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A STUDY OF FUZZY $\mathcal{W}$-CLOSED SUBMODULES AND RELATED CONCEPTS

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Abstract. In this paper we define fuzzy weak essential submodules to introduce the concept of fuzzy $\mathcal{W}$-closed submodules of an R-module M. Further, we define fuzzy fully semiprime module. We use the condition of fuzzy fully semiprime module to show that a non-constant fuzzy closed submodule is $\mathcal{W}$-closed in M. Also, the chain condition on fuzzy $\mathcal{W}$-closed submodules is studied.

Keywords: fuzzy $\mathcal{W}$-closed submodules, fuzzy weak essential submodules, fuzzy fully semi-prime submodules.

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1. Introduction

In 1965, Zadeh [20] proposed the concept of a fuzzy set. Soon after fuzzy concept was extended in algebra and other branches of mathematics. Moderson and Malik [11] studied the concept of fuzzy submodules. Saikia and Kalita [16] studied various properties on fuzzy essential submodules. Atani [3] studied L-fuzzy multiplication modules. Nimbhorkar and Khubchandani [14] studied fuzzy semi-essential submodules and fuzzy semi-closed submodules. Also, Nimbhorkar and Khubchandani [15] studied L-fuzzy hollow modules and L-fuzzy multiplication modules, Nimbhorkar and Khubchandani [12] studied fuzzy essential-small submodules and fuzzy small-essential submodules.

In [16] Saikia and Kalita studied fuzzy closed submodules and Hassan and Hatam [5] studied weak essential fuzzy submodules. This motivated us to define fuzzy $\mathcal{W}$-closed submodules and study its properties. Also its wide applications in cryptography, coding theory, decision-making, automata and soft computing.

In present paper in section 3, we define fuzzy weak essential submodules. In section 4, we study some examples and properties of fuzzy $\mathcal{W}$-closed submodules. In section 5, we study some properties of fuzzy fully semiprime submodules. Lastly in section 6, we study chain conditions on fuzzy $\mathcal{W}$-closed submodules.

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2. Preliminaries

Throughout this paper R denotes a commutative ring with identity, M a unitary R-module with zero element θ. We use the notations "$\subseteq$"and "$\leq$"to denote inclusion and submodule respectively. We recall some definitions and results.

Definition 2.1. [20] *Let S be a nonempty set. A mapping $\mathcal{C} : S \to [0, 1]$ is called a fuzzy subset of S.*

Remark 2.1. [20] *If $\mathcal{C}$ and $\mathcal{F}$ are two fuzzy subsets of M, then*
(i) $\mathcal{C} \subseteq \mathcal{F}$ *if and only if* $\mathcal{C}(x) \leq \mathcal{F}(x)$;
(ii) $(\mathcal{C} \cup \mathcal{F})(x) = \max\{\mathcal{C}(x), \mathcal{F}(x)\}$;
(iii) $(\mathcal{C} \cap \mathcal{F})(x) = \min\{\mathcal{C}(x), \mathcal{F}(x)\}$; *for all* $x \in M$.

Let $N \leq M$, then the characteristic function, χ_N, of N is defined as,

$$\chi_N(x) = \begin{cases} 1, & \text{if } x \in N, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. [11] *Let M be an R-module. A fuzzy subset $\mathcal{C}$ of M is said to be a fuzzy submodule, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:*
(i) $\mathcal{C}(\theta) = 1$;
(ii) $\mathcal{C}(x - y) \geq \min\{\mathcal{C}(x), \mathcal{C}(y)\}$;
(iii) $\mathcal{C}(rx) \geq \mathcal{C}(x)$.

The set of all fuzzy submodules of M is denoted by $F(M)$ and we denote $\mathcal{C}_*$ the set $\mathcal{C}_* = \{x \in M \mid \mathcal{C}(x) = 1\}$.

Definition 2.3. [16] *A fuzzy submodule $\mathcal{C}$ of M is called an essential fuzzy submodule of M, denoted by $\mathcal{C} \trianglelefteq M$, if for every nonzero fuzzy submodule $\mathcal{F}$ of M, $\mathcal{C} \cap \mathcal{F} \neq \chi_\theta$.*

Definition 2.4. [16] *A fuzzy submodule $\mathcal{C}$ of M is said to be a closed submodule of M if μ has no non-constant (proper) essential extension, i.e, the only solution of the relation $\mathcal{C} \trianglelefteq \mathcal{F} \subseteq M$ is $\mathcal{C} = \mathcal{F}$.*

Definition 2.5. *Let X be a fuzzy module of an R-module M then X is called a chained fuzzy module if for each fuzzy submodules A, B of X either $A \subseteq B$ or $B \subseteq A$.*

