TWMS J. App. and Eng. Math. V.16, N.4, 2026, pp. 521-535

ECCENTRICITY SPECTRA OF SOME GRAPH OPERATIONS IN REGULAR GRAPHS

SURYA S[1,*], P. RAMACHANDRAN[1], §

Abstract. The eccentricity matrix of a graph G is derived from its distance matrix by letting the ij^{th} entry be equal to the distance between two vertices i and j, if the distance is the minimum of their eccentricities and zero otherwise. The eigenvalues of the eccentricity matrix of G are called ε-eigenvalues. Its ε-spectrum is the set of ε-eigenvalues together with its multiplicity and ε-energy is the sum of the absolute values of the ε-eigenvalues. In this paper, we study the ε-spectra of certain operations on regular graphs. We also established some bounds on ε-energy of graphs and characterize the extreme graphs.

Keywords: eccentricity matrix, distance matrix, spectrum, energy, eigenvalue.

AMS Subject Classification: 05C50, 05C76.

1. Introduction

Throughout this paper, we consider simple, connected, undirected graphs without loops or parallel edges. Let $G = (V(G), E(G))$ be a graph with order $n = |V(G)|$. For a graph G there are several matrices with it, including adjacency matrix $A(G)$, Laplacian matrix $L(G)$, distance matrix $D(G)$ and many more. The distance $d(u, v)$ between two vertices u and v is the minimum length of the paths connecting them. The eccentricity of a vertex $u \in V(G)$ is defined as $\varepsilon(u) = max\{d(u, v) : v \in V(G)\}$. The radius $rad(G)$ and diameter $diam(G)$ are defined as $rad(G) = min\{\varepsilon(u) : u \in V(G)\}$ and $diam(G) = max\{\varepsilon(u) : u \in V(G)\}$. The maximum degree $\max_{v \in V(G)} d(v)$ of a graph G is denoted by $\triangle(G)$. For a graph G its adjacency matrix is an $n \times n$ matrix defined as $A(G) = (a_{ij})$ where $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. The distance matrix $D(G) = (d_{ij})$ is defined as $d_{ij} = d(v_i, v_j)$. These matrices have been widely studied and have applications in Chemistry, Physics, Computer science, etc., see[2][3].For a real number x, $[x]$ denotes the integer part of x or greatest integer less than or equal to x. In 2013 Randić [6] introduced a new graph matrix called D_{MAX} matrix, which is obtained

[1] Department of Mathematics, St. Paul's College, Kalamassery, 683503, Kerala, India.
e-mail: suryas1116@gmail.com; ORCID: 0009-0003-7976-4299.
e-mail: pramada@stpauls.ac.in; ORCID: 0000-0001-8647-0000.
[*] Corresponding author.
§ Manuscript received: March 13, 2025; accepted: June 23, 2025.

from the distance matrix $D(G)$ by retaining the largest distance in each row and each column and setting the rest of the entries of $D(G)$ to zero. Later Wang et al. [5] renamed it as eccentricity matrix $\varepsilon(G)$. The eccentricity matrix of a graph G is defined as follows

$$\epsilon_{ij} = \begin{cases} d(v_i, v_j) & \text{if } d(v_i, v_j) = min\,\{\varepsilon(v_i), \varepsilon(v_j)\} \\ 0 & \text{otherwise} \end{cases}$$

Note that $\varepsilon(G)$ is a non-negative real symmetric matrix with diagonal entries 0. To distinguish the eigenvalues corresponds to various graph matrices, the eigenvalues of $A(G)$ and $D(G)$ are denoted by A-eigenvalues and D-eigenvalues, respectively. The eigenvalues corresponding to $\varepsilon(G)$ are denoted by ε-eigenvalues and the corresponding spectrum is the ε-spectrum. Let $\xi_1 > \xi_2 > ... > \xi_k$ be the distinct ε-eigenvalues. Then the ε-spectrum is given by

$$Spec_{\varepsilon}(G) = \begin{Bmatrix} \xi_1 & \xi_2 & ... & \xi_k \\ m_1 & m_2 & ... & m_k \end{Bmatrix}$$

where m_i is the multiplicity of each eigenvalue $1 \leq i \leq k$. The energy is defined by

$$E_{\varepsilon}(G) = \sum_{i=1}^{n} \mid \xi_i \mid$$

Unlike the adjacency and distance matrices of a connected graph, the eccentricity matrix is not always irreducible [5]. The eccentricity matrix of a complete bipartite graph on n vertices with maximum degree less than $n-1$ is reducible while the eccentricity matrices of trees of order $n \geq 2$ are irreducible [5]. We can find application of eccentricity matrix on the branching pattern of molecular graphs[4], in terms of molecular descriptor [5] [6] and in the study of boiling point of hydrocarbons [7]. Motivated by the concepts and results of other graph matrices, several spectral properties have been studied for the eccentricity matrix in literature; see[4]-[10]

For a connected graph G of order n with $diam(G) = 2$ and $\triangle(G) < n-1$, the entries of its eccentricity matrix are given by $\epsilon_{ij} = 2$ if $v_iv_j \in E(\overline{G})$ and $\epsilon_{ij} = 0$ if $v_iv_j \in E(G)$, where $\overline{G}$ is the complement of G, then $\varepsilon(G) = 2A(\overline{G})$. The relation between the eigenvalues of $A(G)$ and $E(G)$ has been investigated for certain graphs in [10]. Motivated by the above works we continue to investigate the relation between eigenvalues of $A(G)$ and $E(G)$ in some graph operations. Moreover we also calculate some bounds of the energy of eccentric matrix of a graph G.

This paper is organised as follows: Section 2 contains some definitions and previously known results. In section 3, we determine ε-spectra of the Cartesian product, strong product, splice and link of regular graphs. Moreover we calculated ε-spectra of Mycielskian graph, double graph and strong double graph of regular graphs. In section 4, we determine some bounds of ε-energy and characterize the extreme graphs.

2. Preliminaries

Definition 2.1. [11] *Let $M = [M_{ij}]$ be a complex block matrix of order n, where the blocks M_{ij} are $n_i \times n_j$ matrices for any $1 \leq i, j \leq t$. That is, the $n \times n$ matrix M has t-row partitions and t-column partitions. Let q_{ij} denote the average row sum of M_{ij} for $1 \leq i, j \leq t$. Then $Q_M = (q_{ij})$(simply Q)is called quotient matrix of M. In addition, for each pair (i, j), if M_{ij} has a constant row sum, then Q is called equitable quotient matrix of M.*

Lemma 2.1. [11] *Let Q be a quotient matrix of a square matrix M corresponding to an equitable partition. Then the spectrum of M contains the spectrum of Q.*

Lemma 2.2. [1] *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. If $Spec(A)$ and $Spec(B)$ are the spectra of matrices A and B, respectively then the spectrum of their Kronecker product $A \otimes B$ is given by $SpecA \otimes B = \{\lambda\mu : \lambda \in Spec(A),\ \mu \in Spec(B)\}$.*

Lemma 2.3. [22] *Let $B = \begin{pmatrix} B_0 & B_1 \\ B_1 & B_0 \end{pmatrix}$ be a symmetric 2×2 block matrix. Then the spectrum of B is the union of the spectra of $B_0 + B_1$ and $B_0 - B_1$.*

