

$\delta\beta$ -OPEN SETS IN PYTHAGOREAN FUZZY NANO TOPOLOGICAL SPACES

MOHANARAO NAVULURI¹, K. SHANTHA LAKSHMI^{2*}, A. VADIVEL^{3*}, V. SIVAKUMAR³, §

ABSTRACT. The purpose of this paper is to define and study a new class of sets called Pythagorean fuzzy nano δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-open sets in Pythagorean fuzzy nano topological spaces. Analyse the basic properties of Pythagorean fuzzy nano δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-open (resp. closed) sets. We also used them to introduce the new notions like Pythagorean fuzzy nano δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$)-closure (resp. interior) and investigate their relations with already existing well known sets. We apply entropy measure for decision making problem of selecting the optimum wastewater treatment method for the dying factories based on the required criteria.

Keywords: Pythagorean fuzzy nano open set, Pythagorean fuzzy nano δ open set, Pythagorean fuzzy nano δ interior, Pythagorean fuzzy nano δ closure.

AMS Subject Classification: 03E72, 54A05, 54A40.

1. INTRODUCTION

Zadeh in [19] established the idea of fuzzy in 1965, which is a generalization of usual set using fuzzy where each element has a membership degree in $[0, 1]$. The subsequent advancement of fuzzy subsets was the intuitionistic fuzzy set published by Atanassov, [3] in 1983, which has elements having membership and non-membership degree. In 1968, Chang in [4] defined fuzzy topological space and fundamental results such as continuity, open and closed set. Following this, Lowen in [8] defined fuzzy topological space in other form. Coker introduced the idea of intuitionistic fuzzy topological space with few properties, see [5]. The concept of Pythagorean fuzzy subset which is a typical fuzzy subset was presented by Yager, see [17, 18]. Pythagorean fuzzy topological space was introduced by Olgun in [8] by taking the lead as from Chang.

¹ Department of Mathematics, Government College of Engineering, Bodinayakanur- 625582, Theni, India.

e-mail: mohanaraonavuluri@gmail.com; ORCID: <https://orcid.org/0009-0001-5654-5486>.

² Department of Mathematics, M. Kumarasamy College of Engineering, Karur - 639 113, India.

e-mail: kslakshmi20@gmail.com; ORCID: <https://orcid.org/0009-0006-0532-0793>.

³ PG and Research Department of Mathematics, Arignar Anna Government Arts College, Namakkal - 637 002, India.

e-mail: avmaths@gmail.com; ORCID: <https://orcid.org/0000-0001-5970-035X>.

e-mail: mathsshiva@gmail.com; ORCID: <https://orcid.org/0009-0003-9713-0388>.

* Corresponding authors.

§ Manuscript received: December 29, 2024; accepted: October 13, 2025.

TWMS Journal of Applied and Engineering Mathematics, Vol.16, No.4; © Işık University, Department of Mathematics, 2026; all rights reserved.

In 2013, a new topology called Nano topology was introduced by Lellis Thivagar [6] which is an extension of rough set theory. He also introduced Nano topological spaces which were defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it. The elements of a Nano topological space are called the Nano open sets and its complements are called the Nano closed sets. Nano means something very small. Nano topology, thus literally means the study of very small surface. The fundamental ideas in Nano topology are those of approximations and indiscernibility relation. Furthermore nano δ open sets in nano topological space was studied in [12]. Recently, Lellis Thivagar et. al [7] explored a new concept of neutrosophic nano topology. The notion of Z -closed sets in double fuzzy topological spaces were introduced by Shiventhiradevi Sathaananthan et. al. [14, 15] in 2020 and studied some of their properties, and also M -open sets in a nano topological spaces by Padma et. al. [9], δ -open sets in a nano topological spaces by Pankajam and Kavitha[12] and neutrosophic nano topological space by [16].

Research Gap: No investigation on some stronger and weaker forms of Pythagorean fuzzy nano open sets such as Pythagorean fuzzy nano δ open set, Pythagorean fuzzy nano δ -semi open set, Pythagorean fuzzy nano δ -pre open set, Pythagorean fuzzy nano $\delta\alpha$ open set and Pythagorean fuzzy nano $\delta\beta$ open sets on Pythagorean fuzzy nano topological space has been reported in the Pythagorean fuzzy literature.

In this paper some preliminary concepts required in our work are briefly recalled in section 2. In section 3, we introduce the concept of $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}o$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{S}o$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{P}o$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\alpha o$ and $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta o$ sets and studied some of their properties. Also, we discuss on Pythagorean fuzzy nano δ -interior and Pythagorean fuzzy nano δ -closure operators in Pythagorean fuzzy nano topological spaces.

2. PRELIMINARIES

Definition 2.1. [19] A function λ from X into the unit interval I is called a fuzzy set in X . For every $x \in X$, $\lambda(x) \in I$ is called the grade of membership of x in λ . Some authors say that λ is a fuzzy subset of X instead of saying that λ is a fuzzy set in X . The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X .

Definition 2.2. [19] If λ and ξ are any two fuzzy subsets of a set X , then λ is said to be included in ξ or λ is contained in ξ or λ is less than or equal to ξ iff $\lambda(x) \leq \xi(x)$ for all x in X and is denoted by $\lambda \leq \xi$. Equivalently, $\lambda \leq \xi$ iff $\mu_\lambda(x) \leq \mu_\xi(x)$ for all x in X .

Note that every fuzzy subset is included in itself and empty fuzzy subset is included in every fuzzy subset.

Definition 2.3. [19] Two fuzzy subsets λ and μ of a set X are said to be equal, written $\lambda = \mu$, if $\lambda(x) = \mu(x)$ for every x in X .

Definition 2.4. [19] The complement of a fuzzy subset λ in a set X , denoted by $1 - \lambda$, is the fuzzy subset of X defined by $1 - \lambda(x)$ for all x in X . Note that $1 - (1 - \lambda) = \lambda$.

Definition 2.5. [19] The union of two fuzzy subsets λ and μ in a set X , denoted by $\lambda \vee \mu$, is fuzzy subset in X defined by $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$, for all x in X .

In general, the union of a family of fuzzy subsets $\{\xi_i : i \in I\}$ is a fuzzy subset denoted by $\bigvee_{i \in I} \xi_i$ and defined by $(\bigvee_{i \in I} \xi_i)(x) = \sup\{\xi_i(x) : i \in I\}$, for all x in X .

Definition 2.6. [19] The intersection of two fuzzy subsets λ and μ in a set X , denoted by $\lambda \wedge \mu$, is fuzzy subset in X defined by $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$, for all x in X .

In general, the intersection of a family of fuzzy subsets $\{\xi_i : i \in I\}$ is a fuzzy subset denoted by $\bigwedge_{i \in I} \xi_i$ and defined by $(\bigwedge_{i \in I} \xi_i)(x) = \inf\{\xi_i(x) : i \in I\}$, for all x in U .

