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UNIQUENESS OF CERTAIN FIXED POINT THEOREMS ON MULTIPLICATIVE METRIC SPACE WITH APPLICATION

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Abstract. In this paper, we establish few fixed point results in multiplicative metric space and prove the existence of fixed points along with its uniqueness, by employing contraction mappings in complete multiplicative metric space. We have provided few examples to validate our obtained results. Furthermore, we prove some theorems involving solutions of nonlinear integral equation as an application of our fixed point theorems.

Keywords: Contraction mapping, Complete multiplicative metric space, Fixed point, Weakly commuting mappings.

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1. Introduction

The Theory of fixed point is essential for several fields of study comprising topology, mathematics of fractals, optimal control, and logic programming. Fixed point theory is a mathematical discipline focused on characterizing all self-maps within a given context in which at least one element remains unchanged. Specifically, fixed point techniques have proven helpful in research, related to the theories of integral equations, differential equations, and mathematical programming. Additionally, these techniques have proven valuable in a variety of other domains, including biology, chemistry, and economics. Metric space is typically a non-empty set including distance function. A standard result referred to as 'Banach Contraction Principle', that is prevalently recognized as the foundation of "Metric Fixed Point Theory", was pioneered by Stefan Banach in 1922. Then, numerous fixed point theorems were established utilizing the mappings concerning the contraction principle in several classes of metric spaces.

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After defining the notion of metric space, many authors have developed a number of generalizations of metric spaces in abundance, by removing or relaxing some conditions, modifying the metric function or abstracting the notion. As a result, numerous extensions of metric spaces were constructed. A Non-Newtonian calculus called "Multiplicative Calculus" (also known as exponential calculus) formulated by Michael Grossman and Robert Katz [16], is a family of non-linear calculi, which doesn't have additive operators. Multiplicative calculus is characterized by two operations called the "multiplicative derivative" and "multiplicative integral", which maintain an inverse connection similar to the relationship between differentiation and integration in classical calculus, as established by Newton and Leibniz. Bashirov et al. asserted that multiplicative calculus proves to be more efficient than Newtonian calculus in solving various problems across diverse disciplines. Also an application of Multiplicative Calculus in the field of Biomedical Image analysis is obtained [14]. By defining multiplicative distance in 2008, Bashirov et al. established the space called "Multiplicative Metric Space"(MMS) [5]. Also, Ozavsar and Cevikel have proved numerous topological conditions [20] in the setting of MMS. Fixed Point Theory is extensively categorized into a few fixed point theories in which our result falls under the metric fixed point theory, where this theory depends on Banach fixed point theorem. The key aspect of this theorem is that it ensures both the existence of fixed points along with its uniqueness, and furnishes a contractive method to obtain those fixed points. This theorem was initially formulated by Stefan Banach in 1922. Now we recall some preliminaries that support our key results.

2. Preliminaries

Definition 2.1. [5] *A mapping $\rho_m : M \times M \to [1, \infty)$ on non-empty set M is called multiplicative metric with the provided below conditions:*

(1) $\rho_m(\mu, \xi) \geq 1$, *for all* $\mu, \xi \in M$.
(2) $\rho_m(\mu, \xi) = \rho_m(\xi, \mu)$, *for all* $\mu, \xi \in M$.
(3) $\rho_m(\mu, \xi) \leq \rho_m(\mu, \nu).\rho_m(\nu, \xi)$ *for all* $\mu, \xi, \nu \in M$.

Hence the ordered pair (M, ρ_m) is called MMS.

Example 2.1. *Let $\rho_m : \mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined as $\rho_m(\mu, \xi) = 6^{min\{1,|\mu-\xi|\}}$, where $\mu, \xi \in \mathbb{R}$. Then ρ_m satisfies multiplicative metric conditions and so, the pair $(\mathbb{R}, \rho_m)$ is called MMS.*

Definition 2.2. [20] *Let (M, ρ_m) be MMS. A mapping $Q : M \to M$ is called a multiplicative contraction mapping if*

$$\rho_m(Qx_1, Qx_2) \leq [\rho_m(x_1, x_2)]^\lambda, \text{where } \lambda \in [0, 1),$$

for all $x_1, x_2 \in M$.

Lemma 2.1. [20] *A sequence $\{\kappa_n\}$ in MMS (M, ρ_m), is said to be a multiplicative convergent to an element $\xi \in M$, iff*

$$\lim_{n \longrightarrow \infty} \rho_m(\kappa_n, \xi) = 1.$$

Lemma 2.2. [20] *A sequence $\{\kappa_n\}$ in MMS (M, ρ_m) is called multiplicative Cauchy iff*

$$\lim_{n,m \longrightarrow \infty} \rho_m(\kappa_m, \kappa_n) = 1.$$

Lemma 2.3. [20] *Suppose each multiplicative Cauchy sequence in M is multiplicative convergent to an element $x \in M$, then we say (M, ρ_m) is complete MMS.*

We demonstrate results that establish a Unique Fixed Point (UFP) using self mapping and Unique Common Fixed Point (UCFP) using weakly commuting maps in a complete MMS.
Our obtained results incorporates a number of corollaries, supported by the theorems. Furthermore, few examples were attained at the end of theorem, for the validation of our results.

3. Main results

In this section, for the given complete MMS, we demonstrate the existence concerning fixed point together with its uniqueness using self mapping, by employing the multiplicative contraction condition.

Theorem 3.1. *Suppose (M, ρ_m) be a complete MMS & if $Z : M \to M$ is continuous function such that*

$$\rho_m(Z\kappa, Z\xi) \leq \left[\left(\frac{\rho_m(\kappa, Z\kappa).\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa).\rho_m(\xi, Z\xi)}{\rho_m(\kappa, \xi)}\right)\right]^\alpha .[\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)]^\beta.[\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa)]^\gamma.[\rho_m(\kappa, \xi)]^\delta$$

$\forall \kappa, \xi \in M$, where $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $3\alpha+2\beta+2\gamma < 1-\delta$ and $\alpha+2\gamma+\delta < 1$, then there exists a UFP κ_0 for self mapping Z in the given complete MMS (M, ρ_m).

Proof. Let $\{\kappa_n\}$ be a sequence in M defined by $\kappa_n = Z\kappa_{n-1}$
Here $\kappa_1 = Z\kappa_0$
then $\kappa_2 = Z\kappa_1 = Z^2\kappa_0$
$\kappa_3 = Z\kappa_2 = Z^3\kappa_0$
In general, $\kappa_n = Z\kappa_{n-1} = Z^n\kappa_0$, then $\{\kappa_n\}$ is an iterative sequence.