Lemma 2.4. [12] *If the A-eigenvalues of an r-regular graph G with order n are $r, \lambda_2, ..., \lambda_n$, then the A-eigenvalues of $\overline{G}$ are $n - r - 1, -(\lambda_2 + 1), ..., -(\lambda_n + 1)$.*

Lemma 2.5. [13] *Let G and H be two connected graphs and let $(u, v) \in V(G) \times V(H)$. Let $G\square H$ denote their Cartesian product. Then $e_{G\square H}(u, v) = e_G(u) + e_H(v)$.*

Lemma 2.6. [14] *Let G and H be two connected graphs and $G\square H$ denote their Cartesian product. Let $u = (u_1, u_2); v = (v_1, v_2) \in V(G) \times V(H)$, then $d_{G\square H}(u, v) = d_G(u_1, v_1) + d_H(u_2, v_2)$.*

Lemma 2.7. [16] *Let $x_1, x_2, ..., x_N$ be non-negative numbers and let $\alpha = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\gamma = \left(\prod_{i=1}^{N} x_i\right)^{\frac{1}{N}}$ be their arithmetic and geometric means. Then*

$$\frac{1}{N(N-1)}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2 \leq \alpha - \gamma \leq \frac{1}{N}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2$$

and equality holds if $x_1 = x_2 = \cdots = x_N$.

Lemma 2.8. *(Cauchy Interlace Theorem)*[1] *Let B be a $p \times p$ symmetric matrix and let B_k be its leading $k \times k$ submatrix, that is, B_k is a matrix obtained from B by deleting its last $p - k$ rows and columns. Then for $i = 1, 2, ..., k$*

$$\rho_{p-i+1}(B) \leq \rho_{k-i+1}(B_k) \leq \rho_{k-i+1}(B)$$

where $\rho_i(B)$ is the i^{th} largest eigenvalues of B.

3. On Regular graphs with diameter at most 2

Almost all connected graphs of order n have diameter 2 and $\triangle < n-1$[10]. For a regular graph with diameter 2 and $\triangle < n - 1$; $\varepsilon(G) = A(\overline{G})$[5]. In subsection 3.1 we find the ε-spectra of the Cartesian product and strong product of regular graphs. In subsection 3.2 we study ε-spectra of splice and link of regular graphs and subsection 3.2 explores the ε-spectra of some graph operations.

3.1. ε-spectra of Graph Products.

Here regular graphs with diameter 2 and $\triangle < n-1$ are considered.

Definition 3.1. ***Cartesian product*** [12] *The Cartesian product of two graphs G and H denoted by $G\square H$ is a graph with vertex set $V(G) \times V(H)$. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G\square H$ if and only if either $u_1 = v_1$ and $u_2v_2 \in V(H)$ or $u_2 = v_2$ and $u_1v_1 \in V(G)$.*

Theorem 3.1. *Let G be a r-regular graph with order n and $\lambda_i \in Spec(A(\overline{G}))$ then the ε-spectrum of $G\ \square P_m$ is given by*
(i) If $m = 2k$, then $Spec(\varepsilon(G\square P_{2k})) =$

$$\left\{0, \lambda_i(\frac{-(2k+1)+\sqrt{s}}{2}), \lambda_i(\frac{-(2k+1)-\sqrt{s}}{2}), \lambda_i(\frac{(2k+1)+\sqrt{s}}{2}), \lambda_i(\frac{(2k+1)-\sqrt{s}}{2})\right\}$$

where $s = \sqrt{\dfrac{56k^3 + 12k^2 - 20k + 8}{6}}$.

(ii) If $m = 2k + 1$*, then* $Spec(\varepsilon(G\square P_{2k+1})) =$

$$\left\{0, \lambda_i(\frac{-(2k+2)+\sqrt{t_1}}{2}), \lambda_i(\frac{-(2k+2)-\sqrt{t_1}}{2}), \lambda_i(\frac{(2k+2)+\sqrt{t_2}}{2}), \lambda_i(\frac{(2k+2)-\sqrt{t_2}}{2})\right\}$$

where $t_1 = \sqrt{\dfrac{28k^3 + 54k^2 + 2k - 36}{3}}$ *and* $t_2 = \sqrt{\dfrac{28k^3 + 78k^2 + 98k + 60}{3}}$.

Proof. (i)By suitable labelling on vertices of $G\,\square P_{2k}$ the eccentricity matrix of $G\,\square P_{2k}$ is given by

$$\varepsilon(G\square P_{2k}) = Q \otimes A(\overline{G})$$

where

$$Q = \begin{bmatrix} 0 & 0 & . & . & . & k+2 & . & . & . & 2k & 2k+1 \\ 0 & 0 & . & . & . & 0 & . & . & . & 0 & 2k \\ \vdots & \vdots & & & & \vdots & & & & \vdots & \vdots \\ 0 & 0 & & \cdots & & & & \cdots & & 0 & k+2 \\ k+2 & 0 & & \cdots & & & & \cdots & & 0 & \\ \vdots & \vdots & & & & \vdots & & & & \vdots & \vdots \\ 2k & 0 & . & . & . & & . & . & . & 0 & 0 \\ 2k+1 & 2k & . & . & . & k+2 & . & . & . & 0 & 0 \end{bmatrix} \tag{1}$$

Let $\xi \neq 0$ be an eigenvalue of Q with eigenvector $x^t = (x_1, x_2, ..., x_{2k})$ such that $Qx = \xi x$. Then

$$(k+2)x_{k+1} + (k+3)x_{k+2} + \cdots + 2kx_{2k-1} + (2k+1)x_2k = \xi x_1 \tag{2}$$

$$\begin{cases} 2kx_{2k} & = \xi x_2 \\ (2k-1)x_{2k} & = \xi x_3 \\ & \vdots \\ (k+2)x_{2k} & = \xi x_k \end{cases} \tag{3}$$

$$\begin{cases} (k+2)x_1 & = \xi x_{k+1} \\ (k+3)x_1 & = \xi x_{k+2} \\ & \vdots \\ 2kx_1 & = \xi x_{2k-1} \end{cases} \tag{4}$$

$$(2k+1)x_1 + 2kx_2 + \cdots + (k+2)x_k = \xi x_{2k} \tag{5}$$

Combining (1) and (3), (2) and (4), we get

$$\left(\xi^2 - \left[(k+2)^2 + (k+3)^2 + \cdots + (2k)^2\right]\right) x_1 = \xi(2k+1)x_{2k} \tag{6}$$

$$\left(\xi^2 - \left[(k+2)^2 + (k+3)^2 + \cdots + (2k)^2\right]\right) x_{2k} = \xi(2k+1)x_1 \tag{7}$$