Definition 2.7. [17, 18] Let X be a universal set. Then, a Pythagorean fuzzy set A , which is a set of ordered pairs over X , is defined by the following: $A = \{ \langle x, \mu_A(x), \lambda_A(x) \rangle \mid x \in X \}$ or $A = \left\{ \left\langle \frac{\mu_A(x), \lambda_A(x)}{x} \right\rangle \mid x \in X \right\}$, where the functions $\mu_A(x) : X \rightarrow [0, 1]$ and $\lambda_A(x) : X \rightarrow [0, 1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A , which is a subset of X , and for every $x \in X$, $0 \leq (\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$. Supposing $(\mu_A(x))^2 + (\lambda_A(x))^2 \leq 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\lambda_A(x))^2]}$ and $\pi_A(x) \in [0, 1]$. In what follows, $(\mu_A(x))^2 + (\lambda_A(x))^2 + (\pi_A(x))^2 = 1$. Otherwise, $\pi_A(x) = 0$ whenever $(\mu_A(x))^2 + (\lambda_A(x))^2 = 1$. We denote the set of all Pythagorean fuzzy sets over X by $pfs(X)$.

Definition 2.8. [18] Let A and B be pfs 's of the forms $A = \{ \langle a, \mu_A(a), \lambda_A(a) \rangle \mid a \in X \}$ and $B = \{ \langle a, \mu_B(a), \lambda_B(a) \rangle \mid a \in X \}$. Then

- (i) $A \subseteq B$ if and only if $\mu_A(a) \leq \mu_B(a)$ and $\lambda_A(a) \geq \lambda_B(a)$ for all $a \in X$.
- (ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- (iii) $\bar{A} = \{ \langle a, \lambda_A(a), \mu_A(a) \rangle \mid a \in X \}$.
- (iv) $A \cap B = \{ \langle a, \mu_A(a) \wedge \mu_B(a), \lambda_A(a) \vee \lambda_B(a) \rangle \mid a \in X \}$.
- (v) $A \cup B = \{ \langle a, \mu_A(a) \vee \mu_B(a), \lambda_A(a) \wedge \lambda_B(a) \rangle \mid a \in X \}$.
- (vi) $0_P = \{ \langle a, 0, 1 \rangle \mid a \in X \}$ and $1_P = \{ \langle a, 1, 0 \rangle \mid a \in X \}$.
- (vii) $1_P = 0_P$ and $0_P = 1_P$.

Definition 2.9. [1] Let U be a non-empty set and R be an equivalence relation on U . Let A be a Pythagorean fuzzy set in U with the membership function $\mu_A(x)$ and non membership function $\lambda_A(x)$, $\forall x \in U$. The Pythagorean fuzzy nano lower, Pythagorean fuzzy nano upper approximation and Pythagorean fuzzy nano boundary of A in the approximation (U, R) denoted by $\underline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A)$, $\overline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A)$ and $B_{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A)$ are respectively defined as follows:

- (i) $\underline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A) = \left\{ \langle x, \mu_{\underline{R}(A)}(x), \lambda_{\underline{R}(A)}(x) \rangle \mid y \in [x]_R, x \in U \right\}$
- (ii) $\overline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A) = \left\{ \langle x, \mu_{\overline{R}(A)}(x), \lambda_{\overline{R}(A)}(x) \rangle \mid y \in [x]_R, x \in U \right\}$
- (iii) $B_{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A) = \overline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A) - \underline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A)$

$$\begin{aligned} \text{where } \mu_{\underline{R}(A)}(x) &= \bigwedge_{y \in [x]_R} \mu_A(y) \\ \lambda_{\underline{R}(A)}(x) &= \bigwedge_{y \in [x]_R} \lambda_A(y), \\ \mu_{\overline{R}(A)}(x) &= \bigvee_{y \in [x]_R} \mu_A(y), \\ \lambda_{\overline{R}(A)}(x) &= \bigvee_{y \in [x]_R} \lambda_A(y). \end{aligned}$$

Definition 2.10. [1] Let U be an universe of discourse, R be an equivalence relation on U and A be a Pythagorean fuzzy set in U and if the collection $\tau_{\mathcal{R}}(A) = \{0_P, 1_P, \underline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A), \overline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A), B_{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}}(A)\}$ forms a topology then it is said to be a Pythagorean fuzzy nano topology. We call $(U, \tau_{\mathcal{R}}(A))$ (or simply U) as the Pythagorean fuzzy nano topological space. The elements of $\tau_{\mathcal{R}}(A)$ are called Pythagorean fuzzy nano open (briefly, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}o$) sets.

Remark 2.1. [1] $[\tau_{\mathcal{R}}(A)]^c$ is called the dual fuzzy nano topology of $\tau_{\mathcal{R}}(A)$. Elements of $[\tau_{\mathcal{R}}(A)]^c$ are called Pythagorean fuzzy nano closed (briefly, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}c$) sets. Thus, we note that a Pythagorean fuzzy set G of U is Pythagorean fuzzy nano closed in $\tau_{\mathcal{R}}(A)$ if and only if $1_P - G$ is Pythagorean fuzzy nano open in $\tau_{\mathcal{R}}(A)$.

Definition 2.11. [1, 2] Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PYFY}\mathcal{N}ts$ with respect to A where A is a Pythagorean fuzzy subset of U . Let S be a Pythagorean fuzzy subset of U . Then Pythagorean fuzzy nano

- (i) interior of S (briefly, $\mathcal{PYFY}\mathcal{N}int(S)$) is defined by $\mathcal{PYFY}\mathcal{N}int(S) = \cup\{I : I \subseteq S \ \& \ I \text{ is a } \mathcal{PYFY}\mathcal{N}o \text{ set in } U\}$.
- (ii) closure of S (briefly, $\mathcal{PYFY}\mathcal{N}cl(S)$) is defined by $\mathcal{PYFY}\mathcal{N}cl(S) = \cap\{A : S \subseteq A \ \& \ A \text{ is a } \mathcal{PYFY}\mathcal{N}c \text{ set in } U\}$.
- (iii) regular open (briefly, $\mathcal{PYFY}\mathcal{N}ro$) set if $S = \mathcal{PYFY}\mathcal{N}int(\mathcal{PYFY}\mathcal{N}cl(S))$.
- (iv) regular closed (briefly, $\mathcal{PYFY}\mathcal{N}rc$) set if $S = \mathcal{PYFY}\mathcal{N}cl(\mathcal{PYFY}\mathcal{N}int(S))$.

3. PYTHAGOREAN FUZZY NANO δ (RESP. δ PRE, δ SEMI, $\delta\alpha$ AND $\delta\beta$)-OPEN SETS

Definition 3.1. Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PYFY}\mathcal{N}ts$ with respect to A where A is a pf s of U . Let S be a pf s of U . Then Pythagorean fuzzy nano δ interior (resp. closure) of S (briefly, $\mathcal{PYFY}\mathcal{N}\delta int(S)$ (resp. $\mathcal{PYFY}\mathcal{N}\delta cl(S)$)) is defined by $\mathcal{PYFY}\mathcal{N}\delta int(S) = \cup\{I : I \subseteq S \ \& \ I \text{ is a } \mathcal{PYFY}\mathcal{N}ro \text{ set in } U\}$ (resp. $\mathcal{PYFY}\mathcal{N}\delta cl(S) = \cap\{A : S \subseteq A \ \& \ A \text{ is a } \mathcal{PYFY}\mathcal{N}rc \text{ set in } U\}$).

