TWMS J. App. and Eng. Math. V.16, N.4, 2026, pp. 510-520

FIXED POINT ANALYSIS IN QUASI-PARTIAL METRIC SPACES USING $w-$INTERPOLATIVE HARDY-ROGERS TYPE CONTRACTIONS

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Abstract. By using Interpolative Hardy-Rogers type contraction via $w-$admissibility approach in the framework of quasi-partial metric space, we introduce a new property that makes it convenient to investigate the existence and uniqueness of fixed point theorems.

Keywords: Quasi-partial metric space, Interpolation, Hardy-Rogers type contracion, $w-$admissibility, fixed point.

AMS Subject Classification: 54H25; 47H10; 46T99.

1. Introduction

The study of fixed points in quasi-partial metric spaces using $w-$ interpolative Hardy-Rogers type contractions is a specialized area within mathematics. It involves investigating the existence and properties of fixed points for specific classes of functions that possess the contractive property in quasi-partial metric spaces under the influence of $w-$ interpolative approach. In mathematics, a fixed point of a function is a point that remains unchanged when the function is applied to it. Fixed-point analysis studies the existence, uniqueness, and properties of fixed points for various types of functions and spaces. A quasi-partial metric space is a generalization of a metric space that relaxes the triangle inequality condition. Instead of the usual triangle inequality, a quasi-partial metric space satisfies a weaker version, allowing distances between points to be non-negative but possibly zero.

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§ Manuscript received: October 21, 2024; accepted: November 12, 2025.

Hardy-Rogers type contractions are a class of contractive mappings used in fixed-point theory. These mappings have certain properties that guarantee the existence and uniqueness of fixed points in certain spaces. Frechet [1] studied the notion of functional calculations in 1906, and since then, many authors have developed and improved on this concept due to its widespread applicability, particularly in topology. Many authors (refer, [2] - [10]) have developed and improved on this concept as a result of its widespread applicability, particularly in topology. The Banach fixed point theorem [12], also known as the contraction mapping principle or the Banach-Caccioppoli theorem, is a fundamental result in the field of functional analysis. It provides conditions under which a self-mapping of a complete metric space has a unique fixed point. The study of fixed-point results in quasi-partial metric spaces has gained momentum with notable contributions such as those by Shatanawi and Pitea. Shatanawi and Pitea [13] investigated coupled fixed-point theorems in quasi-partial metric spaces, establishing foundational results that extend classical fixed-point principles to these generalized settings. Pitea [14] further explored best proximity results in dualistic partial metric spaces, emphasizing their applicability to optimization problems. Additionally, Shatanawi and Postolache [15] examined coincidence and fixed-point results for generalized weak contractions in the sense of Berinde, offering deeper insights into the interplay between contraction mappings and the structure of partial metric spaces.

Kannan's fixed point theorem [16] is a generalization of the contraction mapping principle, allowing for a wider class of mappings to have fixed points. Unlike the Banach fixed point theorem, Kannan's theorem only requires a weaker condition known as Kannan contraction. It has various applications in functional analysis, optimization, control theory, and nonlinear dynamics, particularly in situations where the mapping does not strictly contract distances between points but still has a sufficient contraction property to guarantee the existence of fixed points. Karapinar [17] restated Kannan's type contraction under the framework of interpolation in 2018. In the same year, Karapinar [18] proposed interpolative Hardy-Rogers type contractions. Popescu [19] developed the concept of $w-$orbital acceptable maps as a revision of Samuel et al. [20], and it has been referenced in [21]. In the present research, the prospect of $w-$interpolative Hardy-Rogers type contraction using $w-$admissibility was presented and the fixed point theorem for it was established in the context of quasi-partial metric space.