Assume $x_1 = 0$, then by (2),(3) and (5) we get $x_2 = \cdots = x_{2k} = 0$. Hence $x_1 \neq 0$, similarly $x_{2k} \neq 0$.
Together with (5) and (6) and $\left[(k+2)^2 + (k+3)^2 + \cdots + (2k)^2\right] = \dfrac{(k-1)(14k^2+17k+6)}{6}$ we get

$$\xi^2 \pm (2k+1)\xi - \frac{(k-1)(14k^2+17k+6)}{6} = 0 \tag{8}$$

Therefore, $Spec(Q) =$

$$\left\{\begin{matrix} \frac{-(2k+1)+\sqrt{s}}{2} & \frac{-(2k+1)-\sqrt{s}}{2} & \frac{(2k+1)+\sqrt{s}}{2} & \frac{(2k+1)-\sqrt{s}}{2} & 0 \\ 1 & 1 & 1 & 1 & 2k-4 \end{matrix}\right\}$$

where $s = \sqrt{\dfrac{56k^3+12k^2-20k+8}{6}}$. Hence by Lemma 2.2, $Spec(\varepsilon(G\Box P_{2k}))$ is given by

$$\left\{0, \lambda_i(\frac{-(2k+1)+\sqrt{s}}{2}), \lambda_i(\frac{-(2k+1)-\sqrt{s}}{2}), \lambda_i(\frac{(2k+1)+\sqrt{s}}{2}), \lambda_i(\frac{(2k+1)-\sqrt{s}}{2})\right\}$$

where $\lambda_i \in spec(A(\overline{G})$.
(ii) We have

$$\varepsilon(G\Box P_{2k+1}) = Q' \otimes A(\overline{G})$$

where

$$Q' = \begin{bmatrix} 0 & 0 & . & . & . & k+2 & . & . & . & 2k+1 & 2k+2 \\ 0 & 0 & . & . & . & 0 & . & . & . & 0 & 2k+1 \\ \vdots & \vdots & & & & \vdots & & & & \vdots & \vdots \\ k+2 & 0 & & \cdots & & & & \cdots & & & k+2 \\ \vdots & \vdots & & & & \vdots & & & & \vdots & \vdots \\ 2k+1 & 0 & . & . & . & & . & . & . & 0 & 0 \\ 2k+2 & 2k+1 & . & . & . & k+2 & . & . & . & 0 & 0 \end{bmatrix} \tag{9}$$

Proceeding as in (i), we get $Spec(Q') =$

$$\left\{\begin{matrix} \frac{-(2k+2)+\sqrt{t_1}}{2} & \frac{-(2k+2)-\sqrt{t_1}}{2} & \frac{(2k+2)+\sqrt{t_2}}{2} & \frac{(2k+2)-\sqrt{t_2}}{2} & 0 \\ 1 & 1 & 1 & 1 & 2k-3 \end{matrix}\right\}$$

where $t_1 = \sqrt{\dfrac{28k^3+54k^2+2k-36}{3}}$ and $t_2 = \sqrt{\dfrac{28k^3+78k^2+98k+60}{3}}$.
Hence for $\lambda_i \in spec(A(\overline{G})$, $Spec(\varepsilon(G\Box P_{2k+1}))$ is given by

$$\left\{0, \lambda_i(\frac{-(2k+2)+\sqrt{t_1}}{2}), \lambda_i(\frac{-(2k+2)-\sqrt{t_1}}{2}), \lambda_i(\frac{(2k+2)+\sqrt{t_2}}{2}), \lambda_i(\frac{(2k+2)-\sqrt{t_2}}{2})\right\}$$

□

Theorem 3.2. *Let G be an r- regular graph and H be a k-regular graph with order n and m, respectively. Then* $Spec_\varepsilon(G\Box H) = \{\lambda\mu : \lambda \in Spec_\varepsilon(G),\ \mu \in Spec_\varepsilon(H)\}$. .

Proof. For $u, v \in V(G) \times V(H)$ by Lemma 2.5 and Lemma 2.6

$$\varepsilon(G\Box H) = [\epsilon_{uv}] = \begin{cases} 4 & \text{if } d_{G\Box H}(u,v) = 4 \\ 0 & \text{Otherwise} \end{cases}$$

Let $(a_k, b_s), (a_k, b_r) \in V(G) \times V(H),\ s \neq r$.
Then by Lemma 2.6, $d\left((a_k, b_s), (a_k, b_r)\right) = d(b_s, b_r) \leq min\left\{e(b_s), e(b_r)\right\} = 2$. Similarly $d\left((a_p, b_k), (a_q, b_k)\right)_{p\neq q} \leq 2$. Therefore for $u = (a_i, b_p), v = (a_j, b_q) \in V(G) \times V(H)$,

$$\epsilon_{uv} = \begin{cases} 0 & \text{if } a_i = a_j \text{ or } b_p = b_q \\ 0 & \text{if } a_i \text{ adj } a_j \text{ in G or } \ b_p \text{ adj } b_q \text{ in H} \\ 4 & \text{if } a_i \text{ not adj } a_j \text{ in G and } \ b_p \text{ not adj } b_q \text{ in H} \end{cases} \tag{10}$$

Let $\varepsilon(G) = [a_{ij}]$ and $\varepsilon(H) = [b_{ij}]$ be the eccentricity matrices of G and H. Label the vertices in $G\Box H$ as $(a_1, b_1), ..., (a_n, b_1), ..., (a_1, b_m), ..., (a_n, b_m)$. So the diagonal block matrix of $\varepsilon(G\Box H)$ is $[A]_{ij} = 0_{n\times n}$ by equation (10) and the non-diagonal block matrix corresponding to vertices $(a_1, b_i), ..., (a_n, b_i)$ and $(a_1, b_j), ..., (a_n, b_j)$ $i \neq j$ is given by

$$(A_{ij}) = \begin{cases} 0_{n\times n} \text{ if } b_i \text{ adj } b_j \text{ in H} \\ 2[a_{ij}] \text{ if } b_i \text{ not adj } b_j \text{ in H} \end{cases}$$

Therefore $\varepsilon(G\Box H) = \varepsilon(H) \otimes \varepsilon(G)$. □

Next we discuss the ε-spectrum of regular graphs with $\triangle = n - 1$. For the complete graph K_n , $\varepsilon(K_n) = A(K_n)$. However, while taking the Cartesian product, both matrices are not the same. The next theorem is about the ε-spectrum of Cartesian product of complete graphs.

Theorem 3.3. $\varepsilon(K_n\Box K_m) = 2\varepsilon(K_n) \otimes \varepsilon(K_m)$

Proof. By suitable labelling the eccentricity matrix of $K_n\Box K_m$ is given by

$$\varepsilon(K_n\Box K_m) = \begin{bmatrix} 0 & 2\varepsilon(K_m) & \cdots & 2\varepsilon(K_m) \\ 2\varepsilon(K_m) & 0 & \cdots & 2\varepsilon(K_m) \\ \vdots & \vdots & & \vdots \\ 2\varepsilon(K_m) & 2\varepsilon(K_m) & \cdots & 2\varepsilon(K_m) \end{bmatrix} = 2\varepsilon(K_n) \otimes \varepsilon(K_m)$$

□

Corollary 3.1. $Spec_\varepsilon(K_n\Box K_n) = \left\{2, -2(n-1), 2(n-1)^2\right\}$

