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ON STRONGLY *-GRAPHS AND STRONGLY MULTIPLICATIVE GRAPHS

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Abstract. This paper studies strongly *-graphs as a variation of strongly multiplicative graphs. We show that every strongly multiplicative graph induces a strongly *-graph. We establish a relationship between the upper bounds on the number of edges of strongly *-graphs $\lambda^*(n)$ and strongly multiplicative graphs, and identify a condition under which the upper bound for strongly *-graphs exceeds that of strongly multiplicative graphs, we show that this condition holds for infinitely many values of n. We derive explicit formulas for $\lambda^*(n)$. Finally, we prove the independence of several necessary conditions for graphs that do not admit strongly*-labeling.

Keywords: strongly *-graphs, strongly multiplicative, Graph labeling.

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1. Introduction

Graph labeling is a central topic in graph theory with applications in areas such as computer science, communication networks, biology, and information systems. In many practical contexts, labeling schemes are used to encode structural information, model interactions, or optimize network performance. Among the various labeling methods studied in the literature, multiplicative-type labelings are of particular interest due to their close connection with number theory.

The concept of strongly multiplicative graphs was introduced by Beineke and Hegde in [1]. In this labeling, the vertices of a graph of order n are assigned the labels $1, 2, \ldots, n$ in such a way that the products of the labels of adjacent vertices are all distinct. Several classes of graphs, including trees, cycles, and wheels, were shown to admit such labelings. An important problem arising from this definition is determining the maximum possible number of edges in a strongly multiplicative graph of order n, denoted by $\lambda(n)$. This problem was further investigated by Adiga, Ramaswamy, and Somashekara in [2], who obtained improved bounds for $\lambda(n)$. Additional families of strongly multiplicative graphs

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and necessary conditions for their existence were later studied by Seoud and Zid [3] and Seoud and Mahran [4]. Another labeling scheme is strongly*-graphs. In this case, a graph of order n is labeled with the integers $1, 2, \ldots, n$, and each edge joining vertices labeled i and j receives the label $i+j+ij$. The labeling is said to be strongly* graph if all induced edge labels are distinct. This notion was introduced by Adiga and Somashekara in [5], who showed that several graph families, such as trees, cycles, and grids, admit strongly*-labelings. Further necessary conditions and additional results on strongly*-graphs were obtained in [6, 7].
In the literature, the relationship between strongly multiplicative graphs and strongly *-graphs has not been fully explored. In particular, existing studies treat the two labeling schemes as independent, and consequently no relationship between the maximum number of edges in strongly*-graphs $\lambda^*(n)$ and the maximum number of edges in strongly multiplicative graphs $\lambda(n)$ has been investigated. In this paper, we formally establish a fundamental relationship between strongly*-labeling and strongly multiplicative labeling, showing that every strongly multiplicative graph of order n induces a strongly*-graph of order $n-1$. We further prove that $\lambda^*(n) = \lambda(n+1) - \left(\pi(n+1) + \pi\left(\sqrt{n+1}\right)\right)$, where $\pi(n)$ is the number of prime numbers less than or equal to n, and verify our theoretical results computationally. In addition, we introduce and prove a necessary and sufficient condition under which $\lambda^*(n) > \lambda(n)$, and we show that this condition holds whenever $n+1$ is prime. We also derive explicit formulas for $\lambda^*(n)$. Finally, we demonstrate that some necessary conditions used to identify graphs that do not admit strongly*-labeling are independent.

2. Preliminaries

In this section, we recall the basic definitions of several families of graphs, as well as the definitions of multiplicative, strongly multiplicative, and strongly* graphs. We then provide an example of a graph that is strongly multiplicative but not strongly* to illustrate the difference between the two concepts.

Definition 2.1. *[11] A complete graph K_n is simple undirected graph in which every pair of distinct vertices is connected by a unique edge.*

Definition 2.2. *[11] A graph $G(V, E)$ is called a complete bipartite graph $K_{m,n}$ if its vertices can be partitioned into two subsets V_1 and V_2 such that no edges have both end points in the same subset, and each vertex of $V_1(V_2)$ is connected to all vertices of $V_2(V_1)$.*

Definition 2.3. *[10] A star graph S_n is a complete bipartite graph with one internal node and n leaves.*

Definition 2.4. *[10] An umbrella graph $U_{m,n}$ is the graph obtained by joining a path P_n with the central vertex of a fan F_m.*

Definition 2.5. *[1] A graph $G(V, E)$ with n vertices is said to be multiplicative if the vertices of G can be labeled with distinct positive integers such that the label induced on the edges by the products of labels of end vertices are all distinct.*

Definition 2.6. *[1] A graph $G(V, E)$ of order n is said to be strongly multiplicative if the vertices of G can be labeled with n distinct integers $1, 2, \cdots, n$ such that the labels induced on the edges by the product of the labels of the end vertices are all distinct.*

Definition 2.7. *[5] A graph $G(V, E)$ of order n is said to be a strongly *-graph if its vertices can be assigned the values $1, 2, \cdots, n$ in such a way that, when an edge is labeled with the value $i + j + ij$, where i and j are the labels of its vertices, all edge labels are distinct.*

Although the definitions of strongly multiplicative graphs and strongly*-graphs are closely related, the two concepts differ in several respects: in particular, there exist graphs that are strongly multiplicative but not strongly* (see Fig. 1), and vice versa, as we will prove in the next sections.