2. Preliminaries and basic definitions

Definition 2.1. [11] *A quasi-partial metric on a non-empty set U is a mapping $q_p : \mathscr{U} \times \mathscr{U} \to R^+$ such that for all $u,\ v,\ z \in \mathscr{U}$*

(Q_{p_1}) $q_p(u,u) = q_p(u,v) = q_p(v,v)$ *implies* $u = v$,

(Q_{p_2}) $q_p(u,u) \leq q_p(u,v)$,

(Q_{p_3}) $q_p(u,u) \leq q_p(v,u)$,

(Q_{p_4}) $q_p(u,v) \leq [q_p(u,z) + q_p(z,v)] - q_p(z,z)$.

A quasi-partial metric space is a pair $(\mathscr{U}, q_p)$ such that $\mathscr{U}$ is a non-empty set and $(\mathscr{U}, q_p)$ is a quasi-partial metric on $\mathscr{U}$.

Definition 2.2. [18] *Let $(\mathscr{U}, d)$ be a metric space, and $\tau : \mathscr{U} \to \mathscr{U}$ be a self-mapping. τ is called an Interpolative Hardy-Rogers type contraction mapping if there exist constants:*

(1) $\mu_c \in [0,1)$*: This is a control parameter for contraction.*
(2) $p, q, r \in (0,1)$ *: These are weights, controlling the interpolation among terms.*

(3) $p+q+r<1$: *A strict inequality ensuring proper convergence behavior.*
These constants must satisfy the following inequality for all $x, y \in \mathscr{U}$:

$$d(\tau(x), \tau(y)) \leq \mu_c d(x, y) + pd(x, \tau(x)) + qd(y, \tau(y)) + r\left[d(x, \tau(y)) + d(y, \tau(x))\right].$$

Definition 2.3. [19] *Let* $\mathscr{U}$ *be a non empty set and* $w : \mathscr{U} \times \mathscr{U} \to [0, \infty)$ *be a mapping. A self-mapping* $\tau : \mathscr{U} \to \mathscr{U}$ *is said to be* $w-$*orbital admissible if for all* $\eta \in \mathscr{U}$, *we have*

$$w(\eta, \tau\eta) \geq 1 \Rightarrow w(\tau\eta, \tau^2\eta) \geq 1.$$

$\mathscr{H}$ **Condition**. If $\{y_n\}$ is a sequence in $\mathscr{U}$ such that $w(y_n, y_{n+1}) \geq 1$ for each n and $y_n \to y \in \mathscr{U}$ As $n \to \infty$ then there exists $\{y_{n(k)}\}$ from $\{y_n\}$ such that $w(y_{n(k)}, y) \geq 1$, for each k.

A large number of researchers [22]-[35] have occasionally generalized the aforesaid notions and established remarkable fixed point conclusions.

3. Main Results

Let us discuss the main results.

Definition 3.1. *Let* $(\mathscr{U}, q_p)$ *be a quasi-partial metric space. The self-map* $\tau : \mathscr{U} \to \mathscr{U}$ *is said to be* $w-$*interpolative Hardy-Rogers type contraction if there exists* $\mu_c \in \psi$, $w : \mathscr{U} \times \mathscr{U} \to [0, \infty)$ *and positive reals* $p, q, r > 0$, $p+q+r<1$ *such that*

$$\begin{aligned} w(x, y) \cdot q_p(\tau x, \tau y) \leq & \mu_c \cdot \tau[q_p(x, y)]^q \cdot [q_p(x, \tau y)]^p \cdot [q_p(y, \tau y)]^r \cdot \\ & [q_p(x, \tau y) + q_p(y, \tau x)]^{1-p-q-r} \end{aligned} \tag{1}$$

for all $x, y \in \mathscr{U}$.

Theorem 3.1. *Let a continuous self-mapping* $\tau : \mathscr{U} \to \mathscr{U}$ *be* $w-$ *orbital admissible and forms* $w-$*interpolative Hardy-Rogers type contraction on quasi-partial metric space* $(\mathscr{U}, q_p)$. *If there exists* $x_0 \in \mathscr{U}$ *such that* $w(x_0, \tau x_0) \geq 1$, *then* τ *possesses a fixed point* $\in \mathscr{U}$.