Corollary 3.2. $E_\varepsilon(K_n\Box K_n) = 4n(n-1)$

Definition 3.2. ***Strong Product***[20] *The strong product of two graphs G and H denoted by $G \boxtimes H$ is a connected graph with vertex set is $V(G) \times V(H)$. Any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \boxtimes H$ if and only if $u_1 = v_1$ and $u_2v_2 \in V(H)$ or $u_2 = v_2$ and $u_1v_1 \in V(G)$ or $u_1u_2 \in V(G)$ and $v_1v_2 \in V(H)$.*

Lemma 3.1. [20] *Let G and H be graphs. Then for every vertex $(u, v) \in V(G \boxtimes H)$ we have $e_{G\boxtimes H} = max\left\{e_G(u), e_H(v)\right\}$.*

Theorem 3.4. *Let G be an r-regular graph and H be a k-regular graph with order n and m, respectively. If $\lambda_1, ..., \lambda_n$ and $\mu_1, ..., \mu_m$ are eigenvalues of the adjacency matrices of G and H, respectively then*
$Spec(G \boxtimes H) = \{2(r+1)(m-k-1) + 2m(n-r-1), -2(\lambda_i+1)(k+1), (r+1)\mu_j\}$
$i = 1, ..., n$, $j = 1, ..., m$ *and* $2(r+1)(m-k-1)+2m(n-r-1)$ *with multiplicity* $nm-n-m$.

Proof. By Lemma 3.1, $e_{G\boxtimes H}(u,v) = 2 \;;\forall\; (u,v) \in V(G\boxtimes H)$.
Let $u = (a_i, b_p), v = (a_j, b_q) \in V(G) \times V(H)$ the entries of $\varepsilon(G\boxtimes H) = (\epsilon_{uv})$ are given by

$$\epsilon_{uv} = \begin{cases} 0 & \text{if } i = j, b_pb_q \in E(H) \\ 0 & \text{if } p = q, a_ia_j \in E(G) \\ 0 & \text{if } a_ia_j \in E(G), b_pb_q \in E(H) \\ 2 & \text{Otherwise} \end{cases}$$

Label the vertices in $G \boxtimes H$ as $(a_1, b_1), ..., (a_1, b_m), ..., (a_n, b_1), ..., (a_n, b_m)$. The entries in the block diagonal matrix of $\varepsilon(G \boxtimes H)$ corresponding to vertices $(a_i, b_1), ..., (a_i, b_m)$ are given by

$$\epsilon_{(a_i,b_p)(a_i,b_q)} = \begin{cases} 2 & \text{if } b_pb_q \notin E(H) \\ 0 & \text{Otherwise} \end{cases}$$

Likewise, the entries in the block non-diagonal matrix corresponding to the vertices $(a_i, b_1), ..., (a_i, b_m)$ and $(a_j, b_1), ..., (a_j, b_m)$ for $i \neq j$ are given by
(i) If a_i adjacent to a_j in $V(G)$ then

$$\epsilon_{(a_i,b_p)(a_j,b_q)} = \begin{cases} 2 & \text{if } b_pb_q \notin E(H) \\ 0 & \text{Otherwise} \end{cases}$$

(ii) If a_i is not adjacent to a_j in $V(G)$, then

$$\epsilon_{(a_i,b_p)(a_j,b_q)} = 2 \;;\forall\; (a_i, b_p)(a_j, b_q) \in V(G) \times V(H)$$

Therefore

$$\begin{aligned} \varepsilon(G \boxtimes H) &= A(G) \otimes \varepsilon(H) + \varepsilon(G) \otimes J_m + I_n \otimes \varepsilon(H) \\ &= (A(G) + I_n) \otimes \varepsilon(H) + \varepsilon(G) \otimes J_m \end{aligned}$$

Let 1_n and 1_m be $n \times 1$ and $m \times 1$ matrices with all entries one. Then

$$\begin{aligned} &((A(G) + I_n) \otimes \varepsilon(H) + \varepsilon(G) \otimes J_m) (1_n \otimes 1_m) \\ &= ((A(G) + I_n) \otimes \varepsilon(H)) (1_n \otimes 1_m) + (\varepsilon(G) \otimes J_m) (1_n \otimes 1_m) \\ &= (r1_n + 1_n) \otimes 2(m - k - 1)1_m + 2(n - r - 1)1_n \otimes m1_m \\ &= 2(m - k - 1)(r + 1) (1_n \otimes 1_m) + 2m(n - r - 1) (1_n \otimes 1_m) \\ &= (2(r + 1)(m - k - 1) + 2m(n - r - 1)) (1_n \otimes 1_m) \end{aligned}$$

Let X_i be eigenvector corresponding to eigenvalue λ_i of $A(G)$. Then

$$\begin{aligned} &((A(G) + I_n) \otimes \varepsilon(H) + \varepsilon(G) \otimes J_m) (X_i \otimes 1_m) \\ &= (A(G)X_i + I_nX_i) \otimes \varepsilon(H)1_m + \varepsilon(G)X_i \otimes J_m1_m \\ &= (\lambda_i + 1) \otimes 2(m - k - 1)1_m + (-2 (\lambda_i + 1)) X_i \otimes m1_m \\ &= (-2(k + 1) (\lambda_i + 1)) (X_i \otimes 1_m) \end{aligned}$$

Similarly, for an eigenvector Y_i corresponds to eigenvalue μ_i of $\varepsilon(H)$,

$$((A(G) + I_n) \otimes \varepsilon(H) + \varepsilon(G) \otimes J_m) (1_n \otimes Y_i) = ((r + 1)\mu_i) (1_n \otimes Y_i)$$

Therefore $(2(r+1)(m-k-1)+2m(n-r-1), -2(k+1)(\lambda_i+1)$ and $(r+1)\mu_i$ are the eigenvalues of $\varepsilon(G\boxtimes H)$ corresponding to eigenvectors $(1_n\otimes 1_m), (X_i\otimes 1_m)$ and $(1_n\otimes Y_i)$ respectively. □

3.2. ε-spectra of Splice and Link of Graphs.

Definition 3.3. [21] *Suppose G and H are graphs with disjoint vertex sets. For given vertices $u\in V(G)$ and $v\in V(H)$, a splice of G and H by vertices u and v denoted by $(G.H)(u,v)$ is defined by identifying the vertices u and v in the union of G and H. Similarly, a link of G and H by vertices u and v denoted by $(G\sim H)(u,v)$ is defined as the graph obtained by joining u and v by an edge in the union of these graphs.*

Remark 3.1. *If $\mid V(G)\mid = n_1$; $\mid V(H)\mid = n_2$; $\mid E(G)\mid = m_1$ and $\mid E(H)\mid = m_2$ then it follows that $\mid V((G.H)(u,v))\mid = n_1+n_2-1$, $\mid V((G\sim H)(u,v))\mid = n_1+n_2$; $\mid E((G.H)(u,v))\mid = m_1+m_2$, $\mid E((G\sim H)(u,v))\mid = m_1+m_2+1$.*

Theorem 3.5. *Let G be an r-regular graph and H be a k-regular graph with vertices n_1 and n_2, respectively. Then the ε-spectrum of a splice of G and H by vertices u and v, $(G.H)(u,v)$ is the roots of the polynomial*

$$\lambda^4-(16(n-r-1)(m-k-1)+(n-r-1)(9k+4)+(m-k-1)(9k+4))\,\lambda^2- \\ 32(n-r-1)(m-k-1)\lambda+3(n-r-1)(m-k-1)(27kr+12)(k+r))=0$$