3. From Strongly Multiplicative Graphs to Strongly *-Graphs

In this section, we study the relationship between strongly multiplicative graphs and strongly *-graphs, and prove that every strongly multiplicative graph of order n induces a strongly *-graph of order $n-1$.

Theorem 3.1. *A graph G of order n is a strongly *-graph if and only if there exists an injective labeling $f : V(G) \to \{2, 3, \dots, n+1\}$ such that for every edge $uv \in E(G)$, the products $f(u)f(v)$ are distinct.*

Proof. Let $g : V(G) \to \{1, \dots, n\}$ be a strongly *-labeling. The induced edge weights are defined as $w^*(uv) = g(u) + g(v) + g(u)g(v)$. Notice the algebraic identity

$$g(u) + g(v) + g(u)g(v) = (g(u) + 1)(g(v) + 1) - 1.$$

Define a new labeling $f(v) = g(v) + 1$. Since the range of g is $\{1, \dots, n\}$, the range of f is $\{2, \dots, n+1\}$. The condition that all edge weights $w^*(uv)$ are distinct is equivalent to saying that all values $f(u)f(v) - 1$ are distinct. This, in turn, holds if and only if all products $f(u)f(v)$ are distinct. □

Theorem 3.2. *Every strongly multiplicative graph of order n induces a strongly *-graph of order $n-1$.*

Proof. Let G be a strongly multiplicative graph of order n. Then there exists a labeling

$$f : V(G) \to \{1, 2, \dots, n\}$$

such that all induced edge products $f(u)f(v)$, for $uv \in E(G)$, are distinct.

Remove the vertex labeled 1 together with all edges incident to it, and denote the resulting graph by G'. Define a new labeling

$$g : V(G') \to \{1, 2, \dots, n-1\} \quad \text{by} \quad g(v) = f(v) - 1.$$

For any edge $uv \in E(G')$, the induced edge weight under the labeling g is

$$w^*(uv) = g(u) + g(v) + g(u)g(v) = f(u)f(v) - 1.$$

Since all products $f(u)f(v)$ are distinct, it follows that all values $w^*(uv)$ are also distinct. Therefore, g is a strongly*-labeling of G'. Hence, the graph obtained from a strongly multiplicative graph by removing the vertex labeled 1 admits a strongly*-labeling, and thus G' is a strongly*-graph. □

Corollary 3.1. *The following families of graphs are strongly *-graphs:*

(1) *The complete graph K_n, for $n \leq 4$.*
(2) *The complete bipartite graph $K_{r,r-1}$, for $r \leq 4$.*
(3) *The path graph P_n for any positive integer n.*
(4) *The fan graph F_n for any positive integer n.*
(5) *The star graph S_n for any positive integer n.*
(6) *The umbrella graph $U_{m,n}$ for any positive integers m, n.*

Proof. The proof is directly follows from Theorem 3.2 and the fact that these graphs were established as strongly multiplicative in [1, 9, 10] and these graphs retain their shape after the removal of the vertex labeled 1. □

4. On the upper bound of the number of edges of a strongly *-graph

In this section, we establish a fundamental relationship between the upper bounds of strongly *-graphs and strongly multiplicative graphs, verify this relationship computationally, and analyze conditions under which one bound is strictly larger than the other. Then, we derive two explicit formulas for the upper bound of the number of edges in a strongly * graph of order n.

The problem of determining an upper bound for the number of edges of a strongly multiplicative graph with n vertices is fundamentally a number-theoretic problem. More precisely, given a positive integer n, one must determine how many distinct positive integers can be expressed as a product of two distinct positive integers, each not exceeding n. Set $\lambda(n) = \left|\{ rs : 1 \le r, s \le n, r \ne s\}\right|$ and $\lambda^*(n) = |\{kl + k + l,\ 1 \le k, l \le n, k \ne l\}|$. Note that $\lambda(n)$ is the number of edges of any maximal strongly multiplicative graph with n vertices and $\lambda^*(n)$ is the number of edges of any maximal strongly *-graph with n vertices. In fact, we have:

Observation 4.1.

$$\begin{aligned}\lambda^*(n) &= |\{kl + k + l,\ 1 \le k, l \le n, k \ne l\}| \\ &= |\{(k+1)(l+1) - 1,\ 1 \le k, l \le n, k \ne l\}| \\ &= |\{rs - 1, 2 \le r, s \le n+1, r \ne s\}| \\ &= |\{rs, 2 \le r, s \le n+1, r \ne s\}| \,.\end{aligned}$$

This observation allows us to connect strongly multiplicative graphs and strongly *-graphs. Moreover, we will derive the exact relationship between $\lambda^*(n)$ and $\lambda(n)$.