Definition 2.6. [16] *A fuzzy submodule $\delta(\neq \chi_\theta)$ is said to be uniform if any two nonzero fuzzy submodules of δ have non-zero intersection, that is, if each non-zero fuzzy submodule of δ is essential in δ.*

Proposition 2.1. [6] *Let X be a fuzzy module of an R-module M and let $A \leq B \leq X$. If A is closed fuzzy in B and B is closed in X, then A is closed fuzzy in X.*

Definition 2.7. [1] *A fuzzy submodule $\mathcal{C}$ of an R-module M is called a fuzzy semi-essential submodule of M if for any nonzero fuzzy prime submodule $\mathcal{F}$ of M, $\mathcal{C} \cap \mathcal{F} \neq \chi_\theta$ and then we write $\mathcal{C} \trianglelefteq_{semi} M$.*

Definition 2.8. [4] *Let μ be a proper fuzzy submodule of a fuzzy module A of an R-module M, then μ is a fuzzy semiprime of A if and only if $\forall r \in R$ and $\forall x_l \subseteq A$ such that $r_t^2 x_l \subseteq \mu$ then $r_t x_l \subseteq \mu$, $t, l \in [0, 1]$.*

Proposition 2.2. [5] *Let A be a fuzzy submodule of a fuzzy module X. Then A is weak essential in X if and only if A_* is weak essential submodule in X_*.*

Proposition 2.3. [5] *Let A and B be non-constant fuzzy submodules of X such that $A \subseteq B$. If A is weak essential fuzzy submodule in B and B is weak fuzzy submodule in X implies A is weak essential fuzzy submodules in X.*

Definition 2.9. [14] *A fuzzy submodule $\mathcal{C}$ of an R-module M is called semi-closed if $\mathcal{C}$ has no proper(non-constant) semi-essential extensions in M, that is, if $\mathcal{C} \trianglelefteq_{semi} \mathcal{F} \leq M$ then $\mathcal{C} = \mathcal{F}$.*

3. FUZZY WEAK-ESSENTIAL SUBMODULES

In this section, we study fuzzy weak essential submodule of a commutative ring R.

Definition 3.1. [5] *A fuzzy submodule $\mathcal{C}$ of an R-module M is called a fuzzy weak essential submodule of M if for any nonzero semiprime submodule $\mathcal{F}$ of M, $\mathcal{C} \bigcap \mathcal{F} \neq \chi_\theta$ and then we write $\mathcal{C} \trianglelefteq_{\mathcal{W}} M$.*

Proposition 3.1. *The sum of two fuzzy weak essential submodules is again a weak essential submodule.*

Proof. Let $\mathcal{C}, \mathcal{F} \leq F(M)$. We know that $\mathcal{C} \leq \mathcal{C} + \mathcal{F}$. Since, $\mathcal{C} \trianglelefteq_{\mathcal{W}} M$, so by Proposition 2.3, $\mathcal{C} + \mathcal{F} \trianglelefteq_{\mathcal{W}} M$. □

Remark 3.1. *The converse of Proposition 2.3 is not true.*

Example 3.1. *Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{36}$ and the fuzzy module define as $X : M \to [0, 1]$.*

$$X(x) = \begin{cases} 1, & if\ x \in <\bar{3}6>, \\ 0, & otherwise. \end{cases}$$

Define fuzzy submodule $\mathcal{C} : M \to [0, 1]$ as follows:

$$\mathcal{C}(x) = \begin{cases} 1, & if\ x \in <\bar{1}8>, \\ 0.5, & otherwise. \end{cases}$$

Here, $\mathcal{C}_* = <\bar{1}8>$ is weak essential submodule of $\mathbb{Z}_{36}$.
Hence, by Proposition 2.2, $\mathcal{C} \trianglelefteq_{\mathcal{W}} \mathbb{Z}_{36}$.
Now, Define fuzzy submodule $\mathcal{F} : M \to [0, 1]$ as follows:

$$\mathcal{F}(x) = \begin{cases} 1, & if\ x \in <\bar{2}>, \\ 0.9, & otherwise. \end{cases}$$

We observe that $\mathcal{C} \subseteq \mathcal{F}$. Also, $\mathcal{C}_* = <\bar{1}8>$, $\mathcal{F}_* = <\bar{2}>$ and $\mathcal{C}_* \subseteq \mathcal{F}_*$.
We observe that $\mathcal{C}_* \ntrianglelefteq_{\mathcal{W}} \mathcal{F}_*$.
Thus, by Proposition 2.2, $\mathcal{C} \ntrianglelefteq_{\mathcal{W}} \mathcal{F}$.