Proof. Consider the following partition of the vertex set of $(G.H)(u,v): V_1=\{u\}=\{v\}$; $V_2=\{u_1,...,u_r\}$ are vertices adjacent to u in G, $V_3=\left\{u'_1,...,u'_{n-r-1}\right\}$ are vertices not adjacent to u in G and $V_4=\{v_1,...,v_k\}$ are vertices adjacent to v in H, $V_5=\left\{v'_1,...,v'_{m-k-1}\right\}$ are vertices not adjacent to v in H such that $V(G)=\{u\}\cup V_2\cup V_3$ and $V(H)=\{v\}\cup V_4\cup V_5$. It is clear that $e(u)=e(v)=2$; $e(u_i)=3=e(v_j)$ for $1\le i\le r$; $1\le j\le k$; $e(u'_i)=4=e(v'_j)$ for $1\le i\le n-r-1$; $1\le j\le m-k-1$. The eccentricity matrix of $(G.H)(u,v)$ corresponding to the above partition is given by

$$\varepsilon((G.H)(u,v))=\begin{bmatrix}
0&0&\cdots&0&0&\cdots&0&2&\cdots&2&2&\cdots&2\\
0&0&\cdots&0&0&\cdots&0&0&\cdots&0&3&\cdots&3\\
\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots\\
0&0&\cdots&0&0&\cdots&0&0&\cdots&0&3&\cdots&3\\
0&0&\cdots&0&0&\cdots&0&3&\cdots&3&0&\cdots&0\\
\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots\\
0&0&\cdots&0&0&\cdots&0&3&\cdots&3&0&\cdots&0\\
2&0&\cdots&0&3&\cdots&3&0&\cdots&0&4&\cdots&4\\
\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots\\
2&0&\cdots&0&3&\cdots&3&0&\cdots&0&4&\cdots&4\\
2&3&\cdots&3&0&\cdots&0&4&\cdots&4&0&\cdots&0\\
\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots&\vdots&\cdots&\vdots\\
2&3&\cdots&3&0&\cdots&0&4&\cdots&4&0&\cdots&0
\end{bmatrix}$$

The quotient matrix corresponding to the above matrix is

$$Q=\begin{bmatrix}
0&0&0&2(n-r-1)&2(m-k-1)\\
0&0&0&0&3(m-k-1)\\
0&0&0&3(n-r-1)&0\\
2&0&3k&0&4(m-k-1)\\
2&3r&0&4(n-r-1)&0
\end{bmatrix}$$

Since $rank(\varepsilon((G.H)(u,v))) = rank(Q) = 4$, we see that 0 is an eigenvalue of $\varepsilon((G.H)(u,v))$ with multiplicity $n_1 + n_2 - 5$ and $Spec(\varepsilon((G.H)(u,v)) = Spec(Q)$. The spectrum of Q is given by $det(\lambda I - Q) = 0$
$\Rightarrow \lambda^4 - (16(n-r-1)(m-k-1) + (n-r-1)(9k+4) + (m-k-1)(9k+4))\,\lambda^2 - 32(n-r-1)(m-k-1)\lambda + 3(n-r-1)(m-k-1)(27kr + 12(k+r)) = 0$ □

Theorem 3.6. *Let G be an r-regular graph and H be a k-regular graph with vertices n_1 and n_2, respectively. Then the ε-spectrum of a link of G and H by vertices u and v; $(G \sim H)(u,v)$ is the roots of the polynomial*

$$\lambda^4 - (25(n-r-1)(m-k-1) + 16(k(n-r-1) + r(m-k-1)) + 9(n-r-1+m-k-1))\lambda + (n-r-1)(m-k-1)(256kr + 144(k+r) + 81) = 0$$

Proof. Consider the same partition of the vertex set of $(G \sim H)(u,v)$ as in Theorem 3.5 such that $V(G) = V_0 \cup V_2 \cup V_3$ and $V(H) = V_1 \cup V_4 \cup V_5$. So $e(u) = e(v) = 3$; $e(u_i) = 4 = e(v_j)$ for $1 \le i \le r$; $1 \le j \le k$; $e(u_i') = 5 = e(v_j')$ for $1 \le i \le n-r-1$; $1 \le j \le m-k-1$. The quotient matrix of $\varepsilon((G \sim H)(u,v))$ corresponding to the above partition is given by

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3(m-k-1) \\ 0 & 0 & 0 & 0 & 3(n-r-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 4(m-k-1) \\ 0 & 0 & 0 & 0 & 4(n-r-1) & 0 \\ 0 & 3 & 0 & 4k & 0 & 5(m-k-1) \\ 0 & 3 & 4r & 0 & 5(n-r-1) & 0 \end{bmatrix}$$

Since $rank(\varepsilon((G \sim H)(u,v))) = rank(Q) = 4$, we see that 0 is an eigenvalue of $\varepsilon((G \sim H)(u,v))$, with multiplicity $n_1 + n_2 - 4$ and $Spec(\varepsilon((G \sim H)(u,v)) = Spec(Q)$ which are exactly roots of $\lambda^4 - (25(n-r-1)(m-k-1) + 16(k(n-r-1) + r(m-k-1)) + 9(n-r-1+m-k-1))\lambda + (n-r-1)(m-k-1)(256kr + 144(k+r) + 81) = 0$

□

3.3. **ε-Spectra of some graph operations.**

Definition 3.4. [17] *The Mycielskian $\mu(G)$ of a graph $G = (V,E)$ is defined as the graph with vertex set $V(\mu(G)) = V \cup V' \cup \{u\}$, where $\{V' = x' : x \in V\}$ and edge set $E(\mu(G)) = E \cup \{yx' : xy \in E\} \cup \{x'u : x' \in V'\}$.*

Theorem 3.7. *Let G be an r- regular graph with n vertices and diameter at most 2. Then the eccentricity spectrum of the Mycielskian graph of G ; $\varepsilon(\mu(G))$ consists of $(((-1)^{k+1}\sqrt{5}-1)-2)\lambda_i(G)$ for $i = 2,3,...,n$ and the roots of the polynomial $\lambda^3 - 2(2n-2-r)\lambda^2 - 4(n^2 - rn + 2n - r^2 - r - 1)\lambda + 4n(n-1) = 0$ where $\lambda_i(G) \in Spec(A(G))$.*

Proof. By suitable labelling on the vertices, $\varepsilon(\mu(G))$ is given by

$$\begin{bmatrix} 0 & 0 & 21_n^t \\ 0 & 2(J_n - I_n) & 2(J_n - A(G)) \\ 21_n & 2(J_n - A(G)) & 2(J_n - I_n - A(G)) \end{bmatrix}$$

The quotient matrix Q corresponding to the equitable partition is given by

$$\begin{bmatrix} 0 & 0 & 2n \\ 0 & 2(n-1) & 2(n-r) \\ 2 & 2(n-r) & 2(n-r-1) \end{bmatrix}$$