Theorem 4.1. *For a positive integer n, we have*

$$\lambda^*(n) = \lambda(n+1) - \left(\pi(n+1) + \pi\left(\sqrt{n+1}\right)\right).$$

Proof. Consider the following table

2×1	3×1	4×1	5×1	$n\times 1$	$(n+1)\times 1$
	3×2	4×2	5×2	$n\times 2$	$(n+1)\times 2$
			$\ddots$	$\vdots$	$\vdots$
				$n\times(n-1)$	$(n+1)\times(n-1)$
					$(n+1)\times n$

Table 1

By definition, $\lambda\,(n+1)$ denotes the number of distinct products in Table 1, whereas, by Observation 4.1, $\lambda^*\,(n)$ denotes the number of distinct products appearing from the second row onward in Table 1. Therefore, the distinction in the evaluation of $\lambda\,(n+1)$ and $\lambda^*(n)$ relies on the numbers in the first row of Table 1. In fact a number $k = k\times 1$ in the first row contributes in the evaluation of $\lambda\,(n+1)$ only if it can not be factored into two integers larger than 1. This condition holds if and only if k is a prime number $p \le n+1$ or the square of a prime $p^2 \le n+1$. Therefore,

$$\lambda^*\,(n) = \lambda\,(n+1) - \sum_{n=p \text{ or } n=p^2}^{n+1} 1$$

$$\lambda^*\,(n) = \lambda\,(n+1) - \left(\pi\,(n+1) + \pi\left(\sqrt{n+1}\right)\right).$$

□

To verify Theorem 4.1, we implemented the following algorithm to compare the exact values of $\lambda^*(n)$ with those derived from Theorem.

Algorithm 1: Verification of the Formula $\lambda^*(n) = \lambda(n+1) - \pi^+(n+1)$

Input: Maximum integer $N_{\max} = 5000$
Output: Table $\mathcal{T}$ validating the formula for $\lambda^*(n)$

Initialize an empty table $\mathcal{T}$ with columns:

$$\langle n,\ \lambda(n+1),\ \pi^+(n+1),\ \lambda(n+1) - \pi^+(n+1),\ \textbf{Exact}\ \lambda^*(n),\ \textbf{Diff}\rangle$$

for $n \leftarrow 1$ **to** $N_{\max} - 1$ **do**

`// 1. Compute exact combinatorial quantities`
$\lambda(n+1) \longleftarrow \big|\{\, ij \mid 1 \le j < i \le n+1 \,\}\big|$
$\lambda^*_{\text{exact}}(n) \longleftarrow \big|\{\, ij \mid 2 \le j < i \le n+1 \,\}\big|$

`// 2. Compute auxiliary prime-counting function`
$\pi^+(n+1) \longleftarrow \pi(n+1) + \pi\big(\lfloor\sqrt{n+1}\rfloor\big)$

`// 3. Compute theoretical value`
$\lambda^*_{\text{theory}}(n) \longleftarrow \lambda(n+1) - \pi^+(n+1)$

`// 4. Define and Compute Diff`
$\textbf{Diff} \longleftarrow \lambda^*_{\text{exact}}(n) - \lambda^*_{\text{theory}}(n)$

`// 5. Verification step (Theory matches Exact)`
if ***Diff*** $= 0$ **then**
Append row to $\mathcal{T}$:

$$\big(n,\ \lambda(n+1),\ \pi^+(n+1),\ \lambda^*_{\text{theory}}(n),\ \lambda^*_{\text{exact}}(n),\ \textbf{Diff}\big)$$

else
return Failure at n (Theory $\neq$ Exact)
end

end
return Success for all $n \le N_{\max}$ **and return table** $\mathcal{T}$

Table 2 presents the results obtained by Algorithm 1. It confirms that our theoretical formula perfectly matches the exact computational counts for all tested values. Note that $\pi^+(n+1) = \pi(n+1) + \pi(\sqrt{n+1})$.

n	$\lambda(n+1)$	$\pi^{+}(n+1)$	$\lambda(n+1)-\pi^{+}(n+1)$	**Exact** $\lambda^{*}(n)$	**Diff**
2	3	2	1	1	0
3	6	3	3	3	0
4	10	4	6	6	0
5	13	4	9	9	0
6	19	5	14	14	0
7	24	5	19	19	0
8	31	6	25	25	0
9	36	6	30	30	0
10	46	7	39	39	0
500	64,658	103	64,555	64,555	0
1000	247,994	179	247,815	247,815	0
1500	547,899	251	547,648	547,648	0
2000	959,472	317	959,155	959,155	0
2500	1,484,282	382	1,483,900	1,483,900	0
3000	2,122,560	447	2,122,113	2,122,113	0
3500	2,865,351	506	2,864,845	2,864,845	0
4000	3,725,731	569	3,725,162	3,725,162	0
4500	4,693,181	629	4,692,552	4,692,552	0
5000	5,770,207	688	5,769,519	5,769,519	0

TABLE 2

Based on Theorem 4.1, a relation between $\lambda^{*}(n)$ and $\lambda(n)$ is obtained. Our aim now is to compare $\lambda^{*}(n)$ and $\lambda(n)$. We derive a necessary and sufficient condition under which $\lambda^{*}(n) > \lambda(n)$.