4. FUZZY $\mathcal{W}$-CLOSED SUBMODULES

In this section, we define fuzzy $\mathcal{W}$-closed submodule, study some of its properties and examples.

Definition 4.1. *A fuzzy submodule $\mathcal{C}$ of an R-module M is called a $\mathcal{W}$-closed in M if $\mathcal{C}$ has no non-constant weak essential extension in M. That is if there exists a submodule $\mathcal{F}$ of M with $\mathcal{C} \trianglelefteq_{\mathcal{W}} \mathcal{F} \leq M$ then $\mathcal{C} = \mathcal{F}$ and we write $\mathcal{C} \leq_{\mathcal{W}} M$.*

Remark 4.1. *Every fuzzy $\mathcal{W}$-closed submodule of an R-module is fuzzy closed submodule in M.*

Proof. Let $\mathcal{F}$ be a fuzzy $\mathcal{W}$-closed submodule of M and $\mathcal{C}$ be a fuzzy submodule in M with $\mathcal{F}$ is essential in $\mathcal{C}$, then by [[5], Remark 2.5], $\mathcal{F}$ is weak essential in $\mathcal{C}$. But $\mathcal{F}$ is $\mathcal{W}$-closed in M, thus, $\mathcal{F} = \mathcal{C}$. Hence, $\mathcal{F}$ is fuzzy closed submodule in M. □

Remark 4.2. *The converse of Remark 4.1 may not be true.*

Example 4.1. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{24}$.*
Define fuzzy submodule $\mathcal{C} : M \to [0,1]$ as follows:

$$\mathcal{C}(x) = \begin{cases} 1, & if\ x \in <\bar{3}>, \\ 0.7, & otherwise. \end{cases}$$

Then $\mathcal{C}$ is closed submodule in $\mathbb{Z}_{24}$, as $\mathcal{C}$ is a direct summand of $\mathbb{Z}$ module $\mathbb{Z}_{24}$, by [[2], Remark 2].

$$\mu(x) = \begin{cases} 1, & if\ x \in <\bar{2}>, \\ 0.2, & otherwise. \end{cases}$$

Here, $<\bar{2}>$ is a prime submodule of $\mathbb{Z}_{24}$ and 0.2 is prime element.
Then by [[2],Theorem 3.6], μ is prime submodule of M.
Now, by [[10], Theorem 2.1], μ is fuzzy semi-prime submodule of $\mathbb{Z}_{24}$.
We observe that $\mathcal{C} \bigcap \mu \neq \chi_\theta$. Implies $\mathcal{C}$ is weak essential submodule of M and thus, $\mathcal{C}$ is not $\mathcal{W}$-closed submodule of $\mathbb{Z}_{24}$.

Proposition 4.1. *If $\mathcal{C}$ is a fuzzy submodule of an R-module M such that $\mathcal{C}$ is weak essential and $\mathcal{W}$-closed submodule of M, then $\mathcal{C} = \chi_M$.*

Proof. As $\mathcal{C}$ is $\mathcal{W}$-closed submodule of M, then $\mathcal{C}$ has no non-constant weak essential extension in M. But given that $\mathcal{C}$ is weak essential submodule of M, then $\mathcal{C} = \chi_M$. □

Remark 4.3. *(i) Every fuzzy submodule of M is $\mathcal{W}$-closed submodule in itself.*
(ii) The fuzzy submodule χ_θ may not be $\mathcal{W}$-closed submodule of an R-module M.
Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_4$. Define a fuzzy submodule $\mathcal{C} : M \to [0,1]$ as $\mathcal{C}(x) = 0$, for all $x \in \mathbb{Z}_4$ is not $\mathcal{W}$-closed submodule of M.

Remark 4.4. [6] *Let X be a fuzzy module of R-module M. A and B are non-trivial fuzzy submodules of X, $A_* \subseteq B_*$ implies $A \subseteq B$.*

Theorem 4.1. *Let $\mathcal{F} \in F(M)$ which satisfies the remark 4.4. Then $\mathcal{F}$ is a $\mathcal{W}$-closed submodule of M if and only if $\mathcal{F}_*$ is $\mathcal{W}$-closed submodule of M.*

Proof. Assume $\mathcal{F}$ is a $\mathcal{W}$-closed submodule of M and $\mathcal{F}_* \trianglelefteq_{\mathcal{W}} N \leq M$.
To show: $\mathcal{F}_* = N$.
Define,

$$\mathcal{C}(x) = \begin{cases} 1, & if\ x \in N, \\ 0.7, & otherwise. \end{cases}$$

It is clear that $\mathcal{C}$ is a fuzzy submodule of M and $\mathcal{C}_* = N$, $\mathcal{F}_* \subseteq \mathcal{C}_* = N$.
So by Remark 4.4, $\mathcal{F} \subseteq \mathcal{C}$. But $\mathcal{F}_* \trianglelefteq_{\mathcal{W}} \leq N = \mathcal{C}_*$, then by Proposition 2.8 of [5], $\mathcal{F}_* \trianglelefteq_{\mathcal{W}} \mathcal{C}_*$. Therefore, $\mathcal{F} = \mathcal{C}$. Since $\mathcal{F}$ is a closed fuzzy submodule in M, so $\mathcal{F}_* = \mathcal{C} = N$, then $\mathcal{F}_* = N$.