Definition 3.2. Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PYFY}\mathcal{N}ts$ with respect to A where A is a pf s of U . Then a $\mathcal{PYFY}\mathcal{N}s$ S in U is said to be Pythagorean:

- (i) fuzzy nano δ -open set (briefly, $\mathcal{PYFY}\mathcal{N}\delta os$) if $S = \mathcal{PYFY}\mathcal{N}\delta int(S)$.
- (ii) fuzzy nano $\delta\mathcal{P}$ -open set (briefly, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}os$) if $S \subseteq \mathcal{PYFY}\mathcal{N}int(\mathcal{PYFY}\mathcal{N}\delta cl(S))$.
- (iii) fuzzy nano $\delta\mathcal{S}$ -open set (briefly, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}os$) if $S \subseteq \mathcal{PYFY}\mathcal{N}cl(\mathcal{PYFY}\mathcal{N}\delta int(S))$.
- (iv) fuzzy nano $\delta\alpha$ or a -open set (briefly, $\mathcal{PYFY}\mathcal{N}\delta\alpha os$ or $\mathcal{PYFY}\mathcal{N}\alpha os$) if $S \subseteq \mathcal{PYFY}\mathcal{N}int(\mathcal{PYFY}\mathcal{N}cl(\mathcal{PYFY}\mathcal{N}\delta int(S)))$.
- (v) fuzzy nano $\delta\beta$ or e^* -open set (briefly, $\mathcal{PYFY}\mathcal{N}\delta\beta os$ or $\mathcal{PYFY}\mathcal{N}e^* os$) if $S \subseteq \mathcal{PYFY}\mathcal{N}cl(\mathcal{PYFY}\mathcal{N}int(\mathcal{PYFY}\mathcal{N}\delta cl(S)))$.
- (vi) δ (resp. δ pre, δ semi, $\delta\alpha$ and $\delta\beta$) dense if $\mathcal{PYFY}\mathcal{N}\delta cl(A)$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}cl(A)$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}cl(A)$, $\mathcal{PYFY}\mathcal{N}\delta\alpha cl(A)$ and $\mathcal{PYFY}\mathcal{N}\delta\beta cl(A)$) $= 1_{\mathcal{P}}$.

The complement of a $\mathcal{PYFY}\mathcal{N}\delta os$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}os$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}os$, $\mathcal{PYFY}\mathcal{N}\delta\alpha os$ & $\mathcal{PYFY}\mathcal{N}\delta\beta os$) is called a $\mathcal{PYFY}\mathcal{N}\delta$ (resp. $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) closed set (briefly, $\mathcal{PYFY}\mathcal{N}\delta cs$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}cs$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}cs$, $\mathcal{PYFY}\mathcal{N}\delta\alpha cs$ and $\mathcal{PYFY}\mathcal{N}\delta\beta cs$)) in U .

The family of all $\mathcal{PYFY}\mathcal{N}\delta os$ (resp. $\mathcal{PYFY}\mathcal{N}\delta cs$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}os$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}cs$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}os$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}cs$, $\mathcal{PYFY}\mathcal{N}\delta\alpha os$, $\mathcal{PYFY}\mathcal{N}\delta\alpha cs$, $\mathcal{PYFY}\mathcal{N}\delta\beta os$ and $\mathcal{PYFY}\mathcal{N}\delta\beta cs$) of U is denoted by $\mathcal{PYFY}\mathcal{N}\delta OS(U)$, (resp. $\mathcal{PYFY}\mathcal{N}\delta CS(U)$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}OS(U)$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}CS(U)$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}OS(U)$, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}CS(U)$, $\mathcal{PYFY}\mathcal{N}\delta\alpha OS(U)$, $\mathcal{PYFY}\mathcal{N}\delta\alpha CS(U)$, $\mathcal{PYFY}\mathcal{N}\delta\beta OS(U)$ and $\mathcal{PYFY}\mathcal{N}\delta\beta CS(U)$).

Definition 3.3. Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{PYFY}\mathcal{N}ts$ with respect to A where A is a pf s of U . Let S be a pf s of U . Then Pythagorean fuzzy nano

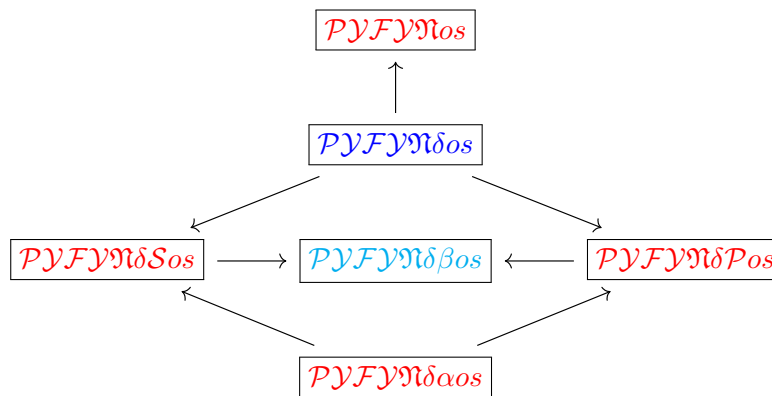
- (i) δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) interior of S (briefly, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}int(S)$, $\mathcal{PYFY}\mathcal{N}\delta\alpha int(S)$ and $\mathcal{PYFY}\mathcal{N}\delta\beta int(S)$)) is defined by $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}int(S)$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}int(S)$, $\mathcal{PYFY}\mathcal{N}\delta\alpha int(S)$ and $\mathcal{PYFY}\mathcal{N}\delta\beta int(S)$) $= \cup\{I : I \subseteq S \ \& \ I \text{ is a } \mathcal{PYFY}\mathcal{N}\delta\mathcal{P}o$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}o$, $\mathcal{PYFY}\mathcal{N}\delta\alpha o$ & $\mathcal{PYFY}\mathcal{N}\delta\beta o$) set in $U\}$.
- (ii) δ pre (resp. δ semi, $\delta\alpha$ and $\delta\beta$) closure of S (briefly, $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}cl(S)$, $\mathcal{PYFY}\mathcal{N}\delta\alpha cl(S)$ and $\mathcal{PYFY}\mathcal{N}\delta\beta cl(S)$)) is defined by $\mathcal{PYFY}\mathcal{N}\delta\mathcal{P}cl(S)$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}cl(S)$, $\mathcal{PYFY}\mathcal{N}\delta\alpha cl(S)$ and $\mathcal{PYFY}\mathcal{N}\delta\beta cl(S)$) $= \cap\{A : S \subseteq A \ \& \ A \text{ is a } \mathcal{PYFY}\mathcal{N}\delta\mathcal{P}c$ (resp. $\mathcal{PYFY}\mathcal{N}\delta\mathcal{S}c$, $\mathcal{PYFY}\mathcal{N}\delta\alpha c$ & $\mathcal{PYFY}\mathcal{N}\delta\beta c$) set in $U\}$.

Theorem 3.1. Let $(U, \tau_{\mathcal{R}}(A))$ be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}ts$. Then, (i) Every $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$ (resp. (ii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$, (iii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$, (iv) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta So$, (v) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta Po$, (vi) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\alpha o$ and (vii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\alpha o$) set is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}o$ (resp. (ii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta Po$, (iii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta So$, (iv) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta o$, (v) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta o$, (vi) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta Po$ and (vii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta So$) set.

Proof.