$$\begin{aligned}
\rho_m(\kappa_n, \kappa_{n+1}) =& \rho_m(Z\kappa_{n-1}, Z\kappa_n) \\
\leq& \left[\left(\frac{\rho_m(\kappa_{n-1}, Z\kappa_{n-1}).\rho_m(\kappa_{n-1}, Z\kappa_n).\rho_m(\kappa_n, Z\kappa_{n-1}).\rho_m(\kappa_n, Z\kappa_n)}{\rho_m(\kappa_{n-1}, \kappa_n)}\right)\right]^\alpha \\
& .[\rho_m(\kappa_{n-1}, Z\kappa_{n-1}).\rho_m(\kappa_n, Z\kappa_n)]^\beta.[\rho_m(\kappa_{n-1}, Z\kappa_n).\rho_m(\kappa_n, Z\kappa_{n-1})]^\gamma \\
& .[\rho_m(\kappa_{n-1}, \kappa_n)]^\delta \\
=& \left[\left(\frac{\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_{n-1}, \kappa_{n+1}).\rho_m(\kappa_n, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})}{\rho_m(\kappa_{n-1}, \kappa_n)}\right)\right]^\alpha \\
& .[\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})]^\beta.[\rho_m(\kappa_{n-1}, \kappa_{n+1}).\rho_m(\kappa_n, \kappa_n)]^\gamma \\
& .[\rho_m(\kappa_{n-1}, \kappa_n)]^\delta \\
\leq& [\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1}).\rho_m(\kappa_n, \kappa_{n+1})]^\alpha \\
& .[\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})]^\beta \\
& .[\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})]^\gamma.[\rho_m(\kappa_{n-1}, \kappa_n)]^\delta \\
\rho_m(\kappa_n, \kappa_{n+1}) \leq& [\rho_m(\kappa_n, \kappa_{n+1})]^{(2\alpha+\beta+\gamma)}.[\rho_m(\kappa_{n-1}, \kappa_n)].^{(\alpha+\beta+\gamma+\delta)} \\
\rho_m(\kappa_n, \kappa_{n+1}) \leq& [\rho_m(\kappa_{n-1}, \kappa_n)]^{\left(\frac{\alpha+\beta+\gamma+\delta}{1-(2\alpha+\beta+\gamma)}\right)}.
\end{aligned}$$

$\rho_m(\kappa_n, \kappa_{n+1}) \leq [\rho_m(\kappa_{n-1}, \kappa_n)]^r$, where $r = \frac{\alpha+\beta+\gamma+\delta}{1-(2\alpha+\beta+\gamma)} < 1$.
(i.e.,) $3\alpha + 2\beta + 2\gamma < 1 - \delta$
In general, we can write $\rho_m(\kappa_n, \kappa_{n+1}) \leq [\rho_m(\kappa_0, \kappa_1)]^{r^n}$.

To prove : $\{\kappa_n\}$ is multiplicative Cauchy sequence .
For $n < m$, $\rho_m(\kappa_n, \kappa_m) \leq \rho_m(\kappa_n, \kappa_{n+1}).\rho_m(\kappa_{n+1}, \kappa_{n+2})....\rho_m(\kappa_{m-1}, \kappa_m)$.
Therefore, $\rho_m(\kappa_n, \kappa_m) \leq \prod_{i=n}^{m-1} \rho_m(\kappa_i, \kappa_{i+1}) \leq \prod_{i=n}^{m-1} [\rho_m(\kappa_0, \kappa_1)]^{r^i}$
we know $\prod_{i=n}^{m-1} [\rho_m(\kappa_0, \kappa_1)]^{r^i} = [\rho_m(\kappa_0, \kappa_1)]^{\sum_{i=n}^{m-1} r^i}$
Since $\sum_{i=n}^{m-1} r^i \leq \sum_{i=n}^{\infty} r^i = \frac{r^n}{1-r}$, we get

$$\rho_m(\kappa_n, \kappa_m) \leq [\rho_m(\kappa_0, \kappa_1)]^{\frac{r^n}{1-r}}$$

Here $\frac{r^n}{1-r} \to 0$ as $n \to \infty$.
So, $\lim_{m,n\to\infty} \rho_m(\kappa_n, \kappa_m) \leq 1$.
Since ρ_m is multiplicative metric, therefore $\lim_{m,n\to\infty} \rho_m(\kappa_n, \kappa_m) = 1$.
Thus ρ_m is multiplicative Cauchy sequence and $\lim_{n\to\infty} \rho_m(\kappa_n, \kappa_0) = 1$.
To prove: κ_0 is fixed point of Z.
Using multiplicative continuity,
$\kappa_0 = \lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} (Z\kappa_{n-1}) = Z\left(\lim_{n\to\infty} \kappa_{n-1}\right) = Z\kappa_0$.
Thus κ_0 is the fixed point of Z.
Now we establish the uniqueness concerning fixed point of Z.
Suppose Z has a couple of fixed points κ_0 and ξ_0.
Then $Z\kappa_0 = \kappa_0$ and $Z\xi_0 = \xi_0$.
Now

$$\begin{aligned}
\rho_m(\kappa_0, \xi_0) =& \rho_m(Z\kappa_0, Z\xi_0) \\
\leq& \left[\left(\frac{\rho_m(\kappa_0, Z\kappa_0).\rho_m(\kappa_0, Z\xi_0).\rho_m(\xi_0, Z\kappa_0).\rho_m(\xi_0, Z\xi_0)}{\rho_m(\kappa_0, \xi_0)}\right)\right]^{\alpha} \\
& .[\rho_m(\kappa_0, Z\kappa_0).\rho_m(\xi_0, Z\xi_0)]^{\beta}.[\rho_m(\kappa_0, Z\xi_0).\rho_m(\xi_0, Z\kappa_0)]^{\gamma}.[\rho_m(\kappa_0, \xi_0)]^{\delta} \\
=& \left[\left(\frac{\rho_m(\kappa_0, \kappa_0).\rho_m(\kappa_0, \xi_0).\rho_m(\xi_0, \kappa_0).\rho_m(\xi_0, \xi_0)}{\rho_m(\kappa_0, \xi_0)}\right)\right]^{\alpha}.[\rho_m(\kappa_0, \kappa_0).\rho_m(\xi_0, \xi_0)]^{\beta} \\
& .[\rho_m(\kappa_0, \xi_0).\rho_m(\xi_0, \kappa_0)]^{\gamma}.[\rho_m(\kappa_0, \xi_0)]^{\delta} \\
\rho_m(\kappa_0, \xi_0) \leq& [\rho_m(\kappa_0, \xi_0)]^{(\alpha+2\gamma+\delta)}
\end{aligned}$$

which is contradiction since $\alpha + 2\gamma + \delta < 1$.
So, $\rho_m(\kappa_0, \xi_0) = 1$ and hence $\kappa_0 = \xi_0$.
Thus the mapping Z holds a UFP κ_0 in a complete MMS (M, ρ_m).
Hence the theorem is proved. □

Corollary 3.1. *Given (M, ρ_m) is a complete MMS & if $Z : M \to M$ is continuous function such that*

$$\begin{aligned}
\rho_m(Z\kappa, Z\xi) \leq& [\rho_m(\kappa, Z\kappa).\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa).\rho_m(\xi, Z\xi)]^{\alpha} \\
& .[\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)]^{\beta}.[\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa)]^{\gamma}
\end{aligned}$$

$\forall \kappa, \xi \in M$, where the values of α, β, γ varies in $[0, 1)$ with $2(2\alpha+\beta+\gamma) < 1$ and $2(\alpha+\gamma) < 1$, then $\exists$ a UFP for self mapping Z in complete MMS (M, ρ_m) .

Theorem 3.2. *Suppose (M, ρ_m) represent a complete MMS, and assume that $Z : M \to M$ is continuous function such that*

$$\rho_m(Z\kappa, Z\xi) \leq [\rho_m(\kappa, \xi).\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)]^{\alpha}.\left[\left(\frac{\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa)}{\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)}\right)\right]^{\beta}$$
$$.[\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)]^{\gamma}.[\rho_m(\kappa, \xi)]^{\delta}$$

$\forall \kappa, \xi \in M$, where the values of $\alpha, \beta, \gamma, \delta$ varies in $[0, 1)$ with $3\alpha+2\gamma+\delta < 1$ and $\alpha+2\beta+\delta < 1$, then $\exists$ a UFP for self mapping Z.