Corollary 4.1. *Let n be a positive integer. Then, we have $\lambda^{*}(n) > \lambda(n)$ if and only if $\delta(n+1) > \pi(n+1) + \pi\left(\sqrt{n+1}\right)$, where $\delta(n) = \lambda(n) - \lambda(n-1), n \geq 2$.*

Proof. We have $\lambda(n+1) = \lambda(n) + \delta(n+1)$. Substituting this into the result from Theorem 4.1:

$$\begin{aligned}\lambda^{*}(n) &= \lambda(n+1) - \left(\pi(n+1) + \pi\left(\sqrt{n+1}\right)\right) \\ &= \lambda(n) + \delta(n+1) - \left(\pi(n+1) + \pi\left(\sqrt{n+1}\right)\right).\end{aligned}$$

Therefore,

$$\lambda^{*}(n) - \lambda(n) = \delta(n+1) - \left(\pi(n+1) + \pi\left(\sqrt{n+1}\right)\right).$$

Then

$$\lambda^{*}(n) > \lambda(n) \text{ if and only if } \delta(n+1) > \left(\pi(n+1) + \pi\left(\sqrt{n+1}\right)\right).$$

□

Computationally, $\lambda^{*}(n)$ generally exceeds $\lambda(n)$. However, the inequality does not hold for all n, as it depends on the difference between the incremental gain $\delta(n+1)$, representing the new distinct products contributed by $n+1$, and the prime-counting term $\pi^{+}(n+1) = \pi(n+1) + \pi(\sqrt{n+1})$. While $\pi^{+}(n+1)$ grows sub-linearly, the incremental gain $\delta(n+1)$ may be smaller than $\pi^{+}(n+1)$ specifically when $n+1$ is a composite with many small factors, resulting in $\lambda^{*}(n) \leq \lambda(n)$, Computational analysis confirms this behavior.

Example 4.1. *For the complete graph K_5, we have $n+1=6$. Calculating the values:*

$$\delta(n+1)=\delta(6)=3, \quad \pi(n+1)=\pi(6)=3, \quad \text{and} \quad \pi(\sqrt{n+1})=\pi(\sqrt{6})=1.$$

Substituting these into the condition from the corollary:

$$\delta(6)=3<(3+1)=4.$$

*Since $\delta(6)<\pi(6)+\pi(\sqrt{6})$, it follows that $\lambda^*(5)<\lambda(5)$. Computationally, $\lambda(5)=10$ and $\lambda^*(5)=9$. Thus, the complete graph K_5 is a strongly multiplicative graph but not a strongly *-graph.*

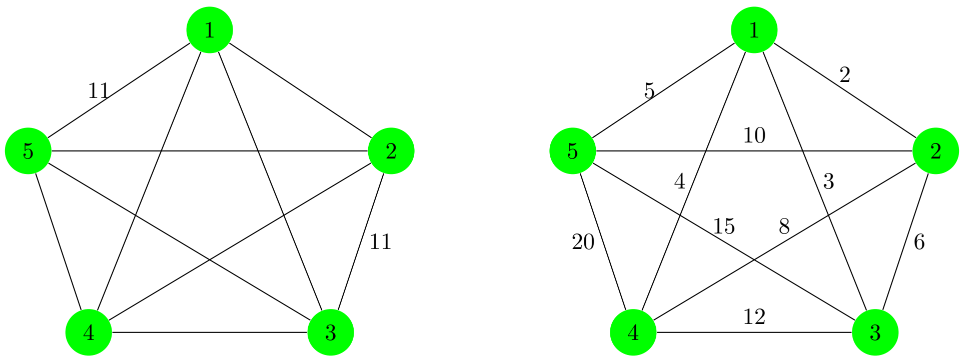


FIGURE 1. The complete graph K_5 is a strongly multiplicative graph but not a strongly *-graph.

To further investigate this behavior, we extended our computational analysis to $n=10000$. The results reveal that while $\lambda^*(n)>\lambda(n)$ holds for the vast majority of integers, there exists a specific subset of n for which $\lambda^*(n)<\lambda(n)$. These exceptions are listed in Table 3.

n	$\lambda(n)$	$\lambda^*(n)$	$\lambda^*(n)-\lambda(n)$
5	10	9	-1
9	31	30	-1
11	46	44	-2
13	63	62	-1
17	103	102	-1
23	183	181	-2
29	280	278	-2
35	390	389	-1
47	700	699	-1
59	1063	1060	-3
71	1513	1512	-1
89	2338	2337	-1
119	4051	4050	-1

TABLE 3

Although, we could not prove $\lambda^*(n) \geq \lambda(n)$ holds for the majority of the positive integers n as it is expected. We could prove it for infinitely many cases as stated in the following theorem.

