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 0, h = \frac{1}{320}$ with exact solution: $y(x) = \frac{-1}{4}e^{x} - \frac{1}{4}e^{-x} + \frac{3}{2}cos(x)$ which has been solved in Areo and Omole [5]
Problem 2: We consider the fourth order IVP $y^{iv} = x, y(0) = 0, y'(0) = 1, y''(0) = y''' = 0, h = 0.1$ with exact solution: $y(x) = \frac{x^5}{120} + x$ which has been solved in Mohammed [13].
Problem 3: We consider the linear differential equation of fourth order IVP $y^{iv} + y'' = 0, y(0) = 0, y'(0) = \frac{1.1}{72-50\pi}, y''(0) = \frac{1}{144-100\pi}, y'''(0) = \frac{1.2}{144-100\pi}, h = 0.01$ with exact solution: $y(x) = \frac{1-x-\cos(x)-1.2\sin(x)}{144-100\pi}$ which has been solved in Ukpebor et al. [17]

Table 1: Comparison of the result of problem 1 with Areo and Omole [5] using $h = \frac{1}{320}$.

X	Exact Solutions	Computed Solutions	Error	Error In [5]	Time	Time in [5]
0.1	0.10000008333333333333	0.10000008333333333333	0	4.440892×10^{-16}	2.29	2.556
0.2	0.20000266666666666667	0.20000266666666666667	0	2.176037×10^{-14}	2.07	2.682
0.3	0.30002025000000000000	0.30002025000000000001	-1×10^{-20}	771916×10^{-13}	1.25	3.078
0.4	0.40008533333333333333	0.40008533333333333334	-1×10^{-20}	7.666090×10^{-13}	2.71	3.006
0.5	0.50026041666666666667	0.50026041666666666668	-1×10^{-20}	2.367773×10^{-12}	2.84	3.444
0.6	0.60064800000000000000	0.60064800000000000003	-3×10^{-20}	5.932477×10^{-12}	2.45	3.384
0.7	0.70140058333333333333	0.70140058333333333341	-8×10^{-20}	1.287681×10^{-11}	1.97	3.600
0.8	0.80273066666666666667	0.80273066666666666683	-1.6×10^{-19}	2.517841×10^{-11}	2.08	3.798
0.9	0.90492075000000000000	0.90492075000000000031	-3.1×10^{-19}	4.546752×10^{-11}	3.02	3.936
1.0	1.0083333333333333333	1.0083333333333333339	-6×10^{-19}	7.712331×10^{-11}	2.87	4.080

Table 2: Comparison of the result of problem 2 with Mohammed [13] using $h = 0.1$.

X	Exact Solutions	Computed Solutions	Error	Error In [13]	Time	Time in [13]
0.003125	0.99999023437897364038	0.99999023437897364040	-2 $\times 10^{-20}$	7.000024 $\times 10^{-10}$	5.25	6.12
0.006250	0.99996093756357812225	0.99996093756357812219	6 $\times 10^{-20}$	8.9999912 $\times 10^{-10}$	6.62	6.25
0.009375	0.99991210969686319582	0.99991210969686319586	-4 $\times 10^{-20}$	771916 $\times 10^{-13}$	6.31	6.55
0.012500	0.99984375101724200780	0.99984375101724200780	0	5.100033 $\times 10^{-09}$	6.27	6.75
0.015625	0.99975586185848644372	0.99975586185848644374	-2 $\times 10^{-20}$	7.799979 $\times 10^{-09}$	6.12	6.85
0.018750	0.99964844264972060958	0.99964844264972060956	2 $\times 10^{-20}$	1.180009 $\times 10^{-08}$	6.18	6.15
0.021875	0.99952149391541244952	0.99952149391541244954	-2 $\times 10^{-20}$	1.80009 $\times 10^{-08}$	5.95	6.77
0.025000	0.99937501627536350208	0.99937501627536350204	4 $\times 10^{-19}$	1.410006 $\times 10^{-08}$	6.17	6.90
0.028125	0.99920901044469679275	0.99920901044469679279	-4.1 $\times 10^{-19}$	1.880000 $\times 10^{-08}$	6.23	6.51
0.031250	0.99902347723384286570	0.99902347723384286566	4 $\times 10^{-19}$	1.008335 $\times 10^{-08}$	6.45	6.83

Table 3: Comparison of the result of problem 3 with Ukpebor et al. [17] using $h = 0.1$.

