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SECURE MONOPHONIC DOMINATION NUMBER OF PRODUCT OF GRAPHS

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ABSTRACT. Let G be a graph which is connected. A monophonic dominating set M is said to be a secure monophonic dominating set S_m(written as SMD set) of G if for each $g \in V\backslash M$ there exists $f \in M$ such that g is adjacent to f and $S_m = (M\backslash\{f\} \cup \{g\}$ is a monophonic dominating set. The smallest cardinality of a secure monophonic dominating set of G is the secure monophonic domination number of G and the notation is $\gamma_{sm}(G)$. In this article, we discuss the secure monophonic domination number of product of graphs.

Keywords: Monophonic path, Monophonic domination number, Secure monophonic domination number, Cartesian product and Corona product.

AMS Subject Classification: 05C69.

1. INTRODUCTION

In this article, consider a graph simple finite and undirected graph $G = (V, E)$. Two vertices f and g are said to be adjacent if fg is an edge of G. We say f is a neighbor of g, if $fg \in E(G)$ and is denoted by $N(g)$. The degree of a vertex $f \in V(G)$ is $deg(f) = |N(f)|$. A vertex of degree one is a leaf and its neighbor is a support vertex. A vertex f is called a universal vertex of G if $deg(f) = n - 1$. Domination concept introduced by O.Ore, is currently receiving much attention in the literature of graph theory. A unique handling of fundamentals of domination in graphs is given by T. W. Haynes et. al.[11]. A set $S \subseteq V$ is said to be a dominating set if every vertex $f \in V\backslash S$ is adjacent to a vertex $g \in S$. The domination number $\gamma(G)$ of G is the minimum cardinality of its dominating sets. The theory of convexity related to a graph structure can be divided into two types, one is geodetic convexity and other one is monophonic convexity. Geodetic convexity is related to the shortest path in a system of paths, which is defined in a connected graph G, but in the monophonic convexity (or an induced path convexity) replaces the shortest

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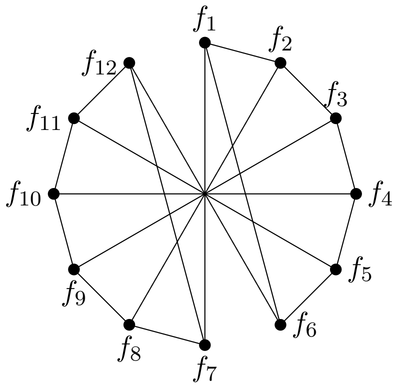
path by a chordless path(or an induced path). A chord is an edge that connects two non-consecutive vertices of path P. An $f-g$ path is called a monophonic path if it is a chordless path. A subset M of V is said to be a monophonic set if every vertex f in V lies in a $f-g$ monophonic path where $f, g \in M$. In ([1], [2], [12], [13], [17]), we studied the concept of the monophonic domination number of a graph. Strategies for protection of a graph $G=(V,E)$ by placing one or more guards at every vertex of a subset S of V, where a guard at g can protect any vertex in its closed neighborhood have resulted in the study of several concepts such as Roman domination, Weak Roman domination and secure domination. The Concept of secure domination is motivated by the following situation. Given a graph $G=(V,E)$ we wish to place one guard at each vertex of a subset S of V in such a way that S is a dominating set of G and if a guard at g moves along an edge to protect an unguarded vertex f, then the new configuration of guards also forms a dominating set. In other words, for each $f \in V \backslash S$ there exists $g \in S$ such that g is adjacent to f and $\{S \backslash g\} \cup \{f\}$ is a dominating set of G. In this case we say that f is $S-$defended by g or g $S-$defends f. A dominating set S is said to be a secure dominating set of G if for each $f \in V \backslash S$ there exists $g \in S$ such that f is adjacent to g and $\{S \backslash g\} \cup \{f\}$ is a dominating set and its minimum cardinality is denoted by $\gamma_s(G)$. This concept was introduced by Cockayne et al.[6] and further investigated in ([5], [9], [14], [15], [16]).
In graph theory, different types of products of two graphs had been studied, e,g., Cartesian product, Tensor Product, Strong Product, Corona product, etc. In this sequel, we discussed the secure monophonic domination number of cartesian and corona product of graphs. The Cartesian product of two graphs is well-known and studied in detail ([7], [19]). The Cartesian product of two graphs H and H' of order m and n is the graph $H \boxdot H'$ with vertex set $V(H \boxdot H') = V(H) \times V(H')$ and the edge set $E(H \boxdot H')$ satisfying the following conditions: $(x,a)(y,b) \in E(H \boxdot H')$ if and only if either $xy \in E(H)$ and $a=b$ or $x=y$ and $ab \in E(H')$. The new vertices are denoted by $\{f_1, f_2, ..., f_{mn}\}$. In 1970, Frucht and Harary [8], introduced the corona of two graphs. The Corona of two graphs H_1 and H_2 is the graph $G = H_1 \odot H_2$ obtained from one copy of H_1 and $|V(H_1)|$ copies of H_2, where the i^{th} vertex of H_1 is adjacent to every vertex in the i^{th} copy of H_2. The Domination parameters of corona was studied in [4]. Also we use the following graph structures to prove our main results. This concept has so many practical application in communication networks and security of buildings
A $Y-tree$ graph Y_{n+1} is obtained from the path P_n by appending an edge to a vertex of the path P_n adjacent to an end point. The Hurdle graph Hd_n is obtained from a path P_n by attaching a pendant edges to every internal vertices of the path and it has $n-2$ hurdles. For basic graph theoretic terminology, we refer [3].