Theorem 4.2. *For $n \geq 6$, if $n+1$ is a prime number, then $\lambda^*(n) > \lambda(n)$.*

Proof. It is easy to verify that $\lambda^*(2) = \lambda(2).$, $\lambda^*(4) = \lambda(4).$ and $\lambda^*(6) > \lambda(6)$. Now, let $n > 6$ and $n+1$ be a prime number. Then, every product $(n+1) \times k$ for $1 \leq k \leq n$ is distinct from all products formed by numbers strictly less than $n+1$. Thus, $\delta\,(n+1) = n$. On the other hand, using standard bounds for the prime counting function, for $n \geq 6$ we have:

$$\pi\,(n+1) + \pi\left(\sqrt{n+1}\right) \leq \frac{n+1}{2} + \frac{\sqrt{n+1}}{2} < n.$$

Then $\delta\,(n+1) > \pi\,(n+1) + \pi\left(\sqrt{n+1}\right)$, which implies $\lambda^*(n) > \lambda(n)$. □

n	$\lambda(n)$	$\lambda^*(n)$	$\lambda^*(n) - \lambda(n)$	n	$\lambda(n)$	$\lambda^*(n)$	$\lambda^*(n) - \lambda(n)$
6	13	14	1	72	1536	1583	47
10	36	39	3	78	1786	1838	52
12	51	55	4	82	1973	2028	55
16	87	94	7	88	2250	2310	60
18	111	119	8	96	2629	2696	67
22	161	172	11	100	2851	2921	70
28	252	267	15	102	2989	3060	71
30	291	307	16	106	3224	3298	74
36	403	424	21	108	3363	3438	75
40	493	517	24	112	3609	3687	78
42	549	574	25	126	4452	4542	90
46	654	682	28	130	4741	4834	93
52	823	855	32	136	5152	5250	98
58	1005	1042	37	138	5340	5439	99
60	1081	1119	38	148	6070	6178	108
66	1287	1330	43	150	6262	6371	109
70	1443	1489	46	156	6721	6835	114

TABLE 4. Comparison between $\lambda^*(n)$ and $\lambda(n)$ when $n+1$ is prime.

Now, we will introduce two different formulas of $\lambda^*\,(n)$. The first formula is obtained by applying Theorem 4.1 and Kafshgarzaferani's theorem in [8], which is

Theorem 4.3. *[8]*

$$\lambda\,(n) = \frac{n(n-1)}{2} + \sum_{m=2}^{n}\sum_{k=1}^{m-1}\left\lfloor -\frac{\theta(m,k)}{\lfloor\sqrt{mk-1}\rfloor - k + 1}\right\rfloor,$$

where

$$\theta(m,k) = \sum_{s=k+1}^{\lfloor\sqrt{mk-1}\rfloor}\left\lfloor \frac{\left\lfloor\frac{mk}{s}\right\rfloor}{\frac{mk}{s}}\right\rfloor.$$

More precisely, we obtained

Theorem 4.4.

$$\lambda^*(n) = \frac{(n+1)n}{2} + \sum_{m=2}^{n+1}\sum_{k=1}^{m-1}\left\lfloor -\frac{\theta(m,k)}{\lfloor\sqrt{mk-1}\rfloor - k + 1}\right\rfloor - \left(\pi(n+1) + \pi\left(\sqrt{n+1}\right)\right),$$

where

$$\theta(m,k) = \sum_{s=k+1}^{\lfloor\sqrt{mk-1}\rfloor}\left\lfloor\frac{\lfloor\frac{mk}{s}\rfloor}{\frac{mk}{s}}\right\rfloor.$$

Proof. The result follows directly from Theorems 4.3 and 4.4. □

The second formula is obtained by applying the same technique in [8] due to Kafshgarzaferani. That's

Theorem 4.5.

$$\lambda^*(n) = \frac{n(n-1)}{2} + \sum_{m=2}^{n}\sum_{k=1}^{m-1}\left\lfloor -\frac{\theta(m+1,k+1)}{\lfloor\sqrt{(m+1)(k+1)-1}\rfloor - k - 1}\right\rfloor,$$

where

$$\theta(m,k) = \sum_{s=k+1}^{\lfloor\sqrt{mk-1}\rfloor}\left\lfloor\frac{\lfloor\frac{mk}{s}\rfloor}{\frac{mk}{s}}\right\rfloor.$$

Proof. Let $\delta^*(n) = \lambda^*(n) - \lambda^*(n-1)$. Then

$$\lambda^*(n) = \sum_{m=2}^{n}\delta^*(m).$$

To obtain a formula of $\delta^*(m)$, consider the edge labels:

$$\begin{array}{ccccc} (2\times 3)-1 & (2\times 4)-1 & (2\times 5)-1 & \times & (2\times(n+1))-1 \\ & (3\times 4)-1 & (3\times 5)-1 & \times & (3\times(n+1))-1 \\ & & & \ddots & \vdots \\ & & & & (n\times(n+1))-1 \end{array}$$

Let A_{k+1} be the set of all elements of the k_{th} row. We count the number of terms in the last column which appear in other rows. If $(n+1)(k+1)$ is divisible by $s\,:\,(k+1 \le s-1)$, then there exists an $m < n+1$ such that $(n+1)(k+1) = s\times m$, hence $(n+1)(k+1)-1 = (s\times m)-1$, so $(n+1)(k+1)-1 \in A_s$. Then, s satisfies

$$k+1 < s < \sqrt{(n+1)(k+1)-1}.$$

Thus the number of repetitions of $((n+1)(k+1))-1$ in these rows is

$$\theta(n+1,k+1) = \sum_{s=k+2}^{\lfloor\sqrt{(n+1)(k+1)-1}\rfloor}\left\lfloor\frac{\left\lfloor\frac{(n+1)(k+1)}{s}\right\rfloor}{(n+1)(k+1)/s}\right\rfloor.$$