Proof. Take a sequence $\{\kappa_n\}$ in M defined by $\kappa_n = Z\kappa_{n-1}$.

$$\begin{aligned}
\rho_m(\kappa_n, \kappa_{n+1}) &= \rho_m(Z\kappa_{n-1}, Z\kappa_n) \\
&\leq [\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_{n-1}, Z\kappa_{n-1}).\rho_m(\kappa_n, Z\kappa_n)]^{\alpha} \\
&\quad .\left[\left(\frac{\rho_m(\kappa_{n-1}, Z\kappa_n).\rho_m(\kappa_n, Z\kappa_{n-1})}{\rho_m(\kappa_{n-1}, Z\kappa_{n-1}).\rho_m(\kappa_n, Z\kappa_n)}\right)\right]^{\beta} \\
&\quad .[\rho_m(\kappa_{n-1}, Z\kappa_{n-1}).\rho_m(\kappa_n, Z\kappa_n)]^{\gamma}.[\rho_m(\kappa_{n-1}, \kappa_n)]^{\delta} \\
&= [\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})]^{\alpha} \\
&\quad .\left[\left(\frac{\rho_m(\kappa_{n-1}, \kappa_{n+1}).\rho_m(\kappa_n, \kappa_n)}{\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})}\right)\right]^{\beta} \\
&\quad .[\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})]^{\gamma}.[\rho_m(\kappa_{n-1}, \kappa_n)]^{\delta} \\
&\leq [(\rho_m(\kappa_{n-1}, \kappa_n))^2.\rho_m(\kappa_n, \kappa_{n+1})]^{\alpha} \\
&\quad .[\rho_m(\kappa_{n-1}, \kappa_n).\rho_m(\kappa_n, \kappa_{n+1})]^{\gamma}.[\rho_m(\kappa_{n-1}, \kappa_n)]^{\delta}. \\
\rho_m(\kappa_n, \kappa_{n+1}) &\leq [\rho_m(\kappa_n, \kappa_{n+1})]^{\alpha+\gamma}.[\rho_m(\kappa_{n-1}, \kappa_n)].^{2\alpha+\gamma+\delta} \\
\rho_m(\kappa_n, \kappa_{n+1}) &\leq [\rho_m(\kappa_{n-1}, \kappa_n)]^{\left(\frac{2\alpha+\gamma+\delta}{1-(\alpha+\gamma)}\right)} \\
\rho_m(\kappa_n, \kappa_{n+1}) &\leq [\rho_m(\kappa_{n-1}, \kappa_n)]^{s}
\end{aligned}$$

where $s = \frac{2\alpha+\gamma+\delta}{1-(\alpha+\gamma)} < 1$.
(i.e.,) $3\alpha + 2\gamma + \delta < 1$.
In general, we can write $\rho_m(\kappa_n, \kappa_{n+1}) \leq [\rho_m(\kappa_0, \kappa_1)]^{s^n}$.
Thereafter, we prove that $\{\kappa_n\}$ is multiplicative Cauchy sequence.
For $n < m$, $\quad \rho_m(\kappa_n, \kappa_m) \leq \rho_m(\kappa_n, \kappa_{n+1}).\rho_m(\kappa_{n+1}, \kappa_{n+2})....\rho_m(\kappa_{m-1}, \kappa_m)$.
Therefore, $\rho_m(\kappa_n, \kappa_m) \leq \prod_{i=n}^{m-1} \rho_m(\kappa_i, \kappa_{i+1}) \leq \prod_{i=n}^{m-1} [\rho_m(\kappa_0, \kappa_1)]^{s^i}$
we know $\prod_{i=n}^{m-1} [\rho_m(\kappa_0, \kappa_1)]^{s^i} = [\rho_m(\kappa_0, \kappa_1)]^{\sum_{i=n}^{m-1} s^i}$
Since $\sum_{i=n}^{m-1} s^i \leq \sum_{i=n}^{\infty} s^i = \frac{s^n}{1-s}$, we get

$$\rho_m(\kappa_n, \kappa_m) \leq [\rho_m(\kappa_0, \kappa_1)]^{\frac{s^n}{1-s}}$$

Here $\frac{s^n}{1-s} \to 0$ as $n \to \infty$.
So, $\lim_{m,n\to\infty} \rho_m(\kappa_n, \kappa_m) \leq 1$.
Since ρ_m is multiplicative metric, therefore $\lim_{m,n\to\infty} \rho_m(\kappa_n, \kappa_m) = 1$.

Thus $\{\kappa_n\}$ is multiplicative Cauchy sequence and $\lim_{n\to\infty} \rho_m(\kappa_n, \kappa_0) = 1$.
To prove: κ_0 is fixed point of Z.
Utilizing multiplicative continuity,
$\kappa_0 = \lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} (Z\kappa_{n-1}) = Z\left(\lim_{n\to\infty} \kappa_{n-1}\right) = Z\kappa_0$.
Therefore, Z possess a fixed point κ_0 for complete MMS (M, ρ_m).
Inorder to demonstrate the uniqueness of fixed point of Z,
assume that Z has two fixed points κ_i and κ_j.
Then $Z\kappa_i = \kappa_i$ and $Z\kappa_j = \kappa_j$.
Now

$$\begin{aligned}\rho_m(\kappa_i, \kappa_j) &= \rho_m(Z\kappa_i, Z\kappa_j)\\ &\leq [\rho_m(\kappa_i, \kappa_j).\rho_m(\kappa_i, Z\kappa_i).\rho_m(\kappa_j, Z\kappa_j)]^{\alpha}\\ &\quad .\left[\left(\frac{\rho_m(\kappa_i, Z\kappa_j).\rho_m(\kappa_j, Z\kappa_i)}{\rho_m(\kappa_i, Z\kappa_i).\rho_m(\kappa_j, Z\kappa_j)}\right)\right]^{\beta}\\ &\quad .[\rho_m(\kappa_i, Z\kappa_i).\rho_m(\kappa_j, Z\kappa_j)]^{\gamma}.[\rho_m(\kappa_i, \kappa_j)]^{\delta}\end{aligned}$$

$$\begin{aligned}\rho_m(\kappa_i, \kappa_j) &= [\rho_m(\kappa_i, \kappa_j).\rho_m(\kappa_i, \kappa_i).\rho_m(\kappa_j, \kappa_j)]^{\alpha}.\left[\left(\frac{\rho_m(\kappa_i, \kappa_j).\rho_m(\kappa_j, \kappa_i)}{\rho_m(\kappa_i, \kappa_i).\rho_m(\kappa_j, \kappa_j)}\right)\right]^{\beta}\\ &\quad .[\rho_m(\kappa_i, \kappa_i).\rho_m(\kappa_j, \kappa_j)]^{\gamma}.[\rho_m(\kappa_i, \kappa_j)]^{\delta}\\ \rho_m(\kappa_i, \kappa_j) &\leq [\rho_m(\kappa_i, \kappa_j)].^{(\alpha+2\beta+\delta)}\end{aligned}$$

which is contradiction since $\alpha + 2\beta + \delta < 1$.
So, $\rho_m(\kappa_i, \kappa_j) = 1$ and hence $\kappa_i = \kappa_j$.
Thus Z possesses a UFP in a complete MMS (M, ρ_m).
Thus the proof is concluded.