The characteristic polynomial of Q is given by $\lambda^3 - 2(2n-2-r)\lambda^2 - 4(n^2 - rn + 2n - r^2 - r - 1)\lambda + 4n(n-1) = 0$. By Lemma 2.1 roots of this polynomial belongs to ε-spectrum of

$\mu(G)$. Let $X_1 = \dfrac{1}{n}(1, ..., 1)$ and $\{X_i\}_{i=1}^{n}$ be set of orthonormal eigenvectors of $A(G)$ and $\gamma_{ik} \neq 0$ be a scalar for $k = 1, 2$. For $i = 2, 3, ..., n$ we have

$$\begin{aligned}\varepsilon(\mu(G)) \begin{bmatrix} 0 \\ \gamma_{ik}X_i \\ X_i \end{bmatrix} &= - \begin{bmatrix} 0 \\ 2(\gamma_{ik} + \lambda_i(G))X_i \\ 2(\gamma_{ik} + \lambda_i(G) + 1)X_i \end{bmatrix} \\ &= -2(1 + \frac{\lambda_i(G)}{\gamma_{ik}}) \begin{bmatrix} 0 \\ \gamma_{ik}X_i \\ X_i \end{bmatrix}\end{aligned}$$

if and only if

$$1 + \frac{\lambda_i(G)}{\gamma_{ik}} = (\gamma_{ik} + 1)\lambda_i(G) + 1$$

if and only if

$$\gamma_{ik}^2 + \gamma_{ik} - 1 = 0 \Rightarrow \gamma_{ik} = \frac{-1 \pm \sqrt{5}}{2}$$

Take $\gamma_{ik} = \dfrac{-1 + (-1)^k\sqrt{5}}{2}$; $k = 1, 2$. Therefore $\lambda_i(G)(((-1)^{k+1}\sqrt{5} - 1) - 2)$ is the eigenvalue of $\varepsilon(\mu(G))$ for $i = 2, 3, ..., n$; $k = 1, 2$. □

Definition 3.5. [18] *The double graph $D(G)$ of a graph G is obtained by taking two copies of G and joining each vertex in one copy with the neighbours of the corresponding vertex in the other copy.*

Theorem 3.8. *Let G be an r-regular graph of order n and diameter at most 2. Then the ε-spectrum of $D(G)$ is given by*

$$Spec(\varepsilon(D(G))) = \begin{Bmatrix} 2(2(n-r)-1) & -2(2\lambda_2+1) & . & . & . & -2(2\lambda_n+1) & -2 \\ 1 & 1 & . & . & . & 1 & n \end{Bmatrix}$$

where $r, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A(G)$.

Proof. By suitable labelling, we get

$$\varepsilon(D(G)) = \begin{bmatrix} \varepsilon(G) & \varepsilon(G) + 2I_n \\ \varepsilon(G) + 2I_n & \varepsilon(G) \end{bmatrix}$$

By Lemmas 2.3 and 2.4 we obtain the desired result. □

Definition 3.6. [19] *The strong double graph $SD(G)$ of a graph G is constructed by taking two copies of graph a G and joining each vertex in one copy with the closed neighbourhood of the corresponding vertex in the other copy.*

Theorem 3.9. *Let G be an r-regular graph of order n and diameter 2. Then the ε-spectrum of $SD(G)$ is given by*

$$Spec(\varepsilon(SD(G))) = \begin{Bmatrix} 4(n-r-1) & -4(\lambda_2+1) & . & . & . & -4(\lambda_n+1) & 0 \\ 1 & 1 & . & . & . & 1 & n \end{Bmatrix}$$

where $r, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A(G)$.

Proof. We have;

$$\varepsilon(SD(G)) = \begin{bmatrix} \varepsilon(G) & \varepsilon(G) \\ \varepsilon(G) & \varepsilon(G) \end{bmatrix} = J_2 \otimes \varepsilon(G)$$

By Lemmas 2.2 and 2.4 gives the desired result. □

Remark 3.2. $Spec_\varepsilon(SD(K_n)) = Spec_\varepsilon(K_{2n})$

4. Some bounds on ε-energy of graphs

In this section, we obtain some upper and lower bounds of ε-energy of a simple connected graph G.

Lemma 4.1. [10] *If G is a connected graph of order n, then $\sum_{i=1}^{n} \xi_i = 0$ and $\sum_{i=1}^{n} \xi_i^2 = 2\sum_{1\le i<j\le n} \epsilon_{ij}^2$.*

Notation 4.1. *Let $\sum_{1\le i<j\le n} \epsilon_{ij}^2$ denoted by K.*

Theorem 4.1. [7] *Let G be a connected graph with order $n \ge 2$. Then $\xi_1 \le \sqrt{\frac{2K(n-1)}{n}}$ where equality holds if and only if $\xi_1 \ne \xi_2 = \cdots = \xi_n$. Furthermore if $\xi_1 = n-1$, the upper bounds is achieved if and only if G is a complete graph.*

Theorem 4.2. *Let A be a non-singular eccentricity matrix of a graph G with n vertices. Then*

$$n \mid detA \mid^{\frac{1}{n}} \le E_\varepsilon(G) \le \frac{2nK}{|detA|^{\frac{1}{n}}}.$$

Proof. By the geometric mean and arithmetic mean inequality

$$\frac{\mid \xi_1 \mid + \cdots + \mid \xi_n \mid}{n} \ge \mid \xi_1 ... \xi_n \mid^{1/n}$$

$$E_\varepsilon(G) \ge n \mid detA \mid^{\frac{1}{n}}$$

Also,$\mid \xi_1 \mid \ge \mid detA \mid^{\frac{1}{n}} \Rightarrow \sum_{i=1}^{n} \mid \xi_i \mid\mid \xi_1 \mid \ge \sum_{i=1}^{n} \mid \xi_i \mid\mid detA \mid^{\frac{1}{n}}$

$$n \mid \xi_1 \mid^2 \ge \sum_{i=1}^{n} \mid \xi_i \mid\mid \xi_1 \mid \ge E_\varepsilon(G) \mid detA \mid^{\frac{1}{n}}$$

Thus, $E_\varepsilon(G) \le \frac{n|\xi_1|^2}{|detA|^{\frac{1}{n}}} \le \frac{2Kn}{|detA|^{\frac{1}{n}}}$. □

The next result is a bound for ε-energy similar to the bound obtained by Milovanović [15] for energy of a graph.

Theorem 4.3. *Let G be a graph with n vertices and m edges. Let $\mid \xi_1 \mid \ge .... \ge \mid \xi_n \mid$ be the non-increasing eigenvalues of $\varepsilon(G)$. Then*

$$E_\varepsilon(G) \ge \sqrt{2Kn - \alpha(n)\left(\mid \xi_1 \mid - \mid \xi_n \mid\right)^2} \tag{11}$$

where $\alpha(n) = n\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\left[\frac{n}{2}\right]$, while $[x]$ denote the integer part of a real number x.