- (i) If S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$ s in U , then $S = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(S)$. Therefore, S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}o$ s.
- (ii) If S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$ s in U , then S is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}o$ s by (i). So, $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(S) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S))$. Therefore, S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta So$ s.
- (iii) If S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$ s in U , then S is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}o$ s by (i). So, $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(S) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta cl(S))$. Therefore, S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta Pos$ s.
- (iv) If S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta So$ s in U , then $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S)) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta cl(S)))$. Therefore, S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta o$ s.
- (v) If S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta Pos$ s in U , then $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta cl(S)) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta cl(S)))$. Therefore, S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta o$ s.
- (vi) If S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\alpha o$ s in U , then $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S)))$. So $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S))) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S))$. Therefore, S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta So$ s.
- (vii) If S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\alpha o$ s in U , then $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S)))$. So, $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S))) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta cl(S))$. Therefore S is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta Pos$ s. It is also true for their respective $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}$ closed sets.

Remark 3.1. The diagram shows $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta o$ s's in $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}ts$.



The converse of the Theorem 3.1 need not to be true. The following examples show it.

Example 3.1. Assume $U = \{s_1, s_2, s_3, s_4\}$ be the universe set and the equivalence relation is $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. Let $A = \left\{ \left\langle \frac{s_1}{0.3, 0.1} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle, \left\langle \frac{s_4}{0.4, 0.25} \right\rangle \right\}$ be a

pfs of U .

$$\begin{aligned} \underline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}}(A) &= \left\{ \left\langle \frac{s_1, s_4}{0.3, 0.25} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle \right\}, \\ \overline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}}(A) &= \left\{ \left\langle \frac{s_1, s_4}{0.4, 0.1} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle \right\}, \\ B_{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}}(A) &= \left\{ \left\langle \frac{s_1, s_4}{0.25, 0.3} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle \right\}. \end{aligned}$$

Thus $\tau_{\mathcal{R}}(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}}(A), \overline{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}}(A), B_{\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}}(A)\}$. Then

- (i) $\left\{ \left\langle \frac{s_1, s_4}{0.4, 0.1} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle \right\}$ is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta o$ (resp. $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\mathcal{P}o$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta o$ and $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\mathcal{P}o$) set but not $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta o$ (resp. $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta o$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta S o$ and $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\alpha o$) set.
- (ii) $\left\{ \left\langle \frac{s_1, s_4}{0.25, 0.3} \right\rangle, \left\langle \frac{s_2}{0.5, 0.1} \right\rangle, \left\langle \frac{s_3}{0.45, 0.2} \right\rangle \right\}$ is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta S o$ set but not $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta o$ (resp. $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\mathcal{P}o$) set.
- (iii) $\left\{ \left\langle \frac{s_1, s_4}{0.25, 0.35} \right\rangle, \left\langle \frac{s_2}{0.25, 0.45} \right\rangle, \left\langle \frac{s_3}{0.3, 0.2} \right\rangle \right\}$ is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta o$ set but not $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\mathcal{P}o$ set.

Theorem 3.2. Let $(U, \tau_{\mathcal{R}}(A))$ be an $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}ts$ and $\{S_{\alpha} | \alpha \in I\}$ be a family of $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta o$ (resp. $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta c$) sets in $(U, \tau_{\mathcal{R}}(A))$. Then $\bigcup\{S_{\alpha} | \alpha \in I\}$ (resp. $\bigcap\{S_{\alpha} | \alpha \in I\}$) is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta os$ (resp. $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cs$).

Proof. Let $\{S_{\alpha} | \alpha \in I\}$ be a family of $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta o$ sets in $(U, \tau_{\mathcal{R}}(A))$. Then for each α , $S_{\alpha} \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta cl(S_{\alpha})))$. Since $S_{\alpha} \subseteq \bigcup S_{\alpha}$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta cl(S_{\alpha}))) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta cl(\bigcup S_{\alpha})))$ and also $S_{\alpha} \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta cl(\bigcup S_{\alpha})))$. Hence $\bigcup S_{\alpha} \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta cl(\bigcup S_{\alpha})))$ which shows that $\bigcup S_{\alpha}$ is an $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta os$. The other case is similar.

Remark 3.2. The Theorem 3.2 is also true for $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta S os$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\mathcal{P} os$ and $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\alpha o$.

Theorem 3.3. Let K be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cs$ in $(U, \tau_{\mathcal{R}}(A))$ then $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) - K$ does not contain any non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cs$ in $(U, \tau_{\mathcal{R}}(A))$.

Proof. Let K be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cs$ in $(U, \tau_{\mathcal{R}}(A))$ and S be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cs$ of $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) - K$. That is, $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) - K$ implies $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) \wedge (1_{\mathcal{P}} - K)$. That is $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K)$ and $S \subseteq (1_{\mathcal{P}} - K)$ which implies $K \subseteq (1_{\mathcal{P}} - S)$ where $1_{\mathcal{P}} - S$ is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}o$ set. Since K is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cs$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) \subseteq (1_{\mathcal{P}} - S)$. That is $S \subseteq (1_{\mathcal{P}} - \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K))$. Thus $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) \wedge (1_{\mathcal{P}} - \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K)) = 0_{\mathcal{P}}$. Hence $S = 0_{\mathcal{P}}$. Therefore $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) - K$ does not contain any non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}(A))$.

Remark 3.3. The converse of the theorem 3.3 need not be true as seen from the following example.

Example 3.2. In Example 3.1, let $K = \left\{ \left\langle \frac{s_1, s_4}{0.4, 0.1} \right\rangle, \left\langle \frac{s_2}{0.1, 0.5} \right\rangle, \left\langle \frac{s_3}{0.2, 0.45} \right\rangle \right\}$ be any subset of U then $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) - K = 1_{\mathcal{P}} - K$ which is a non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}cs$ in U , but K is not a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cs$ in U .

Theorem 3.4. Let K be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}$ subset in $(U, \tau_{\mathcal{R}}(A))$ then K is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cs$ iff $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta cl(K) - K$ is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}c$ set in $(U, \tau_{\mathcal{R}}(A))$.

Proof. Let K be a $\mathcal{PYFY}\mathfrak{N}$ subset in $(U, \tau_{\mathcal{R}}(A))$. Assume that K is $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$ then we have $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) = K$, $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) - K = 0_P$ which is $\mathcal{PYFY}\mathfrak{N}cs$.

Conversely, assume that $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) - K$ be $\mathcal{PYFY}\mathfrak{N}cs$ and K is a $\mathcal{PYFY}\mathfrak{N}\delta\beta c$ set in U . Now $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) - K$ is a $\mathcal{PYFY}\mathfrak{N}c$ subset of itself. Therefore by theorem 3.3, $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) - K = 0_P$. That is $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) = K$, implies K is $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$.

Theorem 3.5. If K is $\mathcal{PYFY}\mathfrak{N}\delta\beta c$ set in $(U, \tau_{\mathcal{R}}(A))$ and $K \subseteq L \subseteq \mathcal{PYFY}\mathfrak{N}\delta\beta cl(K)$, then L is also $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$ in $(U, \tau_{\mathcal{R}}(A))$.