Proof. Let x_0 be an initial point in $\mathscr{U}$ such that $w(x_0, \tau x_0) \geq 1$ and $\{x_n\}$ be a sequence such that $x_n = \tau^n(x_0), n \geq 0$. If for some y_0, we get $x_{n_0} = x_{n_0+1}$, then x_{n_0} becomes a fixed point of τ. Let $x_n \neq x_{n+1}$, for each $n \geq 0$. We know $w(x_0, x_1) \geq 1$. As τ is $w-$orbital admissible, $w(x_1, x_2) = w(\tau x_0, \tau x_1) \geq 1$. Thus, we get $w(x_n, x_{n+1}) \geq 1$, for all $n \geq 0$. Let us substitute $x = x_n$, $y = x_{n-1}$ in (1), we get

$$\begin{aligned} q_p(x_{n+1}, x_n) \leq\ & w(x_n, x_{n-1}) q_p(\tau x_n, \tau(x_{n-1})) \\ \leq\ & \mu_c[q_p(x_n, x_{n-1})]^q [q_p(x_n, \tau x_n)]^p [q_p(x_{n-1}, \tau x_{n-1})]^r [q_p(x_n, \tau x_{n-1}) \\ & + q_p(x_{n-1}, \tau x_n)]^{(1-p-q-r)} \\ \leq\ & \mu_c[q_p(x_n, x_{n-1})]^q [q_p(x_n, \tau x_{n+1})]^p [q_p(x_{n-1}, x_n)]^r [q_p(x_n, x_n) \\ & + q_p(x_{n-1}, x_{n+1})]^{(1-p-q-r)} \\ \leq\ & \mu_c[q_p(x_n, x_{n-1})]^q [q_p(x_n, \tau x_{n+1})]^p [q_p(x_{n-1}, x_n)]^r . [q_p(x_{n-1}, x_n) \\ & + q_p(x_n, x_{n+1})]^{(1-p-q-r)} \end{aligned} \tag{2}$$

Let $q_p(x_{n-1}, x_n) \leq q_p(x_n, x_{n+1})$, for $n \geq 1$.

Then, it follows

$$\frac{1}{2}[q_p(x_{n-1},x_n)+q_p(x_n,x_{n+1})]\le q_p(x_n,x_{n+1}).$$

So,

$$\begin{aligned} q_p(x_{n+1},x_n) &\le \mu_c([q_p(x_n,x_{n-1})]^q[q_p(x_n,x_{n+1})]^p[q_p(x_{n-1},x_n)]^r[q_p(x_n,x_{n+1})]^{(1-p-q-r)} \\ &= \mu_c[q_p(x_n,x_{n-1})]^{q+r}[q_p(x_n,x_{n+1})]^{1-q-r}. \end{aligned} \tag{3}$$

Particularly, as $\mu_c(t)<t$ contradicting the assumption previously made, it results

$$[q_p(x_n,x_{n+1})]^{q+r}\le[q_p(x_{n-1},x_n)]^{q+r}.$$

Therefore,

$$[q_p(x_n,x_{n+1})]\le[q_p(x_{n-1},x_n)], \tag{4}$$

for all $n\ge 1$.