□

Corollary 3.2. *Given (M, ρ_m) is MMS, which is complete and if $Z : M \to M$ is continuous function such that*

$$\rho_m(Z\kappa, Z\xi) \leq [\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)]^{(\alpha+\gamma)}.\left[\left(\frac{\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa)}{\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)}\right)\right]^{\beta}$$

$\forall \kappa, \xi \in M$, in which $\alpha, \beta, \gamma \in [0, 1)$ with $2(\alpha + \gamma) < 1$ and $2\beta < 1$, then $\exists$ a UFP for self mapping Z.

Now we give few examples to validate our obtained results. Note that the following examples holds for both theorems (3.1) and (3.2).

Example 3.1. *Suppose M indicates the set of all real numbers and the multiplicative metric $\rho_m(\omega, \nu) = e^{|\omega-\nu|}$, then (M, ρ_m) is MMS. As the space M is complete concerning the usual metric, the same holds for "ρ_m".*
Let $Z(\omega) = \frac{1}{2}\omega + \kappa$, $\kappa \neq 0$ be a self mapping on M that is continuous .
Note that $\rho_m(Z(\omega), Z(\nu)) = e^{|Z(\omega)-Z(\nu)|} = e^{\frac{1}{2}|\omega-\nu|} = (e^{|\omega-\nu|})^{\frac{1}{2}} = (\rho_m(\omega, \nu))^{\frac{1}{2}}$. On the assumption that $\delta = 1/2$ and $\alpha = \beta = \gamma = 0$,
we conclude $\omega = 2\kappa$ is the UFP.

Example 3.2. *Let $M = \mathbb{R}$ and the multiplicative metric $\rho_m(\omega, \nu) = e^{|\omega-\nu|}$ such that (M, ρ_m) is MMS. Due to the fact that the space M is complete for the usual metric, the same applies to "ρ_m".*

Given the self mapping $Z(\omega) = \alpha'\omega + \kappa$, with $0 < \alpha' < 1$ and non-zero κ, represent a continuous function on M. Consequently, it follows that
$\rho_m(Z(\omega), Z(\nu)) = e^{|Z(\omega)-Z(\nu)|} = e^{\alpha'|\omega-\nu|} = (e^{|\omega-\nu|})^{\alpha'} = (\rho_m(\omega,\nu))^{\alpha'}$.
By presuming $\delta = \alpha'$ and $\alpha = \beta = \gamma = 0$,
it can be concluded that $Z(\omega)$ has UFP, $\forall \alpha' \in (0,1)$ in MMS (M, ρ_m) .

Example 3.3. *Let $M = \mathbb{R}$ and the multiplicative metric $\rho_m(\omega,\nu) = 2^{|\omega-\nu|}$, thereafter (M, ρ_m) is MMS. As the space M is complete pertaining to the usual metric, it is also true for "ρ_m".*
If $Z(\omega) = \kappa$, with $\kappa \neq 0$ is a continuous self map of M ,
then $\rho_m(Z(\omega), Z(\nu)) = 2^{|Z(\omega)-Z(\nu)|} = 2^{|\kappa-\kappa|} = 2^0 = 1$.
Given that $\alpha = \beta = \gamma = \delta = 0$, which implies $\omega = \kappa$ is the UFP.

Example 3.4. *Let (M, ρ_m) be MMS having $M = \mathbb{R}$ and the multiplicative metric $\rho_m(\omega,\nu) = 5^{min\{1,|\omega-\nu|\}}$. Because of the space M is complete, in connection with usual metric, the same is valid for "ρ_m".*
Here $Z(\omega) = a\omega$ is a continuous map from M onto itself, which results in

$$\begin{aligned}\rho_m(Z(\omega), Z(\nu)) = 5^{min\{1,|Z(\omega)-Z(\nu)|\}} &= 5^{min\{1,|a\omega-a\nu|\}}\\ &= 5^{min\{1,a(|\omega-\nu|)\}}\\ &= 5^{min\{1,0\}} = 5^0 = 1.\end{aligned}$$

If we assume that $\alpha = \beta = \gamma = \delta = 0$,
as a result, we have UFP $\omega = 0$.

Next we recall the definition of weak commuting mapping in the space of MMS and establish a fixed point result by utilizing this mapping within a complete MMS.

Definition 3.1. [6] *The mappings f and g such that $f, g : M \to M$ on MMS (M, ρ_m). Consequently, the pair of mappings f and g were called*

- *commuting, if $fg(b) = gf(b), \forall b \in M$.*
- *weakly commuting, if $\rho_m(fg(b), gf(b)) \leq \rho_m(f(b), g(b)), \forall b \in M$.*

Now, we prove another fixed point result using pair of self mappings, which is weakly commuting, to yield a UCFP within a complete MMS.

Theorem 3.3. *If $f, g : M \to M$ be two mappings in a complete MMS (M, ρ_m) in which the below conditions holds:*

(i) *The functions f and g are weakly commuting mappings (i.e.,) $\rho_m(fg(\kappa), gf(\kappa)) \leq \rho_m(f(\kappa), g(\kappa)), \forall \kappa \in M$*
(ii) *$\rho_m(fg(\kappa), \kappa) = \rho_m(f\kappa, \kappa)$ and $\rho_m(\kappa, gf(\kappa)) = \rho_m(\kappa, g(\kappa))$, $\forall \kappa \in M$*
(iii) $$\begin{aligned}\rho_m(fg(\kappa), gf(\xi)) \leq{} & [\rho_m(\kappa, fg(\kappa)).\rho_m(\xi, gf(\xi))]^\alpha.[\rho_m(\kappa, gf(\xi)).\rho_m(\xi, fg(\kappa))]^\beta \\ & .[\rho_m(\kappa,\xi)]^\gamma.\left[\left(\frac{\rho_m(\kappa,fg(\kappa)).\rho_m(\kappa,gf(\xi)).\rho_m(\xi,fg(\kappa)).\rho_m(\xi,gf(\xi))}{\rho_m(\kappa,\xi)}\right)\right]^\delta\end{aligned}$$

where the values of $\alpha, \beta, \gamma, \delta$ varies in $[0, 1)$ with $2\alpha+2\beta+\gamma+3\delta < 1$, $\alpha+\beta+2\delta < 1$ and $2\beta + \gamma + \delta < 1$. Then the pair of mappings f and g possesses UCFP in a complete MMS (M, ρ_m).