Proof. Let K be a $\mathcal{PYFY}\mathfrak{N}\delta\beta c$ set in U and $K \subseteq L \subseteq \mathcal{PYFY}\mathfrak{N}\delta\beta cl(K)$. Let $L \subseteq O$ where O be $\mathcal{PYFY}\mathfrak{N}c$ set in U . Since $K \subseteq L$, implies $K \subseteq O$ and K is $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$, implies $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) \subseteq O$. By hypothesis $L \subseteq \mathcal{PYFY}\mathfrak{N}\delta\beta cl(K)$, implies $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(L) \subseteq \mathcal{PYFY}\mathfrak{N}\delta\beta cl(\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K)) = \mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) \subseteq O$, which implies $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(L) \subseteq O$. Therefore L is $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$ in U .

Theorem 3.6. If a subset K of U is $\mathcal{PYFY}\mathfrak{N}\delta\beta c$ set, then $\mathcal{PYFY}\mathfrak{N}cl(\{x_r\}) \cap K \neq 0_P$ for each $x_r \in \mathcal{PYFY}\mathfrak{N}\delta\beta cl(K)$.

Proof. Suppose K is a $\mathcal{PYFY}\mathfrak{N}\delta\beta c$ set and $x_r \in \mathcal{PYFY}\mathfrak{N}\delta\beta cl(K)$. If possible $\mathcal{PYFY}\mathfrak{N}cl(\{x_r\}) \cap K = 0_P$. Then $K \subseteq 1_P - \mathcal{PYFY}\mathfrak{N}cl(\{x_r\})$ and $1_P - \mathcal{PYFY}\mathfrak{N}cl(\{x_r\})$ is a $\mathcal{PYFY}\mathfrak{N}o$ set containing K . Since K is $\mathcal{PYFY}\mathfrak{N}\delta\beta c$ set, implies $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) \subseteq 1_P - \mathcal{PYFY}\mathfrak{N}cl(\{x_r\})$ which is a contradiction to $x_r \in \mathcal{PYFY}\mathfrak{N}\delta\beta cl(K)$. Therefore, $\mathcal{PYFY}\mathfrak{N}cl(\{x_r\}) \cap K \neq 0_P$.

Theorem 3.7. If $\mathcal{PYFY}\mathfrak{N}\delta\beta O(U, A) = \mathcal{PYFY}\mathfrak{N}\delta\beta C(U, A)$, then $\mathcal{PYFY}\mathfrak{N}\delta\beta C(U, A) = P(U)$ is the power set of U .

Proof. Suppose $K \subseteq O$, where O is $\mathcal{PYFY}\mathfrak{N}\delta o$ in U . Since every $\mathcal{PYFY}\mathfrak{N}\delta o$ set is $\mathcal{PYFY}\mathfrak{N}\delta\beta o$, O is $\mathcal{PYFY}\mathfrak{N}\delta\beta o$. By hypothesis, O is $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$. Hence, $\mathcal{PYFY}\mathfrak{N}\delta\beta cl(K) \subseteq O$. Therefore, K is a $\mathcal{PYFY}\mathfrak{N}\delta\beta c$ set. Since, K is arbitrary, by Theorem 3.6, every subset of U is $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$. Thus $\mathcal{PYFY}\mathfrak{N}\delta\beta C(U, A) = P(U)$.

Remark 3.4. The Theorems 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 are also true for $\mathcal{PYFY}\mathfrak{N}\delta Pos(U)$, $\mathcal{PYFY}\mathfrak{N}\delta Sos(U)$ and $\mathcal{PYFY}\mathfrak{N}\delta\alpha o(U)$.

Proposition 3.1. If S is a $\mathcal{PYFY}\mathfrak{N}\delta o$ and T is a $\mathcal{PYFY}\mathfrak{N}\delta\beta o$, then $S \cap T$ is a $\mathcal{PYFY}\mathfrak{N}\delta\beta o$.

Proof. $S \cap T \subseteq S \cap \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}\delta cl(T))) \subseteq \mathcal{PYFY}\mathfrak{N}cl(S \cap \mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}\delta cl(T))) \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}\delta cl(S \cap T)))$. Therefore $S \cap T$ is a $\mathcal{PYFY}\mathfrak{N}\delta\beta o$.

Remark 3.5. The Proposition 3.1 is also true if T is a $\mathcal{PYFY}\mathfrak{N}\delta Pos$, $\mathcal{PYFY}\mathfrak{N}\delta Sos$ and $\mathcal{PYFY}\mathfrak{N}\delta\alpha o$.

Proposition 3.2. If S is a $\mathcal{PYFY}\mathfrak{N}\delta Pos$ and T is a $\mathcal{PYFY}\mathfrak{N}\delta\alpha o$. Then $S \cap T$ is a $\mathcal{PYFY}\mathfrak{N}\delta Pos$.

Proof. $S \cap T \subseteq \mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}\delta cl(S)) \cap \mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta int(T))) \subseteq \mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}\delta cl(S))) \cap \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta int(T)) \subseteq \mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta cl(S))) \cap \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta int(T)) \subseteq \mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta cl(S \cap T))) = \mathcal{PYFY}\mathfrak{N}int(\mathcal{PYFY}\mathfrak{N}\delta cl(S \cap T))$. Therefore, $S \cap T$ is a $\mathcal{PYFY}\mathfrak{N}\delta Pos$.

Corollary 3.1. If S is a $\mathcal{PYFY}\mathfrak{N}\delta Pcs$ and T is a $\mathcal{PYFY}\mathfrak{N}\delta\alpha cs$. Then $S \cup T$ is a $\mathcal{PYFY}\mathfrak{N}\delta Pcs$.

Proposition 3.3. If S is a pfs of U and T is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{P}os$ of U such that $T \subseteq S \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(T))$. Then S is a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$.

Proof. Since T is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{P}os$, $T \subseteq \mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T))$. Now $S \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(T)) \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T)))) = \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(S)))$. Hence $S \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(S)))$. Therefore S is a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$.

Proposition 3.4. If each T is a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ which is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cs$ is also a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$.

Proof. Let T be a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ and $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cs$. Then, $T \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T)))$ and $\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T)) \subseteq T$. Therefore, $\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T)) \subseteq T$ and so, $\mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T))) \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(T))$. Hence, $T \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T))) \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(T))$. Therefore, T is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$.

Proposition 3.5. If T is a $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$ and $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$. Then T is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cs$.

Proof. Since T is a $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$ and $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$. Then, $1_P - T$ is a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ and $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cs$ and so by Proposition 3.4, $1_P - T$ is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$. Therefore T is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cs$.

Proposition 3.6. If each T is a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ which is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{a}cs$ is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{c}cs$.

Proof. Let T be a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ and $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{a}cs$. Then, $T \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T)))$ and $\mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T))) \subseteq T$. Therefore, $\mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T))) \subseteq T \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T)))$. So, $T = \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\mathit{cl}(T)))$. Therefore, T is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{c}cs$.

Corollary 3.2. If each S is a $\mathcal{PYFY}\mathfrak{N}\delta\beta cs$, which is $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{a}os$, is also a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{o}os$.

Theorem 3.8. Let $(U, \tau_R(A))$ be a $\mathcal{PYFY}\mathfrak{N}\mathit{ts}$, S be a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$ and T be a $\mathcal{PYFY}\mathfrak{N}\mathit{s}$ such that $T \subseteq S \subseteq \mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cl(T)$. Then T is also a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$.

Proof. $S \subseteq \mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cl(T)$, implies that $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cl(S) \subseteq \mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}cl(T)$ and so $S \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(S)) \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(T))$ and so $T \subseteq \mathcal{PYFY}\mathfrak{N}cl(\mathcal{PYFY}\mathfrak{N}\delta\mathit{int}(T))$. Therefore T is a $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{S}os$.