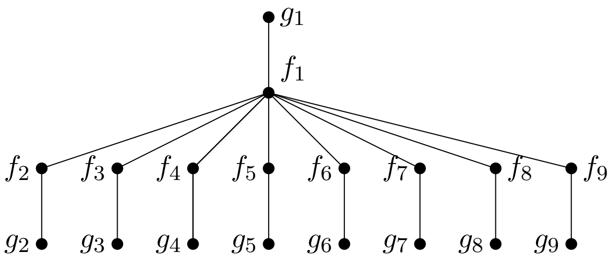
Definition 1.1. *A monophonic dominating set M is said to be a secure monophonic dominating set S_m(written as SMD set) of G if for each $g \in V \backslash M$ there exists $f \in M$ such that g is adjacent to f and $S_m = \{M \backslash f\} \cup \{g\}$ is a monophonic dominating set. The smallest cardinality of a secure monophonic dominating set of G is the secure monophonic domination number of G and the notation is $\gamma_{sm}(G)$.*

The Secure monophonic domination number of the graph $G = P_2 \boxdot C_6$ given in Figure 1.



$S_m = \{f_2, f_4, f_6, f_8, f_{10}\}$.
$\gamma_{sm}(G) = 5$.
Figure 1

The Secure monophonic domination number of the graph $G = K_{1,10-1} \odot K_1$ in Figure 2



$S_m = \{f_1, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9\}$
$\gamma_{sm}(G) = 10$.
Figure 2

2. Observation

Observation 2.1. *In a connected graph G, each end vertex belongs to every SMD set of G.*

Observation 2.2. *In a connected graph G, each pendant vertex belongs to every SMD set of G.*

Observation 2.3. *In a connected graph G, each universal vertex belongs to every SMD set of G.*

3. Main Results

Theorem 3.1. *If $G = P_2 \boxdot C_n (n \geq 3)$, then*

$$\gamma_{sm}(G) = \begin{cases} 3 & if \quad n = 3 \\ 4 & if \quad n = 4, 5 \\ 5 & if \quad n = 6 \\ 2\lceil \frac{3n}{7} \rceil & if \quad n \equiv 0, 2, 4, 6 (mod 7) (n \geq 7) \\ 2\lceil \frac{3n}{7} \rceil - 1 & if \quad n \equiv 1 (mod 7) \\ 2\lceil \frac{3n}{7} \rceil - 2 & if \quad n \equiv 3, 5 (mod 7) \end{cases} .$$

Proof: *Let G be the cartesian product of path graph P_2 and cycle graph $C_n(n \geq 3)$ whose vertices are $\{f_1, f_2, ..., f_{2n}\}$. If $n = 3$ then $G = P_2 \boxdot C_3$,$S_m = \{f_2, f_4, f_6\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 3$.If $n = 4$ then $G = P_2 \boxdot C_4$,$S_m = \{f_2, f_4, f_6, f_8\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 4$.If $n = 5$ then $G = P_2 \boxdot C_5$,$S_m = \{f_2, f_4, f_6, f_8\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 4$.If $n = 6$ then $G = P_2 \boxdot C_6$,$S_m = \{f_2, f_4, f_6, f_8, f_{10}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 5$.*