By definition of $\theta(n+1,k+1)$, it is clear that

$$0 \le \theta(n+1,k+1) \le \lfloor\sqrt{(n+1)(k+1)-1}\rfloor - (k+2) + 1,$$
$$0 \le \theta(n+1,k+1) \le \lfloor\sqrt{(n+1)(k+1)-1}\rfloor - k - 1,$$

Thus

$$0 \leq \frac{\theta(n+1,k+1)}{\lfloor\sqrt{(n+1)(k+1)-1}\rfloor - k - 1} \leq 1.$$

Hence,

$$\left\lfloor -\frac{\theta(n+1,k+1)}{\lfloor\sqrt{(n+1)(k+1)-1}\rfloor - k - 1}\right\rfloor = \begin{cases} -1, & \text{if } (n+1)(k+1)-1 \in \bigcup_{s=k+2}^{\lfloor\sqrt{(n+1)(k+1)-1}\rfloor} A_s, \\ 0, & \text{otherwise.} \end{cases}$$

So,

$$\delta^*(n) = n - 1 + \sum_{k=1}^{n-1} \left\lfloor -\frac{\theta(n+1,k+1)}{\lfloor\sqrt{(n+1)(k+1)-1}\rfloor - k}\right\rfloor.$$

Finally, we get

$$\lambda^*(n) = \frac{n(n-1)}{2} + \sum_{m=2}^{n}\sum_{k=1}^{m-1} \left\lfloor -\frac{\theta(n+1,k+1)}{\lfloor\sqrt{(n+1)(k+1)-1}\rfloor - k}\right\rfloor.$$

□

5. Some necessary conditions of strongly *-graphs

In this section, we recall some necessary conditions of strongly *-graphs introduced by Seoud and Mahran in [6] and then we will show that they are independent.

Condition 1. *If a graph G of n vertices has more than $\lambda^*(n)$ edges, then G is a non-strongly *-graph.*

Condition 2. *If the minimum degree of a graph G of n vertices is greater than the largest minimum degree in all corresponding maximal strongly *-graphs $\mu(n)$, then the graph is a non-strongly *-graph.*

Condition 3. *If G is a graph of n vertices, which has number of vertices of degree $n-1$ more than $t(n)$, where $t(n)$ is the maximum number of vertices of degree $n-1$ in all maximal strongly *-graphs of n vertices, then G is a non-strongly *-graph.*

Lemma 5.1. *If $m+1$ is a prime number, then $t(m) = t(m-1)+1$, where $t(n)$ is the maximum number of vertices of degree $n-1$ in all maximal strongly* graphs of n vertices, and $\mu(m) = \mu(m-1)+1$.*

Proof. Suppose that there exist two edges in the same graph of m vertices having the same edge label, one of them having the vertex labeled m as an end vertex. Then $k\times(m+1)-1 = i\times j-1$, where the labels of the end vertices of the two edges are $k-1, m, i-1, j-1$. So, $k\times(m+1) = i\times j$, and $i, j, k < m+1$. From the previous lemma, since $m+1$ is prime, we get a contradiction. So, the vertex having the label m in a maximal strongly *-graph of m vertices has degree $m-1$. Hence $t(m) = t(m-1)+1$, and $\mu(m) = \mu(m-1)+1$. □

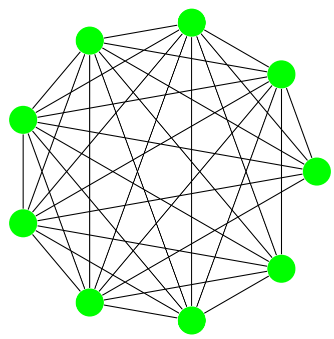
Condition 4. *If G is a graph of n vertices, and $K_{1+\chi(n)} \subseteq G$, where $\chi(n)$ is the order of the largest complete subgraph in all corresponding maximal strongly* graphs, then G is a non-strongly *-graph.*

Now, we show that the four necessary conditions for strongly *-graphs are independent, and they are altogether not sufficient for a graph to be a strongly *-graph by using the following table.

n	$\lambda^*(n)$	$\mu(n)$	$t(n)$	$1+\chi(n)$
2	1	1	2	3
3	3	2	3	4
4	6	3	4	5
5	9	3	3	5
6	14	4	4	6
7	19	5	4	7
8	25	5	5	8
9	30	6	4	8
10	39	7	5	9
11	44	7	5	9

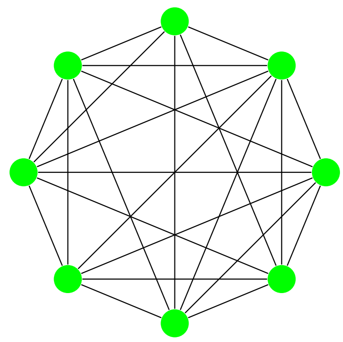
TABLE 5

Example 5.1. *Only Condition 1 proves that the following graph is a non-strongly *-graph.*