Remark 3.6. Theorem 3.8 is also true for $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{P}os$, $\mathcal{PYFY}\mathfrak{N}\delta\mathcal{a}os$ and $\mathcal{PYFY}\mathfrak{N}\delta\beta os$.

Theorem 3.9. Let $(U, \tau_R(A))$ be a $\mathcal{PYFY}\mathfrak{N}\mathit{ts}$ and let S and T be pfs sets. Then the following hold.

- (i) $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(0_P) = 0_P$ and $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(1_P) = 1_P$.
- (ii) S is a $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ iff $S = \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S)$.
- (iii) $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S)$ is the greatest $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ containing S .
- (iv) $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S)) = \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S)$.
- (v) $S \subseteq T$ implies that $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S) \subseteq \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(T)$.
- (vi) $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S \cap T) = \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S) \cap \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(T)$.
- (vii) $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S \cup T) \supseteq \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S) \cup \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(T)$.
- (viii) $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S) \subseteq S$.
- (ix) $(1_P - \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S)) = \mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{cl}(1_P - S)$.
- (x) $\mathcal{PYFY}\mathfrak{N}\delta\beta\mathit{int}(S)$ is the greatest $\mathcal{PYFY}\mathfrak{N}\delta\beta os$ contained in S .

Proof.

- (i) The proofs are directly from Definition 3.2.

- (ii) Suppose S is any $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ of U . Then, the greatest $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ containing S is itself. Therefore $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) = S$.
- (iii) If G is any $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ contained in S , then $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)$. Hence, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)$ is the greatest $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ containing S . Similarly we can prove (x).
- (iv) By (iii) the greatest $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ containing $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)$ is itself. Hence, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)) = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)$.
- (v) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(T) = \cup\{G|G \text{ is a } \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os and } G \subseteq T\} \supseteq \cup\{G|G \text{ is a } \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os and } G \subseteq S\} = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)$. Then $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(T)$.
- (vi) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S \cap T) = \cup\{C|C \text{ is an } \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os and } C \subseteq S \cap T\} = (\cup\{C|C \text{ is a } \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os and } C \subseteq S\}) \cap (\cup\{C|C \text{ is a } \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os and } C \subseteq T\}) = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \cap \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(T)$.
- (vii) $S \subseteq S \cup T$ or $T \subseteq S \cup T$. Hence $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S \cup T)$ or $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(T) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S \cup T)$. Therefore, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S \cup T) \supseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \cup \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(T)$.
- (viii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) = \cup\{G|G \text{ is an } \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os and } G \subseteq S\}$. Thus, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \subseteq S$.
- (ix) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)$ is the greatest $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ containing S . The complement is the smallest $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cs}$ contained in $1_P - S$. Therefore, $1_P - \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(1_P - S)$.

Theorem 3.10. Let $(U, \tau_R(A))$ be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{ts}$ and let S and T be pfs sets. Then the following hold.

- (i) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(0_P) = 0_P$ and $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(1_P) = 1_P$.
- (ii) S is an $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cs}$ iff $S = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$.
- (iii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$ is the smallest $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cs}$ contained in S .
- (iv) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)) = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$.
- (v) $S \subseteq T$ implies that $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(T)$.
- (vi) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S \cap T) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S) \cap \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(T)$.
- (vii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S \cup T) = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S) \cup \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(T)$.
- (viii) $S \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$.
- (ix) $(1_P - \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)) = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(1_P - S)$.
- (x) $x \in \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$ iff $S \cap T \neq 0_P$ for every $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ T containing x .

Proof. (x) Suppose $x \in \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$. Let T be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ containing x . If $S \cap T = 0_P$, then $1_P - T$ is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cs}$ containing S and so $x \notin \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$, a contradiction. Therefore, $S \cap T \neq 0_P$. If $x \notin \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$, then there exists a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cs}$ C containing S such that $x \notin C$. Then $D = 1_P - C$ is a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{os}$ containing x such that $S \cap D = 0_P$, a contradiction. Therefore, $x \in \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)$. The other cases are follows from Theorem 3.9.

Theorem 3.11. Let $(U, \tau_R(A))$ be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{ts}$ and S be a pfs set. Then the following hold.

- (i) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \subseteq S \cap \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S))$.
- (ii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S) \supseteq S \cup \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S))$.
- (iii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \subseteq S \cap \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S))$.
- (iv) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S) \supseteq S \cup \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S))$.
- (v) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{aint}(S) \subseteq S \cap \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)))$.
- (vi) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{acl}(S) \supseteq S \cup \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)))$.
- (vii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S) \subseteq S \cap \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S)))$.
- (viii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(S) \supseteq S \cup \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{cl}(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathfrak{N}\delta\beta\text{int}(S)))$.

Proof. (ii) Since $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{P}cl(S)$ is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{P}cs$, we have $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S)) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{P}cl(S))) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{P}cl(S)$. Thus $S \cup \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}cl(\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(S)) \subseteq \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{P}cl(S)$. The other cases are similar.

Theorem 3.12. Let $(U, \tau_R(A))$ be a $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}ts$ and let S be a pfs set. Then the following are equivalent.

- (i) S is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta$ dense.
- (ii) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cl(S) = 1_P$.
- (iii) If T is any $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cs \ni S \subset T$, then $T = 1_P$.
- (iv) Every non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta os$ has a non-empty intersection with S .
- (v) $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta int(1_P - S) = 0_P$.

Proof. (i) \Rightarrow (ii): Suppose $x \notin \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cl(S)$. Then there exists an $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta os$ G containing $x \ni G \cap S \neq 0_P$. Since G is a non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta os$, there is a non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}os$ $H \ni H \subseteq G$ and so $H \cap S = 0_P$, a contradiction. Therefore, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta cl(S) = U$.

(ii) \Rightarrow (iii): If T is any $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cs \ni S \subset T$, then $1_P = \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cl(S) \subset \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cl(T) = T$ which implies that $T = U$.

(iii) \Rightarrow (iv): If G is a non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta os \ni G \cap S = 0_P$, then $S \subset 1_P - G$ and $1_P - G$ is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cs$. By (iii), it follows that $G = 0_P$, a contradiction.

(iv) \Rightarrow (v): Suppose that $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta int(1_P - S) \neq 0_P$. Then $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta int(1_P - S)$ is a non-empty $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta os \ni \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta int(1_P - S) \cap S \neq 0_P$, a contradiction to the hypothesis.

(v) \Rightarrow (i): $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta int(1_P - S) \neq 0_P$ implies that $1_P - \mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta int(1_P - S) = 1_P$. Thus $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cl(S) = 1_P$. Hence $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta cl(S) = 1_P$ which shows that S is $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\beta$ -dense.

Remark 3.7. The Theorems 3.9, 3.10 and 3.12 are also true for $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta os$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\mathcal{P} os$, $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta S os$ and $\mathcal{P}\mathcal{Y}\mathcal{F}\mathcal{Y}\mathcal{N}\delta\alpha os$ of their respective interior and closure operators.