Conversely, Assume $\mathcal{F}_*$ is $\mathcal{W}$-closed submodule in M and $\mathcal{F} \trianglelefteq_{\mathcal{W}} \mathcal{C} \leq M$.
To show: $\mathcal{F} = \mathcal{C}$.
Since, $\mathcal{F} \trianglelefteq_{\mathcal{W}} \mathcal{C}$, then Proposition 2.8 of [5], $\mathcal{F}_* \trianglelefteq_{\mathcal{W}} \mathcal{C}_*$. But $\mathcal{F}_*$ is $\mathcal{W}$-closed submodule in M so that $\mathcal{F}_* = \mathcal{C}_*$, then by Remark 4.4, $\mathcal{F} = \mathcal{C}$. Hence, $\mathcal{F}$ is a $\mathcal{W}$-closed submodule of M. □

Proposition 4.2. *Let $\mathcal{F}$ be a non-constant fuzzy submodule of M. Then there exists a fuzzy $\mathcal{W}$-closed submodule $\mathcal{C}$ in M with $\mathcal{F}$ is weak essential in $\mathcal{C}$.*

Proof. Let,

$$\tau = \{\omega \in F(M) |\ \mathcal{F}\ is\ weak\ essential\ in\ \omega\},$$

then $\tau \neq \phi$ as $\omega \in \tau$ and $(\tau, \subseteq)$ forms a poset.
Clearly, the union of members of a chain in τ is again a member of τ. Hence by Zorn's lemma, τ has a maximal element say $\mathcal{C}$.
Claim: To prove $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M.
Assume that there exists $\mathcal{H} \in F(M)$ with $\mathcal{C}$ weak essential in $\mathcal{H}$. Since $\mathcal{F}$ is weak essential in $\mathcal{C}$ and $\mathcal{C}$ is weak essential in $\mathcal{H}$ so by Proposition 2.12 of [5], $\mathcal{F}$ is weak essential in $\mathcal{H}$. But this is a contradiction to the maximality of $\mathcal{C}$. Thus, $\mathcal{C} = \mathcal{H}$. Hence, $\mathcal{C}$ is a $\mathcal{W}$-closed submodule in M with $\mathcal{F}$ is weak essential in $\mathcal{C}$. □

Proposition 4.3. *Let $\mathcal{F}, \mathcal{C} \in F(M)$ with $\mathcal{F}$ is any weak essential extensions of $\mathcal{C}$, $\mathcal{C}$ is a $\mathcal{W}$-closed submodule of $\mathcal{F}$ and $\mathcal{F}$ is a $\mathcal{W}$-closed submodule in X, then $\mathcal{C}$ is $\mathcal{W}$-closed submodule in X.*

Proof. Let $\nu \leq F(M)$ such that $\mathcal{C}$ is weak essential in ν.
By hypothesis, $\mathcal{F} \leq \nu$. Since, $\mathcal{C}$ is weak essential in ν and $\mathcal{C}$ is a submodule of $\mathcal{F}$, then by Remark 2.9(2) of [5], $\mathcal{F}$ is weak essential in ν. But $\mathcal{F}$ is a $\mathcal{W}$-closed submodule in X, then $\mathcal{F} = \nu$. That is $\mathcal{C}$ is weak essential $\mathcal{F}$. But $\mathcal{C}$ is $\mathcal{W}$-closed submodule in $\mathcal{F}$. So $\mathcal{C} = \mathcal{F}$.
Hence, $\mathcal{C}$ is $\mathcal{W}$-closed submodule in X. □

Proposition 4.4. *Let $\mathcal{F}_1, \mathcal{F}_2 \leq F(M)$ with $\mathcal{F}_2$ is containing any weak essential extensions of $\mathcal{F}_1$, and $\mathcal{F}_1$ is a $\mathcal{W}$-closed submodule in $\mathcal{F}_2$ and $\mathcal{F}_2$ is $\mathcal{W}$-closed submodule in M, then $\mathcal{F}_1$ is a $\mathcal{W}$-closed submodule in M.*