Proof. Let $x_1, ..., x_n$ and $y_1, ..., y_n$ be real numbers for which there exist constants x, y, X and Y such that $x \le x_i \le X$ and $y \le y_i \le Y \quad \forall \quad i = 1, 2, ..., n$. Then the following inequality is valid [see [15]]

$$\mid n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \mid \;\le\; \alpha(n)(X-x)(Y-y) \tag{12}$$

where $\alpha(n) = n\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\left[\frac{n}{2}\right]$ and equality holds if and only if $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$. If $x_i =\mid \xi_i \mid$, $y_i =\mid \xi_i \mid$, $x = y =\mid \xi_n \mid$, $X = Y =\mid \xi_1 \mid$

$$\mid n\sum_{i=1}^{n} \mid \xi_i \mid^2 - (\sum_{i=1}^{n} \mid \xi_i \mid)^2 \mid \leq \alpha(n)\left(\mid \xi_1 \mid - \mid \xi_n \mid\right)^2$$

$$n\left(2\sum_{i<j} \epsilon_{ij}^2\right) - (E_\varepsilon(G))^2 \leq \alpha(n)\left(\mid \xi_1 \mid - \mid \xi_n \mid\right)^2$$

$$E_\varepsilon(G) \geq \sqrt{2Kn - \alpha(n)\left(\mid \xi_1 \mid - \mid \xi_n \mid\right)^2}$$

Since equality in (12) holds in if and only if $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$, equality in (11) holds if and only if $\mid \xi_1 \mid = \mid \xi_2 \mid = \cdots = \mid \xi_n \mid$.

□

Corollary 4.1. *Since* $\alpha(n) = n\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)\left[\frac{n}{2}\right] \leq \frac{n^2}{4}$, *then by Theorem 4.3* $E_\varepsilon(G) \geq \sqrt{2Kn - \frac{n^2}{4}\left(\mid \xi_1 \mid - \mid \xi_n \mid\right)^2}$.

Theorem 4.4. *Let* G *be a graph with* n *vertices and* m *edges. Let* $\mid \xi_1 \mid \geq \cdots \geq \mid \xi_n \mid$ *be the non-increasing eigenvalues of* $\varepsilon(G)$. *Then* $E_\varepsilon(G) \geq \frac{2K + n|\xi_1||\xi_n|}{|\xi_1| + |\xi_n|}$.

Proof. Let $x_i \neq 0$ y_i, r and R be real numbers $(r \neq R)$ satisfying $rx_i \leq y_i \leq Rx_i$ then the following inequality hold

$$\sum_{i=1}^{n} y_i^2 + rR\sum_{i=1}^{n} x_i \leq (r+R)\sum_{i=1}^{n} x_i y_i$$

Put $y_i =\mid \xi_i \mid$ $x_i = 1$, $r =\mid \xi_n \mid$ $R =\mid \xi_1 \mid$

$$\sum_{i=1}^{n} \mid \xi_i \mid^2 + \mid \xi_1 \mid\mid \xi_n \mid \sum_{i=1}^{n} 1 \leq (\mid \xi_1 \mid + \mid \xi_n \mid)\sum_{i=1}^{n} \mid \xi_i \mid$$

$$2\sum_{i<j} \epsilon_{ij}^2 + n \mid \xi_1 \mid\mid \xi_n \mid \leq (\mid \xi_1 + \mid\mid \xi_n \mid)E_\varepsilon(G)$$

$$E_\varepsilon(G) \geq \frac{2K + n \mid \xi_1 \mid\mid \xi_n \mid}{\mid \xi_1 \mid + \mid \xi_n \mid}$$

Equality holds if and only if $\mid \xi_1 \mid = \mid \xi_2 \mid = \cdots = \mid \xi_n \mid$. □

Theorem 4.5. *Let* G *be a graph with* n *vertices and* m *edges. Then*

$$E_\varepsilon(G) \geq \sqrt{2K + n(n-1)detA^{\frac{2}{n}} + \frac{4}{(n+1)(n-2)}\left(\sqrt{\frac{2\sqrt{K}}{n}} - \sqrt{2K}\right)^2}$$

and equality holds if and only if $G \cong \overline{K_n}$.

Proof. By Lemma 2.7, $\sum_{i=1}^{N} x_i \geq N(\prod_{i=1}^{N} x_i)^{\frac{1}{N}} + \frac{1}{N-1}\sum_{i<j}(\sqrt{x_i} - \sqrt{x_j})^2$. Putting $N = \frac{n(n-1)}{2}$ *and* $(x_1, ..., x_N) = (\mid \xi_1 \mid\mid \xi_2 \mid, \mid \xi_1 \mid\mid \xi_3 \mid, ..., \mid \xi_1 \mid\mid \xi_n \mid, \mid \xi_2 \mid\mid \xi_3 \mid, ..., \mid \xi_2 \mid\mid \xi_n \mid .... \mid \xi_{n-1} \mid\mid \xi_n \mid)$

$$\sum_{1\leq i<j\leq n} \mid \xi_i \mid\mid \xi_j \mid \geq \frac{n(n-1)}{2}(\prod_{i=1}^{n} \mid \xi_i \mid)^{\frac{2}{n}} + \frac{2}{n^2 - n - 2}\sum_{i<j\leq k<l}(\sqrt{\mid \xi_i \mid\mid \xi_j \mid} - \sqrt{\mid \xi_k \mid\mid \xi_l \mid})^2$$

$$2\sum_{1\leq i<j\leq n} \mid \xi_i \mid\mid \xi_j \mid \geq n(n-1)detA^{\frac{2}{n}} + \frac{4}{(n+1)(n-2)}\sum_{i<j\leq k<l}(\sqrt{\mid \xi_i \mid\mid \xi_j \mid} - \sqrt{\mid \xi_k \mid\mid \xi_l \mid})^2$$

Then by Theorem 4.1,

$$\sum_{i<j\leq k<l} (\sqrt{|\xi_i||\xi_j|} - \sqrt{|\xi_k||\xi_l|})^2 \geq (\sqrt{|\xi_1||\xi_n|} - \sqrt{|\xi_{n/2}||\xi_n|})^2$$

$$= |\xi_n| (\sqrt{|\xi_1|} - \sqrt{|\xi_{n/2}|})^2$$

$$\geq \left(\sqrt{\frac{2\sqrt{K}}{n}} - \sqrt{2K}\right)^2$$

Thus

$$2\sum_{1\leq i<j\leq n} |\xi_i||\xi_j| \geq n(n-1)detA^{\frac{2}{n}} + \frac{4}{(n+1)(n-2)} + \left(\sqrt{\frac{2\sqrt{K}}{n}} - \sqrt{2K}\right)^2$$

Adding $\sum_{i=1}^{n} \xi^2 = 2K$ to both sides

$$E_\varepsilon(G) \geq \sqrt{2K + n(n-1)detA^{\frac{2}{n}} + \frac{4}{(n+1)(n-2)}(\sqrt{\frac{2\sqrt{K}}{n}} - \sqrt{2K})^2}.$$

□

Theorem 4.6. *Let A be a non-singular eccentricity matrix of graph G with n vertices and $|\xi_1| \geq .... \geq |\xi_n|$ be the non-increasing eigenvalues of A. Then* $\frac{2\sqrt{K}}{n} + n - 1 + ln\,|detA|$ $-ln\,|\frac{2\sqrt{K}}{n}| \leq E_\varepsilon(G) \leq \sqrt{\frac{2K(n-1)}{n}} + n - 1 + ln\,|detA| - \frac{1}{2}ln\,|\frac{2K(n-1)}{n}|$.