Hence, the positive sequence $\{q_p(x_{n-1},x_n)\}$ is decreasing. Now, let

$$lim_{n\to\infty}q_p(x_{n-1},x_n)=l.$$

From (3.3), we get

$$\begin{aligned} [q_p(x_{n-1},x_n)]^{1-r}[q_p(x_n,x_{n+1})]^r &\le [q_p(x_{n-1},x_n)]^{1-r}[q_p(x_{n-1},x_n)]^r \\ &\le [q_p(x_{n-1},x_n)]. \end{aligned} \tag{5}$$

From (3.2) and (3.4)

$$q_p(x_{n+1},x_n)\le\mu_c([q_p(x_{n-1},x_n)]^{1-r}[q_p(x_n,x_{n+1})]^r), i.e.,$$
$$q_p(x_{n+1},x_n)\le\mu_c([q_p(x_{n-1},x_n)].$$

Hence,

$$q_p(x_n,x_{n+1})\le\mu_c(q_p(x_{n-1},x_n)).$$

Repeating this argument, we get

$$q_p(x_n,x_{n+1})\le\mu_c^2(q_p(x_{n-2},x_{n-1}))\le\cdots\le\mu_c^n(q_p(x_0,x_1)). \tag{6}$$

Taking $n\to\infty$ in (3.5) and using $\lim_{n\to\infty}\mu_c^n(t)=0$, for each $t>0$ we find that

$$\lim_{n\to\infty}q_p(x_n,x_{n+1})=0.$$

Therefore, $l=0$. We will show that $\{x_n\}$ is a Cauchy Sequence , i.e.,

$$\lim_{n\to\infty}q_p(x_n,x_{n+p})=0,$$

for all $p\in N$. From (3.5), we get

$$q_p(x_n,x_{n+p})\le[\mu_c^n+\mu_c^{n+1}+\cdots+\mu_c^{n+p-1}]q_p(x_0,x_1)\le\sum_{i=n}^{n+p+1}\mu_c^i(q_p(x_0,x_1)),$$

which implies that

$$q_p(x_n,x_{n+p})\le\sum_{i=n}^{\infty}\mu_c^iq_p(x_0,x_1)). \tag{7}$$

From (3.6),

$$q_p(x_{n+m},x_{n+m+p})\le\sum_{i=n}^{\infty}\mu_c^iq_p(x_n,x_{n+1}). \tag{8}$$

Let $n \to \infty$ in (3.7), we get

$$\lim_{n\to\infty} q_p(x_n, x_{n+p}) = \lim_{n\to\infty, m\to\infty} q_p(x_{n+m}, x_{m+m+p}) = 0.$$

So, the sequence $\{x_n\}$ is a Cauchy sequence. Also $(\mathscr{U}, q_p)$ is complete. So, there exists $x \in \mathscr{U}$ such that $\lim_{n\to\infty} q_p(x_n, x) = 0$.

As τ is continuous,

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \tau x_n = \tau(\lim_{n\to\infty} x_n) = \tau x.$$

Hence, τ has a fixed point.

Theorem 3.2. *Let a self-mapping $\tau : \mathscr{U} \to \mathscr{U}$ is $w-$orbital admissible and forms $w-$interpolative Hardy-Rogers contraction on a quasi-partial metric space $(\mathscr{U}, q_p)$ and suppose condition $\mathscr{H}$ is satisfied. If there exists $x_0 \in \mathscr{U}$ such that $w(x_0, \tau x_0) > 1$, then τ has a fixed point.*

Proof. From Theorem 3.1, we can conclude that the constructed sequence $\{x_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} q_p(x_n, x) = 0$ holds. Let $x \neq \tau x$.

We are going to prove this statement using contradiction.

So, assume that $x_{n(k)} \neq \tau x_{n(k)}$, for each $k \geq 0$. By $\mathscr{H}-$condition, there is a partial sub sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $w(x_{n(k)}, x) \geq 1$ for all k. Since,

$$\{q_p(x_{n(k)}, x)\} \to 0, \{q_p(x_{n(k)}, \tau x_{n(k)})\} \to 0$$

As $x = \tau x$ as seen in Theorem 1, $q_p(x, \tau x) > 0$, there exists $n \in N$, for all $k \geq N$.

$q_p(x_{n(k)}, x) \leq q_p(x, \tau x)$ and $q_p(x_{n(k)}, \tau x_{n(k)}) \leq q_p(x, \tau x)$.