X	Exact Solutions	Computed Solutions	Error	Error in [17]	Time	Time in [16]
0.01	0.00012899562284403681416	0.00012899562284403681417	-1 $\times 10^{-23}$	0.00 $\times 10^{-00}$	3.32	3.05
0.02	0.00025739654321013573141	0.00025739654321013573144	-3 $\times 10^{-23}$	0.00 $\times 10^{-00}$	1.97	3.15
0.03	0.00038519579791147418337	0.00038519579791147418340	-3 $\times 10^{-20}$	0.00 $\times 10^{-00}$	2.32	2.98
0.04	0.00051238648392729469127	0.00051238648392729469131	-4 $\times 10^{-20}$	0.00 $\times 10^{-00}$	2.99	3.25
0.05	0.00063896175909320118928	0.00063896175909320118935	-7 $\times 10^{-20}$	0.00 $\times 10^{-00}$	3.11	3.11
0.06	0.00076491484278536976241	0.00076491823309728171996	-3.390 $\times 10^{-9}$	0.00 $\times 10^{-00}$	2.54	3.65
0.07	0.00089023901659860537801	0.00089025257750722484229	-1.356 $\times 10^{-8}$	0.00 $\times 10^{-00}$	2.89	3.29
0.08	0.0010149276250181768073	0.0010149581357912481321	-3.051 $\times 10^{-8}$	2.00 $\times 10^{-18}$	3.08	3.87
0.09	0.0011389740760853625514	0.0011390283142956577703	-5.424 $\times 10^{-8}$	2.00 $\times 10^{-18}$	1.88	3.22
0.10	0.0012623718420566412212	0.0012624565829042084180	-8.474 $\times 10^{-8}$	1.00 $\times 10^{-17}$	2.94	3.71

4.2. **Discussion.** Table 1 compares results of solution of problem from exact solution, the newly established method and error in Areo and Omole [5] using step-size h=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 0.10; the outcome shows that the new method compares favourably in terms of accuracy. The CPU time reported in [5] was converted from minutes to seconds for the purpose of comparison. The CPU execution time of the new method showed promising efficiency in comparison with the CPU time reported in [5].

The results of Table 2 compare the newly developed method with error obtained in Mohammed [13] using step-size h=0.003125, 0.006250, 0.009375, 0.012500, 0.015625, 0.018750, 0.021875, 0.025000, 0.028125 and 0.031250; the outcomes show that the new method compares favourably in terms of accuracy. The CPU execution time of the new method perform efficiently in comparison with the CPU time reported in [13].

The results of Table 3 compare the newly developed method with error from Ukpebor et al [17] using step-size h=0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09 and 0.10; the outcomes show that the new method compares favourably in terms of accuracy. In comparison with the CPU time reported in [16], the CPU time of the new method is more efficient.

5. Conclusion

We have developed and applied a new block method for the direct solution of fourth order ordinary differential equations. The generated results from the newly developed method when applied to fourth order initial value problems gave a better performance over existing methods in terms of speed and accuracy.

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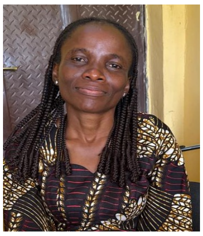

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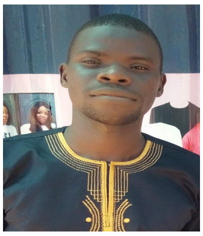

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