Proof. Consider the function $f(x) = x - 1 - lnx$ for $x \geq 0$. It is easy to see that $f(x)$ is an increasing function for $x \geq 1$ and decreasing for $0 \leq x \leq 1$. Then $f(x) \geq f(1) = 0$ implies that $x \geq 1 + lnx$ for $x > 1$.

$$E_\varepsilon(G) = |\xi_i| + \sum_{i=2}^{n} |\xi_i|$$

$$\geq \xi_1 + n - 1 + \sum_{i=2}^{n} ln\,|\xi_i|$$

$$= \xi_i + n - 1 + ln \prod_{i=2}^{n} |\xi_i|$$

$$= \xi_i + n - 1 + ln\,|detA| - ln\,|\xi_i|$$

Define $g(x) = x + n - 1 + ln\,|detA| - lnx$, which is increasing for $1 \leq x \leq n$. For $\xi_1 \geq \frac{2\sqrt{K}}{n}$

$$g(\xi_1) \geq g(\frac{2\sqrt{K}}{n}) = \frac{2\sqrt{K}}{n} + n - 1 + ln\,|detA| - ln\,|\frac{2\sqrt{K}}{n}|$$

Finally, for $\xi_1 \leq \sqrt{\frac{2(n-1)K}{n}}$,

$$g(x) \leq g(\sqrt{\frac{2(n-1)K}{n}}) = \sqrt{\frac{2K(n-1)}{n}} + n - 1 + ln\,|detA| - \frac{1}{2}ln\,|\sqrt{\frac{2(n-1)K}{n}}|$$

$$= \sqrt{\frac{2K(n-1)}{n}} + n - 1 + ln\,|detA| - \frac{1}{2}ln\,|\frac{2K(n-1)}{n}|.$$

□

5. Conclusions

In graph theory, ε-spectra are of particular interest in that they open up new avenues of research in to the properties of chemical compounds. A case in point is how ε-spectra has contributed in studying the boiling point of hydrocarbons and also in calculating the branching pattern of molecular graphs. The operations on graphs play a major role in creating new graphs from pre-existing ones. In this paper we have found the ε-spectra of some graph products, and ε-spectra of Splice and Link of regular graphs with diameter at most 2. We also calculated the ε-spectra of the larger graph acquired from the regular graph with the help of some graph operations. Moreover, we determined some bounds of ε-energy and characterized the extreme graphs.

Acknowledgement. The authors would like to express their sincere gratitude to the anonymous referees for their valuable comments and insightful feedback, which have greatly improved the quality and clarity of this work.

References

[1] Brouwer, A. E. and Haemers, W. H., (2011), Spectra of graphs, Springer Science and Business Media.
[2] Bapat, R. B., (2010), Graphs and matrices, Springer, Vol. 27.
[3] Aouchiche, M. and Hansen, P., (2014), Distance spectra of graphs: A survey, Linear Algebra and Its Applications, 458, pp. 301–386.
[4] Randic, M., Orel, R. and Balaban, A. T., (2013), D-max matrix invariants as graph descriptors graphs having the same balaban index j, MATCH-Communications in Mathematical and in Computer Chemistry, 70(1), pp.239-258.
[5] Wang, J., Lu, M., Belardo, F. and Randić, M., (2018), The anti-adjacency matrix of a graph: Eccentricity matrix, Discrete Applied Mathematics, 251, pp.299-309.
[6] Randić, M., (2013), Dmax–matrix of dominant distances in a graph, MATCH-Communications in Mathematical and in Computer Chemistry, 70(1), pp. 221-238.
[7] Wang, J., Lei, X., Wei, W., Luo, X., Li., S., (2020), On the eccentricity matrix of graphs and its applications to the boiling point of hydrocarbons, Chemometrics and Intelligent Laboratory Systems, 207, pp.104-173.
[8] Wang, J., Lu, M., Lu, L. and Belardo, F., (2020), Spectral properties of the eccentricity matrix of graphs, Discrete Applied Mathematics, 279, pp.168-177.
[9] Mahato, I., Gurusamy, R., Rajesh Kannan, M., Arockiaraj, S., (2023), On the spectral radius and the energy of eccentricity matrices of graphs, Linear and Multilinear Algebra 71 (1), pp.5–15.
[10] Wang, J., Lu, L., Randić, M. and Li, G., (2019), Graph energy based on the eccentricity matrix, Discrete Mathematics, 342(9), pp.2636-2646.
[11] You, L., Yang, M., So, W. and Xi, W., (2019), On the spectrum of an equitable quotient matrix and its application, Linear Algebra and its Applications, 577, pp.21-40.
[12] Cvetkovic, D. M., Doob, M., Sachs, H., (1980), Spectra of graphs: theory and applications, Academic Press, New York, San Francisco, London.
[13] Arora, A. and Mishra, R., (2024), Eccentric graph of trees and their Cartesian products, Discrete Mathematics, 347(9), pp.114062.
[14] Stevanovic, D., (2004), Distance regularity of compositions of graphs, Applied Mathematics Letters, 17(3), pp.337-344.
[15] Milovanovic, I. Z., Milovanovic, E. I. and Zakic, A., (2014), A short note on graph energy, MATCH-Communications in Mathematical and in Computer Chemistry, 72(1), pp.179-182.
[16] Kober, H., (1958), On the arithmetic and geometric means and on Hölder's inequality, Proceedings of the American Mathematical Society, pp.452-459.
[17] Mycielski, J., (1955), Sur le coloriage des graphs, In Colloquium Mathematicae, Vol. 3, No. 2, pp.161-162.
[18] Munarini, E., Cippo, C. P., Scagliola, A. and Salvi, N. Z., (2008), Double graphs, Discrete mathematics, 308(2-3), pp.242-254.
[19] Chishti, T. A., Ganie, H. A. and Pirzada, S., (2014), Properties of strong double graphs, Journal of Discrete Mathematical Sciences and Cryptography, 17(4), pp.311-319.

[20] Tavakoli, M., Rahbarnia, F. and Ashrafi, A. R., (2013), Note on strong product of graphs, Kragujevac Journal of Mathematics, 37(1), pp.187-193.
[21] Došlić, T., (2005), Splices, links and their valence-weighted Wiener polynomials, Graph theory notes of New York, 48, pp.47-55.
[22] Davis, P. J., (1979), Circulant Matrices, Wiley, New York.

Surya S received her MSc degree in Mathematics from SD College, Alappuzha, Kerala, India in 2017 and MPhil degree from Department of Mathematics, Kariyavattom campus, University of Kerala, India in 2019. Currently, she is doing research under the guidance of Dr. Pramada Ramachandran at St Paul's College,Kalamassery, Ernakulam, Kerala, India. Her research interest includes Spectral graph theory and Algebraic graph theory.

Pramada Ramachandran is currently Professor and Principal at St Paul's College, Kalamassery, kerala, India. She received her Ph.D. degree from Mahatma Gandhi University, Kottayam, Kerala. Her research interests include Fuzzy Graph Theory, Graph Energy and Popular Mathematics. She has more than a dozen publications in national and international journals.