Proof. Let $\mathcal{C} \leq F(M)$ with $\mathcal{F}_1$ is a weak essential submodule in $\mathcal{C}$, then by hypothesis we get, $\mathcal{C} \leq \mathcal{F}_2$. Since, $\mathcal{F}_1$ is a $\mathcal{W}$-closed submodule in $\mathcal{F}_2$, then $\mathcal{F}_2 = \mathcal{C}$.
Thus, $\mathcal{F}_1$ is a $\mathcal{W}$-closed submodule in M. □

Proposition 4.5. *Let M be a fuzzy chained module and $\mathcal{C}, \mathcal{F} \leq F(M)$ with $\mathcal{F} \leq \mathcal{C}$, $\mathcal{F} \leq_{\mathcal{W}} \mathcal{C}$ and $\mathcal{C} \leq_{\mathcal{W}} M$, then $\mathcal{F} \leq_{\mathcal{W}} M$.*

Proof. Let $\nu \leq F(M)$ with $\mathcal{F}$ is weak essential in ν. Since M is fuzzy chained module, then either $\nu \leq \mathcal{C}$ or $\mathcal{C} \leq \nu$.
Case(i): If $\nu \leq \mathcal{C}$ and $\mathcal{F} \leq_{\mathcal{W}} \mathcal{C}$, then $\mathcal{F} = \nu$.
Hence, $\mathcal{F} \leq_{\mathcal{W}} M$.
Case(ii): If $\mathcal{C} \leq \nu$ and as $\mathcal{F} \leq_{\mathcal{W}} \nu$, then by Remark (2.9)(2), $\mathcal{C} \trianglelefteq_{\mathcal{W}} \nu$.
But $\mathcal{C} \trianglelefteq_{\mathcal{W}} M$, hence $\mathcal{C} = \nu$. Thus, $\mathcal{F} \trianglelefteq_{\mathcal{W}} \mathcal{C}$.
But $\mathcal{F}$ is $\mathcal{W}$-closed submodule in $\mathcal{C}$, then $\mathcal{F} = \mathcal{C}$.
Hence, $\mathcal{F} \leq_{\mathcal{W}} M$. □

Definition 4.2. *An R-module M is called fuzzy completely essential if each non-constant fuzzy weak essential submodule of M is essential submodule in M.*

Proposition 4.6. *Let $\mathcal{F} \leq F(M)$ such that every essential extensions of $\mathcal{F}$ is completely essential, then $\mathcal{F}$ is closed submodule in M if and only if $\mathcal{F}$ is $\mathcal{W}$-closed submodule in M.*

Proof. Let $\mathcal{F}, \nu \leq F(M)$ such that $\mathcal{F} \trianglelefteq_{\mathcal{W}} \nu$.
By hypothesis as ν is completely essential, therefore $\mathcal{F}$ is essential submodule in ν. But $\mathcal{F}$ is a closed submodule in M, then $\mathcal{F} = \nu$. Thus, $\mathcal{F} \trianglelefteq_{\mathcal{W}} M$.
Conversely, by using Remark 4.1 we get the result. □

Definition 4.3. *A non-constant fuzzy module M is called weak uniform if every non-constant fuzzy submodules of M is weak essential.*

Proposition 4.7. *Let M be an R-module. Then M-module is fuzzy uniform module if and only if M is fuzzy weak uniform and fuzzy completely essential module.*

Proof. Let $\mathcal{C}$ be a non-constant fuzzy submodule of M and since M is uniform module, $\mathcal{C}$ is essential submodule and also every essential submodule is weak essential so $\mathcal{C}$ is weak essential. Let $\mathcal{R}$ be fuzzy weak essential submodule of M. Since, M is uniform module so $\mathcal{R}$ is essential submodule of M. Hence, M is weak uniform and fuzzy completely essential module.

Conversely, let $\mathcal{F}$ be a non-constant fuzzy submodule of M.
Since, M is fuzzy weak uniform module then $\mathcal{F}$ is a fuzzy weak essential submodule of M. But M is fuzzy completely essential module, therefore $\mathcal{F}$ is fuzzy completely essential submodule of M. □

Corollary 4.1. *If M is fuzzy uniform module and $\mathcal{C} \leq F(M)$. Then $\mathcal{C}$ is closed submodule in M if and only if $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M.*

Proof. Assume $\mathcal{C}$ is fuzzy closed submodule in M and let $\mathcal{C}$ be weak essential submodule in $\mathcal{F}$ where $\mathcal{F}$ is fuzzy submodule in M, then $\mathcal{F}$ is uniform. Hence, by Proposition 4.7, $\mathcal{F}$ is completely essential. Thus, $\mathcal{C}$ is essential in $\mathcal{F}$. But $\mathcal{C}$ is closed, then $\mathcal{C} = \mathcal{F}$. Thus, $\mathcal{C}$ is $\mathcal{W}$-closed in M. □