Proof. Let κ_0 be an arbitrary point of M.
Let $\{\kappa_n\}$ be a sequence defined by $\kappa = \kappa_{2n}$ and $\xi = \kappa_{2n-1}$ and

$fg(\kappa_{2n}) = \kappa_{2n+1}, gf(\kappa_{2n-1}) = \kappa_{2n}$.
Now, $\rho_m(fg(\kappa_{2n}), gf(\kappa_{2n-1}))$

$$\begin{aligned}
&\leq [\rho_m(\kappa_{2n}, fg(\kappa_{2n})).\rho_m(\kappa_{2n-1}, gf(\kappa_{2n-1}))]^{\alpha} \\
&\quad .[\rho_m(\kappa_{2n}, gf(\kappa_{2n-1})).\rho_m(\kappa_{2n-1}, fg(\kappa_{2n}))]^{\beta}.[\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\gamma} \\
&\quad .\left[\left(\frac{\rho_m(\kappa_{2n}, fg(\kappa_{2n})).\rho_m(\kappa_{2n}, gf(\kappa_{2n-1})).\rho_m(\kappa_{2n-1}, fg(\kappa_{2n})).\rho_m(\kappa_{2n-1}, gf(\kappa_{2n-1}))}{\rho_m(\kappa_{2n}, \kappa_{2n-1})}\right)\right]^{\delta} \\
&= [\rho_m(\kappa_{2n}, \kappa_{2n+1}).\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\alpha}.[\rho_m(\kappa_{2n}, \kappa_{2n}).\rho_m(\kappa_{2n-1}, \kappa_{2n+1})]^{\beta} \\
&\quad .[\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\gamma}.\left[\left(\frac{\rho_m(\kappa_{2n}, \kappa_{2n+1}).\rho_m(\kappa_{2n}, \kappa_{2n}).\rho_m(\kappa_{2n-1}, \kappa_{2n+1}).\rho_m(\kappa_{2n-1}, \kappa_{2n})}{\rho_m(\kappa_{2n}, \kappa_{2n-1})}\right)\right]^{\delta} \\
&\leq [\rho_m(\kappa_{2n}, \kappa_{2n+1}).\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\alpha}.[\rho_m(\kappa_{2n-1}, \kappa_{2n+1})]^{\beta}.[\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\gamma} \\
&\quad .[\rho_m(\kappa_{2n}, \kappa_{2n+1}).\rho_m(\kappa_{2n-1}, \kappa_{2n+1})]^{\delta} \\
&\leq [\rho_m(\kappa_{2n}, \kappa_{2n+1}).\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\alpha}.[\rho_m(\kappa_{2n-1}, \kappa_{2n}).\rho_m(\kappa_{2n}, \kappa_{2n+1})]^{\beta}.[\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\gamma} \\
&\quad .[\rho_m(\kappa_{2n}, \kappa_{2n+1}).\rho_m(\kappa_{2n-1}, \kappa_{2n}).\rho_m(\kappa_{2n}, \kappa_{2n+1})]^{\delta}.
\end{aligned}$$

$$\begin{aligned}
\rho_m((fg(\kappa_{2n}), gf(\kappa_{2n-1})) &= \rho_m(\kappa_{2n+1}, \kappa_{2n}) \\
&\leq [\rho_m(\kappa_{2n}, \kappa_{2n+1})]^{(\alpha+\beta+2\delta)}.[\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{(\alpha+\beta+\gamma+\delta)}
\end{aligned}$$

Then, we have

$$\rho_m(\kappa_{2n}, \kappa_{2n+1}) \leq [\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\left(\frac{\alpha+\beta+\gamma+\delta}{1-(\alpha+\beta+2\delta)}\right)} \tag{1}$$

Similarly, on taking $\kappa = \kappa_{2n+1}$, $\xi = \kappa_{2n}$ and
$fg(\kappa_{2n+1}) = \kappa_{2n+2}$, $gf(\kappa_{2n}) = \kappa_{2n+1}$, we get

$$\rho_m(\kappa_{2n+1}, \kappa_{2n+2}) \leq [\rho_m(\kappa_{2n}, \kappa_{2n+1})]^{\left(\frac{\alpha+\beta+\gamma+\delta}{1-(\alpha+\beta+2\delta)}\right)} \tag{2}$$

Using Equation (1) in the Equation (2), we obtain

$$\rho_m(\kappa_{2n+1}, \kappa_{2n+2}) \leq [\rho_m(\kappa_{2n-1}, \kappa_{2n})]^{\left(\frac{\alpha+\beta+\gamma+\delta}{1-(\alpha+\beta+2\delta)}\right)^2}.$$

In general, we get
$\rho_m(\kappa_{2n+1}, \kappa_{2n+2}) \leq [\rho_m(\kappa_0, \kappa_1)]^{t^{2n+1}}$, where $t = \frac{\alpha+\beta+\gamma+\delta}{1-(\alpha+\beta+2\delta)} < 1$.
(i.e.,) $2\alpha+2\beta+\gamma+3\delta < 1$.
Thereafter we tend to prove that $\{\kappa_n\}$ is multiplicative Cauchy sequence
For $n < m, \rho_m(\kappa_n, \kappa_m) \leq \rho_m(\kappa_n, \kappa_{n+1}).\rho_m(\kappa_{n+1}, \kappa_{n+2})....\rho_m(\kappa_{m-1}, \kappa_m)$.
$\rho_m(\kappa_n, \kappa_m) \leq \prod_{n=1}^{\infty} \rho_m(\kappa_0, \kappa_1)^{t^n} \longrightarrow 1$ as $n \longrightarrow \infty$
So, $\rho_m(\kappa_n, \kappa_m) \longrightarrow 1$ as $n, m \longrightarrow \infty$.
Thus $\{\kappa_n\}$ is multiplicative Cauchy sequence in M.
As M is complete, $\exists$ a point $c_1 \in M$ such that
$\{\kappa_n\}$ is multiplicative convergent to c_1.
To prove : the mappings f and g holds the common fixed point c_1 .
Take $\kappa = c_1$ & $\xi = \kappa_{2n+1}$, then by condition (iii) of hypothesis,

$$\begin{aligned}
&\rho_m(fg(c_1), gf(\kappa_{2n+1})) = \rho_m(fg(c_1), \kappa_{2n+2})\\
&\quad\leq[\rho_m(c_1, fg(c_1)).\rho_m(\kappa_{2n+1}, gf(\kappa_{2n+1})]^{\alpha}\\
&\quad\quad .[\rho_m(c_1, gf(\kappa_{2n+1})).\rho_m(\kappa_{2n+1}, fg(c_1)]^{\beta}.[\rho_m(c_1, \kappa_{2n+1})]^{\gamma}\\
&\quad\quad .\left[\left(\frac{\rho_m(c_1, fg(c_1)).\rho_m(c_1, gf(\kappa_{2n+1})).\rho_m(\kappa_{2n+1}, fg(c_1)).\rho_m(\kappa_{2n+1}, gf(\kappa_{2n+1}))}{\rho_m(c_1, \kappa_{2n+1})}\right)\right]^{\delta}\\
&\quad=[\rho_m(c_1, fg(c_1)).\rho_m(\kappa_{2n+1}, \kappa_{2n+2})]^{\alpha}\\
&\quad\quad .[\rho_m(c_1, \kappa_{2n+2})).\rho_m(\kappa_{2n+1}, fg(c_1))]^{\beta}.[\rho_m(c_1, \kappa_{2n+1})]^{\gamma}\\
&\quad\quad .\left[\left(\frac{\rho_m(c_1, fg(c_1)).\rho_m(c_1, \kappa_{2n+2}).\rho_m(\kappa_{2n+1}, fg(c_1)).\rho_m(\kappa_{2n+1}, \kappa_{2n+2})}{\rho_m(c_1, \kappa_{2n+1})}\right)\right]^{\delta}.
\end{aligned}$$

Taking $\lim_{n\longrightarrow\infty}$ in the above relation, we get

$$\rho_m(fg(c_1), c_1) \leq [\rho_m(c_1, fg(c_1)).\rho_m(c_1, c_1)]^{\alpha}.[\rho_m(c_1, c_1).\rho_m(c_1, fg(c_1))]^{\beta}$$
$$.[\rho_m(c_1, c_1)]^{\gamma}.\left[\left(\frac{\rho_m(c_1,fg(c_1)).\rho_m(c_1,c_1).\rho_m(c_1,fg(c_1)).\rho_m(c_1,c_1)}{\rho_m(c_1,c_1)}\right)\right]^{\delta}.$$

$\rho_m(fg(c_1), c_1) \leq [\rho_m(fg(c_1), c_1)]^{(\alpha+\beta+2\delta)}$.
Since $\alpha + \beta + 2\delta < 1$ and from condition (ii) of hypothesis,
we have $\rho_m(fg(c_1), c_1) = \rho_m(f(c_1), c_1) = 1$ and thus $f(c_1) = c_1$.
Similarly, by taking $\kappa = \kappa_{2n+1}$ and $\xi = c_1$ in the condition (iii) of hypothesis,
we get $g(c_1) = c_1$.
Thus the mappings f and g possess common fixed point c_1.