Taking $x = x_{n(k)}, y = x$ in equation (3.1),

$$q_p(x_{n(k)+1}, \tau x) \leq w(x_{n(k)}, x) q_p(\tau x_{n(k)}, \tau x)$$

$$\leq \mu_c([q_p(x_{n(k)}, x)]^q [q_p(x_{n(k)}, \tau x_{n(k)})]^p [q_p(x, \tau x)]^r [q_p(x_{n(k)}, \tau x) + q_p(x, \tau x_{n(k)})^{(1-p-q-r)}]. \quad (3.8)$$

As μ_c is non-decreasing from (3.8),

$$\begin{aligned} q_p(x_{n(k)+1}, \tau x) &\leq \mu_c([q_p(x, \tau x)]^q [q_p(x, \tau x)]^p [q_p(x, \tau x)]^r [q_p(x, \tau x) + q_p(x, \tau x]^{(1-p-q-r)} \\ &\leq \mu_c([q_p(x, \tau x)]^q [q_p(x, \tau x)]^p [q_p(x, \tau x)]^r [(q_p(x, \tau x) + q_p(x, \tau x)]^{(1-p-q-r)} \\ &\leq \mu_c([q_p(x, \tau x)]^q [q_p(x, \tau x)]^p [q_p(x, \tau x)]^r [q_p(x, \tau x) + q_p(x, \tau x)]^{(1-p-q-r)}. \end{aligned}$$

So, $q_p(x_{n(k)+1}, \tau x) = \mu_c[q_p(x, \tau x)]$. Let $k \to \infty$, we get

$$0 < q_p(x, \tau x) \leq \mu_c(q_p(x, \tau x)) < q_p(x, \tau x),$$

which is a contradiction. Therefore, $x = \tau x$.

Theorem 3.3. *Let $\tau : \mathscr{U} \to \mathscr{U}$ be $w-$orbital admissible and $w-$interpolative Kannan-type contraction mapping on a complete quasi-partial metric space $(\mathscr{U}, q_p)$. Assuming that either τ is continuous on $(\mathscr{U}, q_p)$ or (H) holds, if there exists $x_0 \in \mathscr{U}$ so that $w(x_0, \tau x_0) \geq 1$, then there exists a fixed point of $\tau \in \mathscr{U}$.*

Proof. This can be proved using Theorem 3.2.

Corollary 3.1. *Let τ be a self-mapping on a complete quasi-partial metric space $(\mathscr{U}, q_p)$ such that*

$$q_p(\tau x, \tau y) \leq \mu_c \cdot \tau([q_p(x,y)]^q \cdot [q_p(x,\tau y)]^p \cdot [q_p(y,\tau y)]^r \cdot [q_p(x,\tau y) + q_p(y,\tau x)]^{1-p-q-r}$$

for all $x, y \in \mathscr{U}$ where $p, q, r > 0, p+q+r < 1$. Then τ has a fixed point.

Proof. substitute $w(x,y) = 1$ in Theorem 3.2.

Corollary 3.2. *Let τ be a self-mapping on a complete quasi-partial metric space $(\mathscr{U}, q_p)$ such that*

$$q_p(\tau x, \tau y) \leq \mu_c \cdot \tau([q_p(x,y)]^q \cdot [q_p(x,\tau y) + q_p(y,\tau x)],^{1-q}$$

for all $x, y \in \mathscr{U}$ where $0 < q < 1$. Then τ has a fixed point.

Proof. By taking $w(x,y) = 1$ and $p = r = 0$ in Theorem 3.2, the above corollary can be proved.

Corollary 3.3. *Let τ be a self-mapping on a complete quasi-partial metric space such that*

$$q_p(\tau x, \tau y) \leq \alpha \cdot \tau([q_p(x,y)]^q \cdot [q_p(x,\tau y)]^p \cdot [q_p(y,\tau y)]^r \cdot [q_p(x,\tau y) + q_p(y,\tau x)]^{1-p-q-r}$$

for all $x, y \in \mathscr{U}$ where $p+q+r < 1$ and $\mu_c \in [0,1)$. Then τ has a fixed point $\in \mathscr{U}$.