Proposition 4.8. *Let M be an R-module with $\mathcal{C}, \mathcal{F} \in M$ such that $\mathcal{C} \leq \mathcal{F}$. Also every weak essential extensions of $\mathcal{C}$ is completely essential submodule of M. If $\mathcal{C} \leq_{\mathcal{W}} \mathcal{F}$ and $\mathcal{F} \leq_{\mathcal{W}} M$, then $\mathcal{C} \leq_{\mathcal{W}} M$.*

Proof. Since $\mathcal{C} \leq_{\mathcal{W}} \mathcal{F}$ and $\mathcal{F} \leq_{\mathcal{W}} M$, then by Remark 4.1, $\mathcal{C}$ is closed submodule in $\mathcal{F}$ and $\mathcal{F}$ is closed submodule in M. Then by Proposition 3.6 of [6], we get $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M and by Proposition 4.6, $\mathcal{C} \leq_{\mathcal{W}} M$. □

Proposition 4.9. *If $\mathcal{C}, \mathcal{F} \in F(M)$ such that $\mathcal{C} \leq \mathcal{F}$ and $\mathcal{C} \leq_{\mathcal{W}} M$, then $\mathcal{C} \leq_{\mathcal{W}} \mathcal{F}$.*

Proof. Let $\mathcal{D} \leq \mathcal{F}$, then $\mathcal{F} \leq F(M)$ and $\mathcal{C}$ is weak essential in $\mathcal{D}$.
But $\mathcal{C} \leq_{\mathcal{W}} M$, then $\mathcal{C} = \mathcal{D}$. Hence, $\mathcal{C} \leq_{\mathcal{W}} \mathcal{F}$. □

Corollary 4.2. *Let $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M, then $\mathcal{C}$ is $\mathcal{W}$-closed submodule in $\sqrt{\mathcal{C}}$.*

Proof. Since, $\mathcal{C} \leq \sqrt{\mathcal{C}} \leq \mathcal{M}$ and $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M, then by Proposition 4.9 $\mathcal{C}$ is $\mathcal{W}$-closed submodule in $\sqrt{\mathcal{C}}$. □

Proposition 4.10. *Let M be an R-module such that every fuzzy semi-prime submodule of M is irreducible. If $\mathcal{C} \in F(M)$ is semi-essential then $\mathcal{C}$ is weak essential submodule of M.*

Proof. Let $\mathcal{F} \in F(M)$ be a non-constant fuzzy semi-prime submodule of M with $\mathcal{C} \cap \mathcal{F} = \chi_\theta$. Since, $\mathcal{F}$ is irreducible then by Theorem 2.4 of [10], $\mathcal{F}$ is prime submodule. But $\mathcal{C}$ semi-essential submodule of M, therefore $\mathcal{F} = \chi_\theta$. □

Proposition 4.11. *Let M be a faithful fuzzy multiplication R-module and $\mathcal{C} \in F(M)$ such that $\mathcal{C} = \mu\lambda_M$ for some fuzzy ideal μ of R. Suppose that every non-constant fuzzy semi-prime submodule of M is irreducible. If μ is weak essential ideal of R then $\mathcal{C}$ is weak essential submodule of M.*

Proof. Suppose that $\mathcal{C} \cap \mathcal{F} = \chi_\theta$ for some non-constant semi-prime submodule $\mathcal{F} \leq F(M)$. Given $\mathcal{F}$ is irreducible submodule of M, so by Theorem 2.4 of [10], $\mathcal{F}$ is prime submodule. But $\mathcal{F}$ is a non-constant submodule of fuzzy multiplication M this implies that there exists

a fuzzy prime ideal ν of R such that $\mathcal{F} = \nu\lambda_M$ by Theorem 17 of [3].
Now consider,

$$\begin{aligned}\chi_\theta &= \mathcal{C} \cap \mathcal{F} \\ &= \mu\lambda_M \cap \nu\lambda_M \\ &= (\mu \cap \nu)\lambda_M\end{aligned}$$

But as M is faithful fuzzy multiplication R-module, therefore $\mu \cap \nu = \chi_\theta$. Since every fuzzy prime submodule is semi-prime and by assumption, we get $\nu = \chi_\theta$ but $\mathcal{F} = \nu\chi_M$ and therefore $\mathcal{F} = \chi_\theta$. □