Inorder to determine the uniqueness,
let us assume that c_2 be another fixed point of mappings f and g.
Then $\rho_m(c_1, c_2) = \rho_m(fg(c_1), gf(c_2))$

$$\begin{aligned}
&\leq[\rho_m(c_1, fg(c_1)).\rho_m(c_2, gf(c_2)]^{\alpha}.[\rho_m(c_1, gf(c_2)).\rho_m(c_2, fg(c_1))]^{\beta}.[\rho_m(c_1, c_2)]^{\gamma}\\
&\quad .\left[\left(\frac{\rho_m(c_1, fg(c_1)).\rho_m(c_1, gf(c_1)).\rho_m(c_1, fg(c_1)).\rho_m(c_2, gf(c_2))}{\rho_m(c_1, c_2)}\right)\right]^{\delta}\\
&\leq[\rho_m(c_1, f(c_1)).\rho_m(c_2, g(c_2))]^{\alpha}.[\rho_m(c_1, g(c_2)).\rho_m(c_2, f(c_1))]^{\beta}.[\rho_m(c_1, c_2)]^{\gamma}\\
&\quad .\left[\left(\frac{\rho_m(c_1, f(c_1)).\rho_m(c_1, g(c_2)).\rho_m(c_2, f(c_1)).\rho_m(c_2, g(c_2))}{\rho_m(c_1, c_2)}\right)\right]^{\delta}\\
&=[\rho_m(c_1, c_1).\rho_m(c_2, c_2)]^{\alpha}.[\rho_m(c_1, c_2).\rho_m(c_2, c_1)]^{\beta}.[\rho_m(c_1, c_2)]^{\gamma}\\
&\quad .\left[\left(\frac{\rho_m(c_1, c_1).\rho_m(c_1, c_2).\rho_m(c_2, c_1).\rho_m(c_2, c_2)}{\rho_m(c_1, c_2)}\right)\right]^{\delta}.
\end{aligned}$$

$\rho_m(c_1, c_2) \leq [\rho_m(c_1, c_2)]^{(2\beta+\gamma+\delta)}$.
Since $2\beta + \gamma + \delta < 1$, therefore $\rho_m(c_1, c_2) = 1$, which implies $c_1 = c_2$.
Hence the mappings f and g possess UCFP in a complete MMS (M, ρ_m).
This concludes the proof. □

Corollary 3.3. *Given $f, g : M \to M$ be two mappings in a complete MMS (M, ρ_m) under which the below conditions exists:*

(i) *The functions f and g are weakly commuting mappings (i.e.,) $\rho_m(fg(\kappa), gf(\kappa)) \leq \rho_m(f(\kappa), g(\kappa)), \forall\kappa \in M$*
(ii) $\rho_m(fg(\kappa), \kappa) = \rho_m(f\kappa, \kappa)$ *and* $\rho_m(\kappa, gf(\kappa)) = \rho_m(\kappa, g(\kappa))$, $\forall\kappa \in M$

(iii) $\rho_m(fg(\kappa), gf(\xi)) \leq [\rho_m(\kappa, fg(\kappa)).\rho_m(\xi, gf(\xi))]^{\alpha}.[\rho_m(\kappa, gf(\xi)).\rho_m(\xi, fg(\kappa))]^{\beta} . \Big[(\rho_m(\kappa, fg(\kappa)).\rho_m(\kappa, gf(\xi)).\rho_m(\xi, fg(\kappa)).\rho_m(\xi, gf(\xi))\Big]^{\gamma}$

where the values of α, β, γ *varies in* $[0, 1)$ *with* $2\alpha + 2\beta + 4\gamma < 1$, $\alpha+\beta+2\gamma < 1$ *and* $2(\beta+\gamma) < 1$. *Then the mappings* f *and* g *must have UCFP in a complete MMS* (M, ρ_m).

4. Application

To establish the validity of our main results Theorem (3.1) and Theorem (3.2), we provide an application of a Nonlinear Integral Equation (NIE) to ensure the existence of a unique solution. Suppose $M = B([i, j], \mathbb{R})$ denotes the Banach space of all continuous mappings on $[i, j]$ along with supremum norm

$$||\kappa|| = \sup_{\tilde{n}\in[i,j]} |\kappa(\tilde{n})|,$$

where $\kappa \in B([i, j], \mathbb{R})$, and the multiplicative metric $\rho_m : M \times M \to \mathbb{R}$ is defined as :

$$\rho_m(\kappa, \xi) = a^{\left(\sup_{\tilde{n}\in[i,j]} |\kappa(\tilde{n})-\xi(\tilde{n})|\right)} = a^{||\kappa-\xi||}, \text{with } a > 1, \text{where } \kappa, \xi \in B([i, j], \mathbb{R}). \quad (3)$$

The NIE is defined as:

$$\kappa(\tilde{n}) = \int_i^j A(\tilde{n}, s, \kappa(s))\, ds \quad (4)$$

with $\tilde{n} \in [i, j] \subset \mathbb{R}$ and $A : [i, j] \times [i, j] \times \mathbb{R} \to \mathbb{R}$.

Theorem 4.1. *Assume that the NIE is defined as in Equation (4) and* $\exists\psi \in (0, 1)$ *so that*

$$a^{||Z\kappa-Z\xi||} \leq (\mathbb{M}(Z, \kappa, \xi))^{\psi}, \quad (5)$$

where

$$\mathbb{M}(Z, \kappa, \xi) = min\Big\{ \Big(a^{(||\kappa-Z\kappa||+||\kappa-Z\xi||+||\xi-Z\kappa||+||\xi-Z\xi||-||\kappa-\xi||)}\Big), \Big(a^{(||\kappa-Z\kappa||+||\xi-Z\xi||)}\Big), \Big(a^{(||\kappa-Z\xi||+||\xi-Z\kappa||)}\Big), \Big(a^{(||\kappa-\xi||)}\Big)\Big\} \quad (6)$$

Then, the NIE (Equation (4)) has unique solution in M.