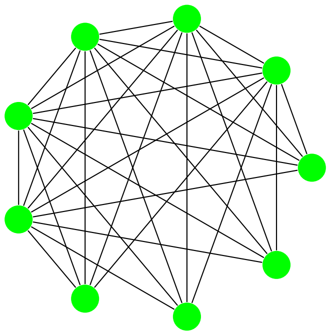
Condition 2: The minimum degree of the graph equals $5 < 6 = \mu(9)$.
Condition 3: The number of vertices of degree 8 is $3 < 4 = t(9)$.
Condition 4: $K_{1+\chi(9)} = K_8 \nsubseteq G$.
However, for Condition 1, the number of edges of G is $m = 31 > 30 = \lambda^*(9)$.

Example 5.2. *Only Condition 2 proves that the following graph is a non-strongly *-graph.*



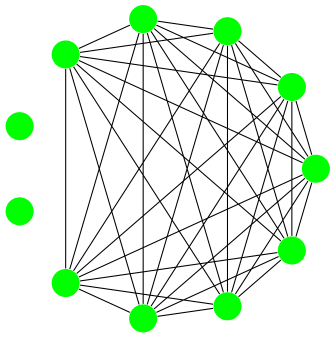
From table 5, for $n = 8$,
Condition 1: The number of edges of G is $m = 24 < 25 = \lambda^*(8)$.
Condition 3: The number of vertices of degree 7 is $0 < 5 = t(8)$.
Condition 4: $K_{1+\chi(8)} = K_8 \neq G$.
However, for Condition 2, the minimum degree of the graph equals $6 > 5 = \mu(8)$.

Example 5.3. *Only Condition 3 proves that the following graph is a non-strongly *-graph.*



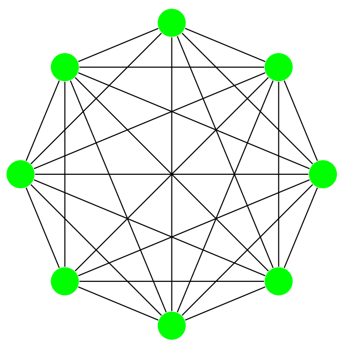
From table 5, for $n = 9$, *and* $G = K_5 + K_4$
Condition 1: The number of edges of G *is* $m = 30 = \lambda^*(9)$.
Condition 2: The minimum degree of the graph equals $5 < 6 = \mu(9)$.
Condition 4: $K_{1+\chi(9)} = K_8 \nsubseteq G$ *However, for Condition 3, the number of vertices of degree 8 is* $5 > 4 = t(9)$.

Example 5.4. *Only Condition 4 proves that the following graph is a non-strongly* **-graph.*



From table 5, for $n = 11$, *and* $G = K_9 \cup K_2$
Condition 1: The number of edges of G *is* $m = 36 < 44 = \lambda^*(11)$.
Condition 2: The minimum degree of the graph equals $0 < 7 = \mu(11)$.
Condition 3: The number of vertices of degree 10 is $0 < 5 = t(11)$.
However, for Condition 4, $K_{1+\chi(11)} = K_9 \subset G$.