Hence $\gamma_{sm}(G) = \begin{cases} 3 & if \quad n = 3 \\ 4 & if \quad n = 4, 5 \\ 5 & if \quad n = 6 \end{cases}$ *Let $n \geq 7$. Consider the following cases.*

Case a Subcase(i): $n \equiv 0(mod7)$

Consider $G = P_2 \boxdot C_7$. Let $S_m = \{f_2, f_4, f_6, ..., f_{2n-1}\}$. For each $g \in V \backslash S_m$ there exists $f \in S_m$ such that g is adjacent to f and $\{S_m \backslash f\} \bigcup \{g\}$ is a monophonic dominating set. In general, $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+2}, f_{7j+4}, f_{7j+6}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 2\lceil \frac{3n}{7} \rceil$.

Subcase(ii): $n \equiv 2(mod7)$

Consider $G = P_2 \boxdot C_9$. Let $S_m = \{f_2, f_4, f_6, ..., f_{2n}\}$. Proceeding the same argument as in Subcase(i), we get $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+2}, f_{7j+4}, f_{7j+6}\} \cup \{f_{2n-2}, f_{2n}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 2\lceil \frac{3n}{7} \rceil$.

Subcase(iii): $n \equiv 4(mod7)$

Consider $G = P_2 \boxdot C_{11}$.Let $S_m = \{f_2, f_4, f_6, ..., f_{2n}\}$. Proceeding the same argument as in Subcase(i), we get $S_m = \{\bigcup_{j=0}^{2k} f_{7j+2}, f_{7j+4}, f_{7j+6}\} \cup \{f_{2n}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 2\lceil \frac{3n}{7} \rceil$.

Subcase(iv): $n \equiv 6(mod7)$

Consider $G = P_2 \boxdot C_{13}$.Let $S_m = \{f_2, f_4, f_6, ..., f_{2n}\}$. Proceeding the same argument as in Subcase(i), we get $S_m = \{\bigcup_{j=0}^{2k} f_{7j+2}, f_{7j+4}, f_{7j+6}\} \cup \{f_{2n-4}, f_{2n-2}, f_{2n}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 2\lceil \frac{3n}{7} \rceil$.

Case b: $n \equiv 1(mod7)$

Consider $G = P_2 \boxdot C_8$. Let $S_m = \{f_1, f_3, f_5, ..., f_{2n-1}\}$. Proceeding the same argument as in Subcase(i), we get $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \cup \{f_{2n-1}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 2\lceil \frac{3n}{7} \rceil - 1$.

Case c Subcase(i) : $n \equiv 3(mod7)$

Consider $G = P_2 \boxdot C_{10}$. Let $S_m = \{f_2, f_4, f_6, ..., f_{2n-2}\}$. Proceeding the same argument as in Case a Subcase(i), we get $S_m = \{\bigcup_{j=0}^{2k-1} f_{7j+2}, f_{7j+4}, f_{7j+6}\} \cup \{f_{2n-4}, f_{2n-2}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 2\lceil \frac{3n}{7} \rceil - 2$.

Subcase(ii): $n \equiv 5(mod7)$

Consider $G = P_2 \boxdot C_{12}$. Let $S_m = \{f_2, f_4, f_6, ..., f_{2n-1}\}$. Proceeding the same argument as in Case a Subcase(i), we get $S_m = \{\bigcup_{j=0}^{2k} f_{7j+2}, f_{7j+4}, f_{7j+6}\} \cup \{f_{2n-1}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = 2\lceil \frac{3n}{7} \rceil - 2$. Finally we conclude that,

$$\gamma_{sm}(G) = \begin{cases} 2\lceil \frac{3n}{7} \rceil & if \quad n \equiv 0,2,4,6(mod7)(n \geq 7) \\ 2\lceil \frac{3n}{7} \rceil - 1 & if \quad n \equiv 1(mod7) \\ 2\lceil \frac{3n}{7} \rceil - 2 & if \quad n \equiv 3,5(mod7) \end{cases}$$

$$Hence\ \gamma_{sm}(G) = \begin{cases} 3 & if \quad n = 3 \\ 4 & if \quad n = 4,5 \\ 5 & if \quad n = 6 \\ 2\lceil \frac{3n}{7} \rceil & if \quad n \equiv 0,2,4,6(mod7)(n \geq 7) \\ 2\lceil \frac{3n}{7} \rceil - 1 & if \quad n \equiv 1(mod7) \\ 2\lceil \frac{3n}{7} \rceil - 2 & if \quad n \equiv 3,5(mod7) \end{cases}$$