Proof. Define a function $Z : M \to M$ by

$$Z\kappa(\tilde{n}) = \int_i^j A(\tilde{n}, s, \kappa(s))\, ds, \ \ \forall\kappa \in M. \quad (7)$$

Next, we employ Theorem (3.1) to the integral operator Z for the validation of our result. The analysis leads to four main possible cases:
(i) If $a^{(||\kappa-Z\kappa||+||\kappa-Z\xi||+||\xi-Z\kappa||+||\xi-Z\xi||-||\kappa-\xi||)}$ is the minimum term in Equation (6), then $\mathbb{M}(Z, \kappa, \xi) = a^{(||\kappa-Z\kappa||+||\kappa-Z\xi||+||\xi-Z\kappa||+||\xi-Z\xi||-||\kappa-\xi||)}$
Now from the Equations (3) and (5), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa-Z\xi||} &\leq (\mathbb{M}(Z, \kappa, \xi))^{\psi} \\ &= \Big(a^{(||\kappa-Z\kappa||+||\kappa-Z\xi||+||\xi-Z\kappa||+||\xi-Z\xi||-||\kappa-\xi||)}\Big)^{\psi} \\ &= \left(\frac{\rho_m(\kappa, Z\kappa).\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa).\rho_m(\xi, Z\xi)}{\rho_m(\kappa, \xi)}\right)^{\psi}, \ \forall\kappa, \xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.1) with $\psi = \alpha$,
$\beta = \gamma = \delta = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)).
(ii) If $a^{(||\kappa - Z\kappa|| + ||\xi - Z\xi||)}$ is the minimum term in Equation (6), then
$\mathbb{M}(Z,\kappa,\xi) = a^{(||\kappa - Z\kappa|| + ||\xi - Z\xi||)}$
Now from the Equations (3) and (5), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa - Z\xi||} &\leq (\mathbb{M}(Z,\kappa,\xi))^{\psi} \\ &= \Big(a^{(||\kappa - Z\kappa|| + ||\xi - Z\xi||)}\Big)^{\psi} \\ &= \Big(\rho_m(\kappa, Z\kappa).\rho_m(\xi, Z\xi)\Big)^{\psi}, \forall \kappa, \xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.1)
with $\psi = \beta, \alpha = \gamma = \delta = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)).
(iii) If $a^{(||\kappa - Z\xi|| + ||\xi - Z\kappa||)}$ is the minimum term in Equation (6), then
$\mathbb{M}(Z,\kappa,\xi) = a^{(||\kappa - Z\xi|| + ||\xi - Z\kappa||)}$
Now from the Equations (3) and (5), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa - Z\xi||} &\leq (\mathbb{M}(Z,\kappa,\xi))^{\psi} \\ &= \Big(a^{(||\kappa - Z\xi|| + ||\xi - Z\kappa||)}\Big)^{\psi} \\ &= \Big(\rho_m(\kappa, Z\xi).\rho_m(\xi, Z\kappa)\Big)^{\psi}, \ \forall \kappa, \xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.1)
with $\psi = \gamma, \alpha = \beta = \delta = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)).
(iv) If $a^{(||\kappa - \xi||)}$ is the minimum term in Equation (6), then
$\mathbb{M}(Z,\kappa,\xi) = a^{(||\kappa - \xi||)}$
Now from the Equations (3) and (5), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa - Z\xi||} &\leq (\mathbb{M}(Z,\kappa,\xi))^{\psi} \\ &= \Big(a^{(||\kappa - \xi||)}\Big)^{\psi} \\ &= \Big(\rho_m(\kappa, \xi)\Big)^{\psi}, \ \forall \kappa, \xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.1)
with $\psi = \delta, \alpha = \beta = \gamma = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)). □

Theorem 4.2. *Assume that the NIE is defined as in Equation (4), and*
$\exists \psi \in (0,1)$ *so that*

$$a^{||Z\kappa - Z\xi||} \leq (\mathbb{M}'(Z,\kappa,\xi))^{\psi}, \tag{8}$$

where

$$\begin{aligned}\mathbb{M}'(Z,\kappa,\xi) = min\Big\{ &\Big(a^{(||\kappa-\xi|| + ||\kappa - Z\kappa|| + ||\xi - Z\xi||)}\Big), \Big(a^{(||\kappa - Z\xi|| + ||\xi - Z\kappa|| - ||\kappa - Z\kappa|| - ||\xi - Z\xi||)}\Big), \\ &\Big(a^{(||\kappa - Z\kappa|| + ||\xi - Z\xi||)}\Big), \Big(a^{(||\kappa - \xi||)}\Big)\Big\}\end{aligned} \tag{9}$$

Then, the NIE (Equation (4)) has unique solution in M.

Proof. Define a function $Z : M \to M$ by

$$Z\kappa(\tilde{n}) = \int_i^j A(\tilde{n}, s, \kappa(s))\, ds, \ \ \forall \kappa \in M. \tag{10}$$

We proceed to apply Theorem (3.2) to the integral operator Z for the justification of our result.
We might encounter the following four main cases :
(i) If $a^{(||\kappa-\xi||+||\kappa-Z\kappa||+||\xi-Z\xi||)}$ is the minimum term in Equation (9), then
$\mathbb{M}'(Z,\kappa,\xi) = a^{(||\kappa-\xi||+||\kappa-Z\kappa||+||\xi-Z\xi||)}$
Now from the Equations (3) and (8), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa-Z\xi||} &\leq (\mathbb{M}'(Z,\kappa,\xi))^{\psi}\\ &= \left(a^{(||\kappa-\xi||+||\kappa-Z\kappa||+||\xi-Z\xi||)}\right)^{\psi}\\ &= \left(\rho_m(\kappa,\xi).\rho_m(\kappa,Z\kappa).\rho_m(\xi,Z\xi)\right)^{\psi}, \ \forall \kappa,\xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.2)
with $\psi = \alpha, \beta = \gamma = \delta = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)).
(ii) If $a^{(||\kappa-Z\xi||+||\xi-Z\kappa||-||\kappa-Z\kappa||-||\xi-Z\xi||)}$ is the minimum term in Equation (9), then
$\mathbb{M}'(Z,\kappa,\xi) = a^{(||\kappa-Z\xi||+||\xi-Z\kappa||-||\kappa-Z\kappa||-||\xi-Z\xi||)}$
Now from the Equations (3) and (8), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa-Z\xi||} &\leq (\mathbb{M}'(Z,\kappa,\xi))^{\psi}\\ &= \left(a^{(||\kappa-Z\xi||+||\xi-Z\kappa||-||\kappa-Z\kappa||-||\xi-Z\xi||)}\right)^{\psi}\\ &= \left(\frac{\rho_m(\kappa,Z\xi).\rho_m(\xi,Z\kappa)}{\rho_m(\kappa,Z\kappa).\rho_m(\xi,Z\xi)}\right)^{\psi}, \ \forall \kappa,\xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.2)
with $\psi = \beta, \alpha = \gamma = \delta = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)).
(iii) If $a^{(||\kappa-Z\kappa||+||\xi-Z\xi||)}$ is the minimum term in Equation (9), then
$\mathbb{M}'(Z,\kappa,\xi) = a^{(||\kappa-Z\kappa||+||\xi-Z\xi||)}$
Now from the Equations (3) and (8), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa-Z\xi||} &\leq (\mathbb{M}'(Z,\kappa,\xi))^{\psi}\\ &= \left(a^{(||\kappa-Z\kappa||+||\xi-Z\xi||)}\right)^{\psi}\\ &= \left(\rho_m(\kappa,Z\kappa).\rho_m(\xi,Z\xi)\right)^{\psi}, \ \forall \kappa,\xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.2)
with $\psi = \gamma, \alpha = \beta = \delta = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)).
(iv) If $a^{(||\kappa-\xi||)}$ is the minimum term in Equation (9), then
$\mathbb{M}'(Z,\kappa,\xi) = a^{(||\kappa-\xi||)}$