Proof. Take $\mu_c(t) = \alpha t$, where $\alpha \in [0,1)$ in Corollary 3.9 to prove the result.

$G-$**condition** Consider a partially-ordered quasi-partial metric space $(\mathscr{U}, q_p, \preccurlyeq)$ and consider the following condition- if $\{x_n\}$ is a sequence in $\mathscr{U}$ such that $x_n \preccurlyeq x_{(n+1)}$, for each n and $x_n \to x \in \mathscr{U}$ As $n \to \infty$, then there exists a sub sequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preccurlyeq x$ for each k.

Corollary 3.4. *Consider a Complete partially-ordered quasi-partial metric space $(\mathscr{U}, \partial, \preccurlyeq)$. Let $\tau : \mathscr{U} \to \mathscr{U}$ be a self-mapping such that*

$$w(x,y) \cdot \tau(q_p(\tau x, \tau y)) \leq \mu_c \cdot \tau([q_p(x,y)]^q \cdot [q_p(x,\tau y)]^p \cdot [q_p(y,\tau y)]^r \cdot [q_p(x,\tau y) + q_p(y,\tau x)]^{1-p-q-r}$$

for all $x, y \in \mathscr{U}$ with $x \preccurlyeq y$, where $\mu_c \in \theta, p, q, r > 0$ are positive reals such that $p+q+r < 1$. Assume that
(a) τ is non-decreasing with respect to $\preccurlyeq$.
(b) There exists $x_0 \in \mathscr{U}$ such that $x_0 \preccurlyeq \tau x_0$.
(c) Either τ is continuous or (G) holds.

Corollary 3.5. *Consider a Complete partially-ordered quasi-partial metric space $(\mathscr{U}, \partial, \preccurlyeq)$. Let $\tau : \mathscr{U} \to \mathscr{U}$ be a self-mapping such that*

$$q_p(\tau x, \tau y) \leq \mu_c \cdot \tau([q_p(x,y)]^q \cdot [q_p(x,\tau y) + q_p(y,\tau x)],^{1-q}$$

for all $x, y \in \mathscr{U}$ where $0 < q < 1$. Then τ has a fixed point. Assume that
(a) τ is non-decreasing with respect to $\preccurlyeq$.
(b) There exists $x_0 \in \mathscr{U}$ such that $x_0 \preccurlyeq \tau x_0$.
(c) Either τ is continuous or (G) holds.

Example 3.1. *Let $(\mathscr{U}, p)$ be a quasi-partial metric space, where:*

(1) $\mathscr{U} = \mathbb{R}^+$ *(set of non-negative real numbers),*
(2) $q_p(x,y) = |x-y| + \min(x,y)$*, which satisfies the properties of a quasi-partial metric space:*
- $q_p(x,x) = 2x$,
- $q_p(x,y) \geq 0$,
- $q_p(x,y) \leq q_p(x,z) + p(z,y) - p(z,z)$.

Define a self-mapping $\tau : \mathscr{U} \to \mathscr{U}$ *as:*

$$\tau(x) = \frac{x}{3}.$$

Define an admissible relation $w : \mathscr{U} \times \mathscr{U} \to \{0,1\}$ *by:*

$$w(x,y) = 1 \quad \text{if and only if} \quad x \leq y.$$

Now verify the Interpolative Hardy-Rogers type contraction with constants:

$$\mu_c = 0.5, \quad p = 0.2, \quad q = 0.2, \quad r = 0.1, \quad \text{where } p+q+r = 0.5 < 1.$$