Theorem 3.2. *If $G = P_3 \boxdot C_n (n \geq 3)$, then*

$$\gamma_{sm}(G) = \begin{cases} n+1 & if \quad n = 3,4,5 \\ n+2 & if \quad n = 6 \\ \lceil \frac{9n}{7} \rceil & if \quad n \equiv 0,2,3(mod7)(n \geq 7) \\ \lceil \frac{9n}{7} \rceil - 1 & if \quad n \equiv 1,4,5(mod7) \\ \lceil \frac{9n}{7} \rceil + 1 & if \quad n \equiv 6(mod7) \end{cases}.$$

Proof: *Let G be the cartesian product of path graph P_3 and cycle graph $C_n(n \geq 3)$ whose vertices are $\{f_1, f_2, ..., f_{3n}\}$. If $n = 3$ then $G = P_3 \boxdot C_3$,$S_m = \{f_2, f_4, f_6, f_9\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 4$.If $n = 4$ then $G = P_3 \boxdot C_4$,$S_m = \{f_2, f_4, f_6, f_9, f_{12}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 5$.If $n = 5$ then $G = P_3 \boxdot C_5$,$S_m = \{f_2, f_4, f_6, f_9, f_{11}, f_{13}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 6$.If $n = 6$ then $G = P_3 \boxdot C_6$,$S_m = \{f_2, f_4, f_6, f_9, f_{11}, f_{13}, f_{15}, f_{17}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = 8$. Hence $\gamma_{sm}(G) = \begin{cases} n+1 & if \quad n = 3,4,5 \\ n+2 & if \quad n = 6 \end{cases}$ Let $n \geq 7$. Consider the following cases.*

Case a Subcase(i): $n \equiv 0(mod7)$

Consider $G = P_3 \boxdot C_7$. Let $S_m = \{f_2, f_4, f_6, ..., f_{3n-1}\}$. For each $g \in V \backslash S_m$ there exists $f \in S_m$ such that g is adjacent to f and $\{S_m \backslash f\} \bigcup \{g\}$ is a monophonic dominating set. In general, $S_m = \{\bigcup_{j=0}^{3k-1} f_{7j+2}, f_{7j+4}, f_{7j+6}\}$ is a minimum SMD set of G.Therefore $\gamma_{sm}(G) = \lceil \frac{9n}{7} \rceil$.

Subcase(ii): $n \equiv 2(mod7)$

Consider $G = P_3 \boxdot C_9$. Let $S_m = \{f_2, f_4, f_6, ..., f_{3n}\}$. Proceeding the same argument as in Subcase(i), we get $S_m = \{\bigcup_{j=0}^{3k} f_{7j+2}, f_{7j+4}, f_{7j+6}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = \lceil \frac{9n}{7} \rceil$.

Subcase(iii): $n \equiv 3(mod7)$

Consider $G = P_3 \boxdot C_{10}$.Let $S_m = \{f_2, f_4, f_6, ..., f_{3n-1}\}$. Proceeding the same argument as in Subcase(i), we get $S_m = \{\bigcup_{j=0}^{3k} f_{7j+2}, f_{7j+4}, f_{7j+6}\} \cup \{f_{3n-1}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = \lceil \frac{9n}{7} \rceil$.

Case b Subcase(i): $n \equiv 1(mod7)$

Consider $G = P_3 \boxdot C_8$.Let $S_m = \{f_1, f_3, f_5, ..., f_{3n-2}\}$. Proceeding the same argument as in Case a Subcase(i), we get $S_m = \{\bigcup_{j=0}^{3k-1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \cup \{f_{3n-2}\}$ is a minimum SMD set of G. Therefore $\gamma_{sm}(G) = \lceil \frac{9n}{7} \rceil - 1$.