4. APPLICATION

Entropy, as a measure of fuzziness, was first introduced by Zadeh [19] in the context of fuzzy set theory. Zadeh's pioneering work provided a quantitative way to capture the uncertainty or vagueness inherent in fuzzy sets, allowing for a more rigorous analysis of imprecise information. Since then, numerous mathematicians and researchers have proposed various entropy measures, each designed to capture different aspects of uncertainty, such as the degree of dispersion, ambiguity, or conflict within a set. These measures have been widely applied in fields ranging from decision-making and pattern recognition to information theory and engineering.

In this section, we focus on defining an entropy measure specifically for Pythagorean fuzzy sets (pfs). Unlike classical fuzzy sets, Pythagorean fuzzy sets are characterized by both a degree of membership and a degree of non-membership, with the constraint that the sum of the squares of these degrees does not exceed one. Therefore, any entropy measure for pfs must account for both membership and non-membership simultaneously, reflecting the overall fuzziness of the information more accurately. By establishing a connection between the degrees of membership and non-membership, the proposed entropy measure provides a comprehensive way to quantify uncertainty in Pythagorean fuzzy sets, making it particularly useful in situations where information is incomplete, imprecise, or contradictory.

To illustrate the applicability of the proposed entropy measure, we have applied it to a practical example in the field of seasons. In this example, seasonal characteristics are

expressed using Pythagorean fuzzy sets, capturing the inherent uncertainty in factors such as temperature, rainfall, and climate variability. The entropy measure then allows us to quantify the fuzziness of these seasonal descriptions, demonstrating the effectiveness of the approach in modeling and analyzing real-world uncertain information. This example highlights how the proposed entropy measure can provide meaningful insights in domains where precise information is difficult to obtain, thereby supporting more informed and robust decision-making.

Definition 4.1. Let $A = \{ \langle x, \mu_A(x), \lambda_A(x) \mid x \in X \rangle \}$ be a *pf*s in X . The new entropy measure for A denoted by $\varepsilon_{pfs}(A)$, is a function, $\varepsilon_{pfs} : \tau_{pfs}(X) \rightarrow [0, 1]$ and is defined as $\varepsilon_{pfs}(A) = 1 - \frac{1}{n} \sum_{i=1}^n (\alpha_A(x_i) - \gamma_A(x_i))^2 (1 - \pi_A(x_i))$; for every $x_i \in A$, where $\tau_{pfs}(X)$ denote the family of all *pf*s's on X .

Example 4.1. With the rapid development of the economy and industrialization, water pollution has become an inevitable consequence of human activity, posing a significant threat to human health and the environment. Contaminated water not only affects aquatic life but also disrupts ecosystems and reduces the availability of safe drinking water for communities. Consequently, effective water pollution treatment has emerged as a critical social and environmental issue, demanding attention from policymakers, industries, and environmental scientists alike.

In particular, the Cauvery River, which is the largest and most significant river in Tamil Nadu, has experienced increasing levels of pollution over the past few decades. This pollution is primarily caused by effluents discharged from dyeing factories and untreated or partially treated industrial wastewater. As the population and industrial activities in the region continue to grow, the severity of the rivers contamination is worsening year by year, leading to adverse ecological and human health impacts. For instance, excess chemical pollutants and heavy metals in water can lead to the bioaccumulation of toxins in aquatic species, which eventually enter the human food chain.

Efforts have been made to reduce industrial wastewater discharge in key districts such as Tirupur, Bhavani, Erode, and Karur, and many factories have gradually brought their effluents closer to legally mandated emission standards. Alongside regulatory enforcement, significant improvements have been made in the facilities and scale of wastewater treatment systems. Modern treatment plants now incorporate advanced filtration, sedimentation, and chemical treatment methods, making the treatment processes more efficient and environmentally friendly. Nevertheless, in order to maintain the stability and effectiveness of wastewater treatment, it is essential for factories to periodically update and upgrade their treatment facilities, adopting the latest technologies and best practices.

Recently, a dyeing factory has faced the challenge of renewing a batch of wastewater treatment facilities. This decision is complex, as it requires careful consideration of multiple factors that affect both operational efficiency and economic feasibility. Five alternative treatment options A_1, A_2, A_3, A_4 , and A_5 have been identified as potential solutions. These alternatives are evaluated based on four critical criteria: C_1, C_2, C_3 , and C_4 , which represent expense, installation time, renewal period or durability, and manpower requirements, respectively.

Given the complexity and uncertainty inherent in such decision-making processes, traditional crisp evaluation methods may fail to capture the vagueness and ambiguity in expert judgments. Pythagorean fuzzy sets offer a powerful tool to model this uncertainty, allowing decision-makers to express their preferences with varying degrees of membership and non-membership. This approach provides a more flexible and realistic representation of

uncertain information, particularly when precise numerical data is unavailable. Additionally, information with lower fuzziness is generally considered more favorable, as it indicates higher reliability and clarity in decision-making. To quantify the level of uncertainty in each alternative, entropy measures are employed, providing a systematic way to evaluate and compare the fuzziness of different treatment options.

Table 1. Pythagorean fuzzy sets of each alternates

	Criteria 1 (C_1)	Criteria 2 (C_2)	Criteria 3 (C_3)	Criteria 4 (C_4)
A_1	$\langle A_1, C_1; 0.8, 0.4 \rangle$	$\langle A_1, C_2; 0.3, 0.8 \rangle$	$\langle A_1, C_3; 0.6, 0.3 \rangle$	$\langle A_1, C_4; 0.6, 0.4 \rangle$
A_2	$\langle A_2, C_1; 0.4, 0.7 \rangle$	$\langle A_2, C_2; 0.3, 0.8 \rangle$	$\langle A_2, C_3; 0.5, 0.6 \rangle$	$\langle A_2, C_4; 0.6, 0.5 \rangle$
A_3	$\langle A_3, C_1; 0.6, 0.5 \rangle$	$\langle A_3, C_2; 0.7, 0.4 \rangle$	$\langle A_3, C_3; 0.3, 0.8 \rangle$	$\langle A_3, C_4; 0.3, 0.5 \rangle$
A_4	$\langle A_4, C_1; 0.7, 0.4 \rangle$	$\langle A_4, C_2; 0.4, 0.7 \rangle$	$\langle A_4, C_3; 0.5, 0.7 \rangle$	$\langle A_4, C_4; 0.6, 0.4 \rangle$
A_5	$\langle A_5, C_1; 0.9, 0.2 \rangle$	$\langle A_5, C_2; 0.4, 0.8 \rangle$	$\langle A_5, C_3; 0.6, 0.3 \rangle$	$\langle A_5, C_4; 0.7, 0.2 \rangle$

Clearly, all values in the Table 1 are pfs 's. Now we calculate the ε_{pfs} of each value.

Table 2. Entropy measure of each item through criteria.

	Criteria 1 (C_1)
A_1	0.939
A_2	0.959
A_3	0.958
A_4	0.974
A_5	0.877

From Table 2, it can be observed that the entropy values of the Pythagorean fuzzy sets corresponding to the different water treatment alternatives follow the order: $\varepsilon_{pfs}(A_5) < \varepsilon_{pfs}(A_1) < \varepsilon_{pfs}(A_3) < \varepsilon_{pfs}(A_2) < \varepsilon_{pfs}(A_4)$. This ranking indicates that among the five alternatives, the water treatment option A_5 exhibits the lowest entropy value. In the context of Pythagorean fuzzy sets, a lower entropy value corresponds to lower fuzziness, meaning that the information related to A_5 is more certain and less ambiguous compared to the other alternatives. Conversely, alternatives with higher entropy values, such as A_4 and A_2 , possess greater fuzziness, reflecting higher uncertainty in their evaluation across the criteria.