Proposition 4.12. *Let M be a faithful fuzzy multiplication R-module and $\mathcal{C} \in F(M)$ such that $\mathcal{C} = \mu\lambda_M$ for some fuzzy ideal μ of R. Suppose that every non-constant fuzzy semi-prime submodule of M is primary. If μ is weak essential ideal of R then $\mathcal{C}$ is weak essential submodule of M.*

Proof. Let $\mathcal{C} \cap \mathcal{F} = \chi_\theta$ for some non-constant semi-prime submodule $\mathcal{F} \leq F(M)$.
By Corollary 2.1 of [10], $[\mathcal{F}, \lambda_M]$ is fuzzy semi-prime ideal of R and also by assumption $\mathcal{F}$ is primary. By Proposition 2.1 of [10], $\mathcal{F}$ is fuzzy prime submodule of M and by Theorem 17 of [3], $\mathcal{F} = \nu\lambda_M$, for some fuzzy prime ideal ν of R.
Now consider,

$$\begin{aligned}\chi_\theta &= \mathcal{C} \cap \mathcal{F} \\ &= \mu\lambda_M \cap \nu\lambda_M \\ &= (\mu \cap \nu)\lambda_M\end{aligned}$$

But M is faithful fuzzy multiplication R-module, therefore $\mu \cap \nu = \chi_\theta$. Since every fuzzy prime submodule is semi-prime and by assumption, we get $\nu = \chi_\theta$ but $\mathcal{F} = \nu\chi_M$ and therefore $\mathcal{F} = \chi_\theta$. □

5. FUZZY FULLY SEMIPRIME SUBMODULES

In this section, we define fuzzy fully semiprime submodule of an R-module M.

Definition 5.1. *An R-module M is called fuzzy fully semiprime, if every non-constant fuzzy submodule of M is fuzzy semiprime.*

Lemma 5.1. *Let $\mathcal{C}, \mathcal{F} \in F(M)$ such that $\mathcal{C} \leq \mathcal{F}$. If $\mathcal{C}$ is a fuzzy semiprime submodule of M, then $\mathcal{C}$ is fuzzy semiprime submodule in $\mathcal{F}$.*

Proof. Let for any $r_t \subseteq R$ and $for\ all\ x_l \subseteq M$, t, $l \in [0, 1]$ such that $r_t^2 x_l \subseteq \mathcal{C}$. But $\mathcal{C}$ is a fuzzy semiprime submodule of M,then $r_t x_l \subseteq \mathcal{C}$. Hence, $\mathcal{C}$ is a fuzzy semiprime submodule in $\mathcal{F}$. □

Proposition 5.1. *Let M be a fuzzy fully semi-prime R-module. Let $\mathcal{C}, \mathcal{F} \in F(M)$. Then $\mathcal{C}$ is weak essential in $\mathcal{F}$ if and only if $\mathcal{C}$ is essential in $\mathcal{F}$.*

Proof. As $\mathcal{C} \in F(M)$ and μ be a fuzzy submodule of $\mathcal{C}$ such that $\mathcal{F} \cap \mu = \chi_\theta$. Since, M is fuzzy fully semi-prime module then both the fuzzy submodules $\mathcal{C}$ and μ are semiprime submodules of M and by Lemma 5.1, $\mathcal{F}$ is fuzzy semiprime submodule of $\mathcal{C}$. But $\mathcal{C}$ is weak essential submodule of $\mathcal{F}$, therefore $\mu = \chi_\theta$, hence $\mathcal{C}$ is weak essential in $\mathcal{F}$.

Converse is straight forward. □

Corollary 5.1. *If M is fuzzy fully semiprime module, then every non-zero weak essential submodule of M is fuzzy essential submodule in M.*

Proposition 5.2. *Let M be a fuzzy fully semiprime module and $\mathcal{C} \in F(M)$. Then $\mathcal{C}$ is a non-constant fuzzy closed in M if and only if $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M.*

Proof. Assume $\mathcal{C}$ is a non-constant fuzzy closed in M, and $\nu \in F(M)$ such that $\mathcal{C}$ is weak essential in ν. Then by Corollary 5.1, $\mathcal{C}$ is an fuzzy essential submodule in ν. But $\mathcal{C}$ is a non-constant fuzzy closed in M, hence $\mathcal{C} = \nu$. Thus, $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M.
The converse is direct. □

Proposition 5.3. *Let M be a fuzzy fully semiprime module and let $\mathcal{C}$ be a non-constant $\mathcal{W}$-closed submodule in $\mathcal{F}$ and $\mathcal{F}$ is $\mathcal{W}$-closed submodule in M. Then $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M.*