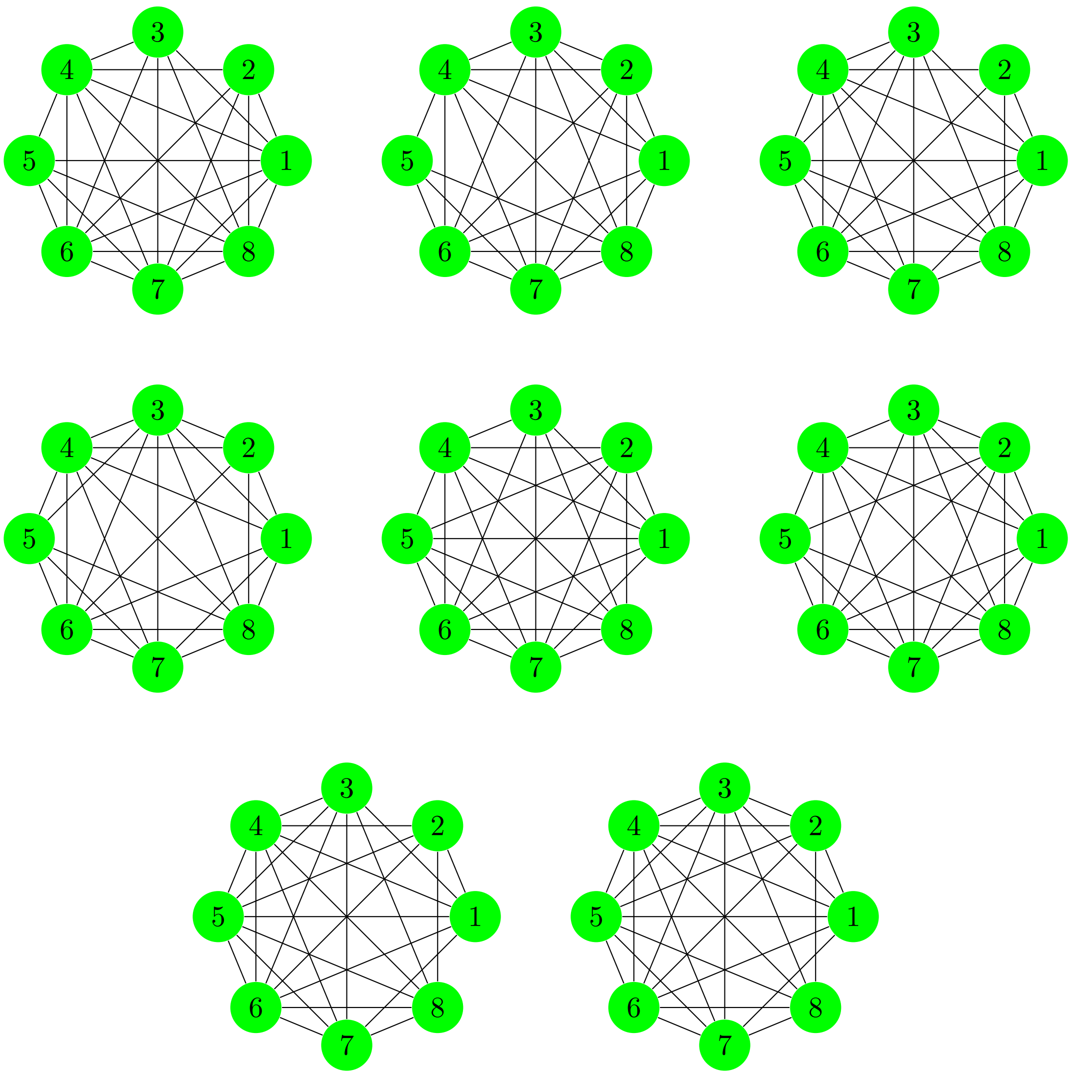
Example 5.5. *Here we give an example of a non-strongly* **-graph, but the three conditions fail to decide that it is a non-strongly* **-graph, i.e., they are altogether not sufficient for a graph to be a strongly* **-graph.*



From table 5, for $n = 8$, *and* $G = K_5 + (K_2 \cup K_1)$
Condition 1: The number of edges of G *is* $m = 24 < 25 = \lambda(8)$.
Condition 2: The minimum degree of the graph equals $5 = \delta(8)$.
Condition 3: The number of vertices of degree 7 is $5 = t(8)$.

Condition 4: $K_{1+\chi(8)} = K_8 \neq G$.
It remains to show that the graph G is a non-strongly *-graph.

The following graphs are all the maximal strongly *-graphs of 8 vertices.



The previous maximal strongly *-graphs of 8 vertices are obtained by the following way: First we determine the edges which have the same label, i.e., the edge connecting the vertices having the labels 1 and 5 and the edge connecting the vertices having the labels 2 and 3 have the same edge label 11, the edge connecting the vertices having the labels 2 and 7 and the edge connecting the vertices having the labels 3 and 5 have the same edge label 23 and the edge connecting the vertices having the labels 1 and 8 and the edge connecting the vertices having the labels 2 and 5 have the same edge label 17. Second, by deleting one edge of each pair in all possible ways, we get the maximal strongly *-graphs of 8 vertices.

From the previous graphs, we notice that there are 7 graphs in which the number of vertices of degree equal to 7 is less than 5, so it follows that our graph, which has 5 vertices of degree equal to 7, is not a subgraph of those graphs. Also, in the remaining graph, the number of vertices of degree equal to 6 is zero, since the number of vertices of degree equal to 6 in our graph equals 2. It follows that our graph is not a subgraph of any maximal strongly *-graph of 8 vertices. Hence, it is a non-strongly *-graph.

6. Conclusions

In this paper, we studied the strongly *-labeling as a variation of strongly multiplicative labeling. We showed that every strongly multiplicative graph induces a strongly *-graph and established an explicit relation between the upper bounds on the number of edges of these two classes. In particular, we derived an explicit formula for $\lambda^*(n)$, and identified a sufficient condition under which $\lambda^*(n) > \lambda(n)$. Computational results were provided to support the theoretical findings and to illustrate the behavior of $\lambda^*(n)$ and its relation to $\lambda(n)$. Moreover, We demonstrated the independence of several necessary conditions used to show that certain graphs or families of graphs do not admit strongly *-labeling. Future research may focus on a complete characterization of the integers n for which $\lambda^*(n) \leq \lambda(n)$, as well as on the development of sharper upper bounds for $\lambda^*(n)$. Additional directions include investigating the asymptotic behavior of $\lambda^*(n)$, identifying further structural properties of strongly*-graphs, and developing more efficient algorithms for detecting graphs that admit strongly*-labeling.

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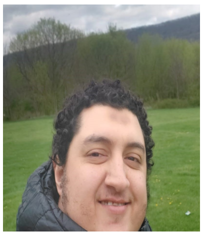

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