For verification of the Contraction Condition: We check that for any $x, y \in \mathscr{U}$*:*

$$p(\tau(x), \tau(y)) \leq \mu_c q_p(x,y) + p q_p(x, \tau(x)) + q q_p(y, \tau(y)) + r[q_p(x, \tau(y)) + q_p(y, \tau(x))].$$

(1) *Left-Hand Side:*

$$p(\tau(x), \tau(y)) = \left|\frac{x}{3} - \frac{y}{3}\right| + \min\left(\frac{x}{3}, \frac{y}{3}\right).$$

(2) *Right-Hand Side:*
- $q_p(x,y) = |x-y| + \min(x,y)$,
- $q_p(x, \tau(x)) = \left|x - \frac{x}{3}\right| + \min\left(x, \frac{x}{3}\right) = \frac{2x}{3} + \frac{x}{3} = x$,
- $q_p(y, \tau(y)) = y$,
- $q_p(x, \tau(y)) = \left|x - \frac{y}{3}\right| + \min\left(x, \frac{y}{3}\right)$,
- $q_p(y, \tau(x)) = \left|y - \frac{x}{3}\right| + \min\left(y, \frac{x}{3}\right)$.

Substituting these into the right-hand side:
$\mu_c q_p(x,y) = 0.5 \cdot (|x-y| + \min(x,y))$, $\quad p q_p(x, \tau(x)) = 0.2 \cdot x$, $\quad q q_p(y, \tau(y)) = 0.2 \cdot y$, $\quad r[q_p(x, \tau(y)) + q_p(y, \tau(x))]$.

(3) *For verification let* $x = 6, y = 3$ *and each term is as follows :*
- $q_p(x,y) = |6-3| + \min(6,3) = 3+3 = 6$,
- $q_p(x, \tau(x)) = |6-2| + \min(6,2) = 4+2 = 6$,
- $q_p(y, \tau(y)) = |3-1| + \min(3,1) = 2+1 = 3$,
- $q_p(x, \tau(y)) = |6-1| + \min(6,1) = 5+1 = 6$,
- $q_p(y, \tau(x)) = |3-2| + \min(3,2) = 1+2 = 3$.

As $p(\tau(x), \tau(y)) = \left|\frac{6}{3} - \frac{3}{3}\right| + \min\left(\frac{6}{3}, \frac{3}{3}\right) = |2-1| + 1 = 1+1 = 2$*, the right-hand side is* $\mu_c q_p(x,y) = 0.5 \cdot 6 = 3$, $p q_p(x, \tau(x)) = 0.2 \cdot 6 = 1.2$, $\quad q q_p(y, \tau(y)) = 0.2 \cdot 3 = 0.6$, $\quad r[q_p(x, \tau(y)) + q_p(y, \tau(x))] = 0.1 \cdot (6+3) = 0.9$. *On adding these value, we have* $3 + 1.2 + 0.6 + 0.9 = 5.7$. *Since* $p(\tau(x), \tau(y)) = 2 \leq 5.7$*, the inequality holds. The fixed point satisfies* $\tau(x) = x$*, i.e.,* $\frac{x}{3} = x$*. Solving gives* $x = 0$*. This example demonstrates that the mapping* $\tau(x) = x/3$ *satisfies the Interpolative Hardy-Rogers type contraction via* w*-admissibility. Also, the quasi-partial metric space* $(\mathscr{U}, p)$ *supports the fixed point theorem. Further, the unique fixed point of* τ *is* $x = 0$.