Proof. Since, $\mathcal{C}$ be a non-constant $\mathcal{W}$-closed submodule in $\mathcal{F}$ and $\mathcal{F}$ is $\mathcal{W}$-closed submodule in M, then by Remark 4.1, $\mathcal{C}$ is $\mathcal{W}$-closed submodule in $\mathcal{F}$ and $\mathcal{F}$ is $\mathcal{W}$-closed submodule in M. Hence by Proposition 2.1, $\mathcal{C}$ is closed in M. Hence, by Proposition 5.2, $\mathcal{C}$ is $\mathcal{W}$-closed submodule in M. □

6. Chain conditions on fuzzy $\mathcal{W}$-closed submodule

In this section, we introduce the definitions of a module that have ascending (descending) chain conditions on fuzzy $\mathcal{W}$-closed submodules of an R-module M.

Definition 6.1. *A fuzzy module M is said to have ascending chain condition (acc) on fuzzy $\mathcal{W}$-closed submodules, if every ascending chain $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq ...$ of $\mathcal{W}$-closed submodules of M is finite. That is there exists some $n \in N$ such that $\mathcal{C}_n = \mathcal{C}_m$, for all $n \geq m$.*

Definition 6.2. *A fuzzy module M is said to have descending chain condition (dcc) on fuzzy $\mathcal{W}$-closed submodules, if every descending chain $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq ...$ of $\mathcal{W}$-closed submodules of M is finite. That is there exists some $n \in N$ such that $\mathcal{C}_n = \mathcal{C}_m$, for all $n \geq m$.*

Proposition 6.1. *Let M be fuzzy module and satisfy (acc) on closed submodules, then M satisfy (acc) on fuzzy $\mathcal{W}$-closed submodules.*

Proof. Let $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq ...$ be an increasing chain of fuzzy $\mathcal{W}$-closed submodules of M. But by Remark 4.1, each $\mathcal{C}_i$ is fuzzy closed submodule for each $i = 1, 2, ...$ Since, M satisfy (acc) on closed submodules, then there exists $n \in N$ such that $\mathcal{C}_n = \mathcal{C}_m$, for all $n \geq m$.
Thus, M satisfy (acc) on fuzzy $\mathcal{W}$-closed submodules. □

Proposition 6.2. *Let M be fuzzy module and satisfy (dcc) on closed submodules, then M satisfy (dcc) on fuzzy $\mathcal{W}$-closed submodules.*

Proposition 6.3. *If M be a fuzzy fully semiprime module, then M satisfy (acc) on $\mathcal{W}$-closed submodules if and only if M satisfy (acc) on closed submodules.*

Proof. Let $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq ...$ be ascending chain of fuzzy closed submodules. Then by Proposition 5.2, $\mathcal{C}_i$ is fuzzy $\mathcal{W}$-closed submodules for each $i = 1, 2, ...$ But M satisfy (acc) on $\mathcal{W}$-closed submodules, so there exists $n \geq m$ such that $\mathcal{C}_n = \mathcal{C}_m$, for all $n \geq m$. Hence, M satisfies (acc) on fuzzy closed submodules.
Conversely, by Proposition 6.1. □

Proposition 6.4. *If M be a fuzzy fully semiprime module, then M satisfy (dcc) on $\mathcal{W}$-closed submodules if and only if M satisfy (dcc) on closed submodules.*

Remark 6.1. *A direct summand of fuzzy submodules of M may not be necessary $\mathcal{W}$-closed in M as shown in Example 6.1.*

Example 6.1. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{24}$.*
Define fuzzy submodules $\mathcal{C}, \mathcal{F} : M \to [0,1]$ as follows:

$$\mathcal{C}(x) = \begin{cases} 1, & if\ x \in <\bar{3}>, \\ 0, & otherwise. \end{cases}$$

$$\mathcal{F}(x) = \begin{cases} 1, & if\ x \in <\bar{8}>, \\ 0, & otherwise. \end{cases}$$

We observe that $\chi_M = \mathcal{C} \oplus \mathcal{F}$ as $\mathcal{C} + \mathcal{F} = \chi_M$ and $\mathcal{C} \cap \mathcal{F} = \chi_\theta$.
Here, $\mathcal{C}$ is not $\mathcal{W}$-closed submodules in $\mathbb{Z}_{24}$ as $\mathcal{C}$ is weak essential in $\mathbb{Z}_{24}$.

7. Conclusion

In this paper, we have studied fuzzy weak-essential submodules and fuzzy $\mathcal{W}$-closed submodules. Moreover, we studied chain conditions on fuzzy $\mathcal{W}$-closed submodules.

This work can be extended and generalised in the environment of q-rung orthopair fuzzy sets. It can be applied in order to solve multi-criteria decision making problems.

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