Now from the Equations (3) and (8), we have

$$\begin{aligned}\rho_m(Z\kappa, Z\xi) = a^{||Z\kappa - Z\xi||} &\leq (\mathbb{M}'(Z,\kappa,\xi)))^{\psi} \\ &= \left(a^{(||\kappa-\xi||)}\right)^{\psi} \\ &= \left(\rho_m(\kappa,\xi)\right)^{\psi}, \ \forall \kappa, \xi \in M.\end{aligned}$$

Hence the operator Z satisfies all conditions of Theorem (3.2)
with $\psi = \delta, \alpha = \beta = \gamma = 0$.
Thus the mapping Z has UFP in M, which is a unique solution of NIE (Equation (4)). □

5. Conclusions

Fixed point theory is a crucial field in mathematical analysis and topology, with significant applications in functional analysis. The structure of a multiplicative metric space is especially beneficial for problems that rely on ratio-based distance measurements. In this study, we obtained the fixed point results using self mapping continuous function in the complete MMS and validated our results by providing suitable examples. Moreover, a UCFP have been established in the pair of weakly commuting self mappings. An application of a NIE is presented along with the existence of a unique solution. Different types of contractions can be examined in [8],[24],[27]. Few results concerning MMS could refer to [1],[2],[3],[17],[22],[23],[26] & One can refer [7],[10],[12],[13] and [19] for the recent extension of MMS fixed point results.

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References

[1] Agarwal, R. P., Karapinar, E. Samet, B., (2016), An essential remark on fixed point results on multiplicative metric spaces, Fixed Point Theory and applications, 21, pp. 1-3.

[2] Ansar, A., Idris, M. A., (2022), Some fixed point results on multiplicative metric spaces, Journal of Mathematics : Theory and Applications, 4 (1), pp. 15-20.

[3] Aminat, O. I., Hallowed Olaoluwa, O., Johnson Oladeru, O., (2022), Some fixed points of multivalued maps in multiplicative metric spaces, Heliyon, 8 (12), pp. e12453.

[4] Gupta, A., Maitra, J. K., Rai, V., (2018), Application of fixed point theorem for uniqueness and stability of solutions for a class of nonlinear integral equations, Journal of Applied Mathematics and Informatics, 36 (1-2), pp. 1-14.

[5] Bashirov, A. E., Kurpnar, E. M., Ozyapici, A., (2008), Multiplicative calculus and its applications, Journal of Mathematical Analysis and Applications, 337 (1), pp. 36-48.

[6] Gu, F., Cui, L. M., Wu, Y. H., (2013), Some fixed point theorems for new contractive type mappings, Journal of Qiqihar University, 19, pp. 85-89.

[7] Alshammari, I., Ali, S., Haque Khan, Q., Rashid, T., Ozel, C., (2023), Metrical fixed point results on b-multiplicative metric spaces employing binary relation, European Journal of Pure and Applied Mathematics, 16 (4), pp. 2405-2418.

[8] Kedim, I., Berzig, M., (2021), Fixed point theorems for maia $\alpha - \psi$ contractive type mappings with applications, Journal of Mathematical Analysis and Applications, 504 (1), pp. 1-12.

[9] Rus, I., Serban, M. A., (2013), Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem, Carpathian Journal of Mathematics, 29 (2), PP. 239-258.

[10] Jarvisvivin, J., Mary Priya Dharsini, A., (2024), Application of fixed point theorems to differential equation in b-multiplicative metric spaces, Indian Journal of Science and Technology, 17 (22), pp. 2352-2362.

[11] Wang C., Convergence, Stability and Data Dependence for Fixed Points of Generalized Nonexpansive Mappings with Applications, (2025), TWMS J. Pure Appl. Math. V.16, N.1, 2025, pp.137-148

[12] Joselin Lavino, P., Mary Priya Dharsini, A., (2023), Some fixed point theorems of rational type contraction on b-multiplicative metric spaces, European Chemical Bulletin, 12, pp. 2262-2267.
[13] Yadav, K., Kumar, D., (2024), Multiplicative m-metric space, fixed point theorems with applications in multiplicative integrals equation and numerical results, Journal of Applied Analysis, 30 (1), pp. 173-186.
[14] Florack, L., Assen, H. V., (2011), Multiplicative calculus in biomedical image analysis, Journal of Mathematical Imaging and Vision, 42, pp. 64-75.
[15] Mercy Arokia Rani, J., Leema Maria Prakasam, A., (2024), Fixed points of operators with multiplicative closed graphs on bipolar b-multiplicative metric spaces, Advances in Fixed Point Theory, 14 (2024), pp.1-16.
[16] Grossman, M., Katz, R., (1972), Non-newtonian calculus, Pigeon Cove, Lee press, Massachusats, pp. 23-35.
[17] Abbas, M., Ali, B., Suleiman, Y. I., (2015), Common fixed points of locally contractive mappings in multiplicative metric spaces with application, International Journal of Mathematics and Mathematical Sciences, 2015 (3), pp. 1-7.
[18] Nanware, J. A., Jadhav, N. B., Dhaigude, D. B., (2020), Existence and uniqueness of solutions of nonlinear implicit fractional differential equations, Dynamics of Continuous, Discrete and Impulsive Systems. Series A : Mathematical Analysis, 27, pp. 275-282.
[19] Kayode Adewale, O., Olusola Ayodele, S., Eriwa Oyelade, B., Ehui Aribike, E., (2024), Equivalence of some results and fixed point theorems in S-multiplicative metric spaces, Fixed Point Theory and Algorithms for Sciences and Engineering, 2024 (1), pp. 1-13.
[20] Ozavsar, M., Cevikel, A. C., (2017), Fixed point of multiplicative contraction mappings on multiplicative metric spaces, Journal of Engineering Technology and Applied Sciences, 2 (2), pp. 65-79.
[21] Acar, O., (2023), Some recent and new fixed point results on orthogonal metric-like space, Constructive Mathematical Analysis, 6 (3), pp. 184-197.
[22] Shukla, S., (2016), Some critical remarks on the multiplicative metric spaces and fixed point results, Journal of Advanced Mathematical Studies, 9 (3), pp. 454-458.
[23] Shoaib, A., Saleemullah, M., (2020), Some fixed point results in b-multiplicative metric space, TWMS J. App. and Eng. Math., 10 (4), pp. 846-856.
[24] Sudheer Kumar, P., Jagannadha Rao, G. V. V., Santhi Kumar, R., Satyanarayana, P. E., (2024), Fixed point theorems for a pair of (α, η, ψ)- geraghty contraction type maps in complete metric spaces, Nonlinear Functional Analysis and Applications, 29 (1), pp. 57-67.
[25] Rasham, T., De La Sen, M., (2022), A novel study for hybrid pair of multivalued dominated mappings in b-multiplicative metric space with applications, Journal of Inequalities and Applications, 107, pp. 1-15.
[26] Dosenovic, T., Postolache, M., Radenovic, S., (2016), On multiplicative metric spaces : survey, Fixed Point Theory and Applications, 92 (2016), pp. 1-17.
[27] Dosenovic, T., Radenovic, S., (2018), Multiplicative metric spaces and contractions of rational type, Advances in the theory of Nonlinear Analysis and Applications, 2 (4), pp. 195-201.

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