Since minimizing fuzziness is a desirable property when selecting an efficient treatment method, the alternative A_5 can be considered the most efficient and reliable option among the five evaluated treatments. Its lower entropy implies that decision-makers can have higher confidence in the performance, cost-effectiveness, and overall suitability of A_5 relative to the other alternatives. Therefore, based on the entropy analysis, A_5 emerges as the preferred water treatment solution, providing a balance between operational effectiveness and reduced uncertainty in decision-making.

5. CONCLUSION

In this paper, we have studied some new notions of strongly and weakly open (respective closed) sets called $\mathcal{PYFY}\mathcal{N}\delta o$, $\mathcal{PYFY}\mathcal{N}\delta c$, $\mathcal{PYFY}\mathcal{N}\delta S o$, $\mathcal{PYFY}\mathcal{N}\delta S c$, $\mathcal{PYFY}\mathcal{N}\delta P o$, $\mathcal{PYFY}\mathcal{N}\delta P c$, $\mathcal{PYFY}\mathcal{N}\delta \alpha o$, $\mathcal{PYFY}\mathcal{N}\delta \alpha c$, $\mathcal{PYFY}\mathcal{N}\delta \beta o$ and $\mathcal{PYFY}\mathcal{N}\delta \beta c$ sets and their respective interior and closure operators. Also studied some of their fundamental properties in $\mathcal{PYFY}\mathcal{N}ts$. This can be extended to $\mathcal{PYFY}\mathcal{N}\delta$ continuous respective open and closed mappings and also their stronger and weaker forms in $\mathcal{PYFY}\mathcal{N}ts$. Also we applied

entropy measure for decision making problem of selecting the optimum wastewater treatment method for the dying factories based on the required criteria. In future, we decide to apply entropy measure for decision making in various fields.

REFERENCES

- [1] Ajay, D., Joseline Charisma, J., (2020), Pythagorean nano topological space, International Journal of Recent Technology and Engineering, 8, pp. 3415-3419.
- [2] Ajay, D., Joseline Charisma, J., (2020), On weak forms of Pythagorean nano open sets, Advances in Mathematics: Scientific Journal, 9, pp. 5953-5963.
- [3] Atanassov, K. T., (1983), Intuitionistic fuzzy sets, VII ITKR's Session, Sofia (deposed in Central Sci.-Technical Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).
- [4] Chang, C. L., (1968), Fuzzy topological spaces, J. Math. Anal. Appl., 24, pp. 182-190.
- [5] Coker, D., (1997), An introduction to intuitionistic fuzzy topological spaces, Fuzzy sets and systems, 88, pp. 81-89.
- [6] Lellis Thivagar, M., Richard, C., (2013), On nano forms of weekly open sets, International journal of mathematics and statistics invention, 1(1), pp. 31-37.
- [7] Lellis Thivagar, M., Jafari, S., Sutha Devi, V., Antonyamy, V., (2018), A novel approach to nano topology via neutrosophic sets, Neutrosophic Sets and Systems, 20, pp. 86-94.
- [8] Olgun, M., Unver, M., Yardimci, S., (2019), Pythagorean fuzzy topological spaces, Complex & Intelligent Systems, pp. 177-183. <https://doi.org/10.1007/s40747-019-0095-2>.
- [9] Padma, A., Saraswathi, M., Vadivel, A., Saravanakumar, G., (2019), New Notions of Nano M -open Sets, Malaya Journal of Matematik, S(1), pp. 656-660.
- [10] Pankajam, V., Kavitha, K., (2017), δ -open sets and δ -nano continuity in δ -nano topological spaces, International Journal of Innovative Science and Research Technology, 2(12), pp. 110-118.
- [11] Wu X., Zhu Z., Wang T, Zhang X., (2025), Picture Fuzzy Interactional Aggregation Operators via Strict Triangular Norms and Applications to Multi-Criteria Decision Making, Appl. Comput. Math., Vol.24, No.1, 2025, pp.62-100
- [12] Pankajam, V., Kavitha, K., (2017), δ -open sets and δ -nano continuity in δ -nano topological spaces, International Journal of Innovative Science and Research Technology, 2(12), pp. 110-118.
- [13] Zhou J., Wu X., Chen C., (2025), Investigating Various Intuitionistic Fuzzy Information Measures and Their Relationships, Appl. Comput. Math., Vol.24, No.4, 2025, pp.579-607
- [14] Shiventhiradevi Sathananthan, Tamilselvan, S., Vadivel, A., Saravanakumar, G., (2020), Fuzzy \mathcal{Z} closed sets in double fuzzy topological spaces, AIP Conf Proc., 2277, pp. 090001.
- [15] Shiventhiradevi Sathananthan, Vadivel, A., Tamilselvan, S., Saravanakumar, G., (2020), Generalized fuzzy \mathcal{Z} closed sets in double fuzzy topological spaces, Adv. Math: Sci. J., 9(4), pp. 2107-2112.
- [16] Vadivel, A., John Sundar, C., Kirubadevi, K., Tamilselvan, S., (2022), More on Neutrosophic Nano Open Sets, International Journal of Neutrosophic Science (IJNS), 18(4), pp. 204-222.
- [17] Yager, R. R., (2013), Pythagorean fuzzy subsets, In: Proceedings of the joint *IFSA* world congress *NAFIPS* annual meeting, 57-61.
- [18] Yager, R. R., (2014), Pythagorean membership grades in multicriteria decision making, *IEEE Trans Fuzzy Syst.* 22 (4), 958-965.
- [19] Zadeh, L. A., (1965), Fuzzy sets, Information and control, 8, pp. 338-353.



Mohanarao Navuluri obtained M.Phil and Ph.D at Annamalai University, Annamalai Nagar. Under his guidance 2 scholars obtained their doctoral degrees. He has published 50 research articles both in national and international journals. He serves as referee for 2 peer reviewed international journals. His research interests include general topology, ideal topology and fuzzy topology.



K. Shantha Lakshmi obtained M.Sc.,(Mathematics) and M.Phil.,(Mathematics) degrees from affiliated colleges of Bharathidasan University, Tiruchirappalli, India, also got M.Sc.,(Yoga) degree from Annamalai University, India. Pursuing Ph.D.,(Part-Time) in Annamalai University, India, under the guidance of Dr.Mohanarao navuluri. Her research interests are general topology and fuzzy topology, specific in Pythagorean fuzzy set and Pythagorean fuzzy nano topological spaces.



A. Vadivel obtained his early and collegiate education respectively from Kandaswami Kandar's Boys School and Kandaswami Kandar's College, P-Velur, Namakkal. He obtained Ph.D at Annamalai University, Annamalai Nagar. Under his guidance 8 scholars obtained their doctoral degrees. He has published 250 research articles both in national and international journals. He serves as referee for 5 peer reviewed international journals. His research interests include general topology and fuzzy topology.



V. Sivakumar obtained Ph.D. at Periyar University, Salem-11. Under his guidance 2 scholars obtained their doctoral degrees. He has published 25 research articles both in national and international journals. He serves as referee for 3 peer reviewed international journals. His research interests include Fluid Dynamics and Fuzzy topology.