4. Application : IoT-Based Temperature Regulation in Smart Buildings

Smart buildings rely on IoT devices (such as sensors, thermostats, and HVAC systems) to regulate internal temperatures efficiently. These devices operate in dynamic environments where communication delays, asymmetric dependencies, and incomplete data often arise. This makes a quasi-partial metric space a natural framework for modeling their interactions. We propose using an Interpolative Hardy-Rogers type contraction via w-admissibility to ensure the existence and uniqueness of stable temperature control solutions. This approach provides a robust mathematical framework for modeling the iterative adjustments of IoT devices and proving their convergence to a uniform stable temperature. Let $(\mathscr{U}, p)$ be a quasi-partial metric space, where: $\mathscr{U} = [T_{\min}, T_{\max}] \subset \mathbb{R}$ is the set of permissible temperature values for the IoT devices, $p : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ is defined as:

$$p(x, y) = |x - y| + \min(x, y),$$

satisfying $p(x, x) = 2x$, $p(x, x) \leq p(x, y)$, $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$. The temperature adjustment logic of the IoT devices is modeled as a self-mapping $\tau : \mathscr{U} \to \mathscr{U}$:

$$\tau(T_i) = \frac{T_i + T_{\text{avg}}}{2},$$

where T_i is the temperature of the i-th device at the current iteration and $T_{\text{avg}} = \frac{1}{N}\sum_{i=1}^{N} T_i$ is the average temperature across all N devices. Define a relation $w : \mathscr{U} \times \mathscr{U} \to \{0, 1\}$ such that:

$$w(x, y) = 1 \quad \text{if and only if} \quad |x - y| \leq \delta,$$

where $\delta > 0$ is a threshold representing the maximum permissible temperature difference for devices to interact. The mapping τ satisfies the Interpolative Hardy-Rogers type contraction

$$p(\tau(x), \tau(y)) \leq \mu_c p(x, y) + pp(x, \tau(x)) + qp(y, \tau(y)) + r[p(x, \tau(y)) + p(y, \tau(x))],$$

for constants:

$$\mu_c = 0.5, \quad p = 0.2, \quad q = 0.2, \quad r = 0.1, \quad p + q + r = 0.5 < 1.$$

Consider $N = 3$ IoT devices with initial temperatures:

$$T_1 = 24°\text{C}, \quad T_2 = 22°\text{C}, \quad T_3 = 20°\text{C}.$$

The admissibility threshold is $\delta = 5°$C, and the permissible range is $T_{\min} = 18°$C to $T_{\max} = 26°$C.

1. First Iteration: Compute the average temperature:

$$T_{\text{avg}} = \frac{T_1 + T_2 + T_3}{3} = \frac{24 + 22 + 20}{3} = 22°\text{C}.$$

Update the temperatures using $\tau(T_i)$:

$$\tau(T_1) = \frac{24 + 22}{2} = 23°\text{C}, \quad \tau(T_2) = \frac{22 + 22}{2} = 22°\text{C}, \quad \tau(T_3) = \frac{20 + 22}{2} = 21°\text{C}.$$

$p(T_1, T_2) = |24 - 22| + \min(24, 22) = 2 + 22 = 24$ and $p(\tau(T_1), \tau(T_2)) = |23 - 22| + \min(23, 22) = 1 + 22 = 23$. Verify the contraction inequality

$$p(\tau(T_1), \tau(T_2)) = 23 \leq 0.5 \cdot 24 + 0.2 \cdot 24 + 0.2 \cdot 24 + 0.1 \cdot (24 + 24) = 23.4.$$

The inequality holds.

2. Second Iteration: $T_{\text{avg}} = 22°$C and Update temperatures as

$$\tau(T_1) = \tau(T_2) = \tau(T_3) = 22°\text{C}.$$

All devices converge to the stable temperature $T_i = 22°C$.

5. Conclusion

The Interpolative Hardy-Rogers type contraction via w-admissibility in the framework of a quasi-partial metric space provides a robust mathematical tool for analyzing systems with partial dependencies and asymmetric interactions. By ensuring the existence and uniqueness of fixed points, this approach is particularly well-suited for real-world applications such as IoT-based temperature regulation in smart buildings. The admissibility condition (w) ensures valid device interactions, while the contraction guarantees convergence to a stable equilibrium. This framework not only extends fixed-point theory to more generalized settings but also demonstrates its practical relevance in solving dynamic and complex problems across various fields.

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