Subcase(ii): $n \equiv 4(mod7)$

Consider $G = P_3 \boxdot C_{11}$. *Let* $S_m = \{f_2, f_4, f_6, ..., f_{3n}\}$. *Proceeding the same argument as in Case a Subcase(i), we get* $S_m = \{\bigcup_{j=0}^{3k} f_{7j+2}, f_{7j+4}, f_{7j+6}\} \cup \{f_{3n-3}, f_{3n}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{9n}{7} \rceil - 1$.

Subcase(iii) : $n \equiv 5(mod7)$

Consider $G = P_3 \boxdot C_{12}$. *Let* $S_m = \{f_2, f_4, f_6, ..., f_{3n-2}\}$. *Proceeding the same argument as in Case a Subcase(i), we get* $S_m = \{\bigcup_{j=0}^{3k+1} f_{7j+2}, f_{7j+4}, f_{7j+6}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{9n}{7} \rceil - 1$.

Case c : $n \equiv 6(mod7)$

Consider $G = P_3 \boxdot C_{13}$. *Let* $S_m = \{f_1, f_3, f_5, ..., f_{3n}\}$. *Proceeding the same argument as in Case a Subcase(i), we get* $S_m = \{\bigcup_{j=0}^{3k+1} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \cup \{f_{3n-4}, f_{3n-2}, f_{3n}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{9n}{7} \rceil + 1$. *Finally we conclude that,*

$$\gamma_{sm}(G) = \begin{cases} \lceil \frac{9n}{7} \rceil & if \quad n \equiv 0,2,3(mod7)(n \geq 7) \\ \lceil \frac{9n}{7} \rceil - 1 & if \quad n \equiv 1,4,5(mod7) \\ \lceil \frac{9n}{7} \rceil + 1 & if \quad n \equiv 6(mod7) \end{cases}$$

$$\textit{Hence } \gamma_{sm}(G) = \begin{cases} n+1 & if \quad n = 3,4,5 \\ n+2 & if \quad n = 6 \\ \lceil \frac{9n}{7} \rceil & if \quad n \equiv 0,2,3(mod7)(n \geq 7) \\ \lceil \frac{9n}{7} \rceil - 1 & if \quad n \equiv 1,4,5(mod7) \\ \lceil \frac{9n}{7} \rceil + 1 & if \quad n \equiv 6(mod7) \end{cases}$$

Theorem 3.3. *If* $G = K_{1,3} \boxdot P_n (n \geq 2)$, *then*

$$\gamma_{sm}(G) = \begin{cases} n+2 & if \quad n = 2,3 \\ n+3 & if \quad n = 4 \\ n+4 & if \quad n = 5,6 \\ \lceil \frac{12n}{7} \rceil & if \quad n \equiv 0,1,2,4(mod7)(n \geq 7) \\ \lceil \frac{12n}{7} \rceil - 1 & if \quad n \equiv 3,5,6(mod7) \end{cases}.$$

Proof: *Let* G *be the cartesian product of Star graph* $K_{1,3}$ *and path graph* $P_n (n \geq 2)$ *whose vertices are* $\{f_1, f_2, ..., f_{4n}\}$.*If* $n = 2$ *then* $G = K_{1,3} \boxdot P_2$,$S_m = \{f_2, f_4, f_6, f_8\}$ *is a minimum SMD set of* G.*Therefore* $\gamma_{sm}(G) = 4$. *If* $n = 3$ *then* $G = K_{1,3} \boxdot P_3$,$S_m = \{f_1, f_3, f_5, f_8, f_{11}\}$ *is a minimum SMD set of* G.*Therefore* $\gamma_{sm}(G) = 5$.*If* $n = 4$ *then* $G = K_{1,3} \boxdot P_4$,$S_m = \{f_1, f_4, f_6, f_8, f_{11}, f_{13}, f_{15}\}$ *is a minimum SMD set of* G.*Therefore* $\gamma_{sm}(G) = 7$.*If* $n = 5$ *then* $G = K_{1,3} \boxdot P_5$,$S_m = \{f_1, f_3, f_5, f_8, f_{10}, f_{12}, f_{15}, f_{17}, f_{19}\}$ *is a minimum SMD set of* G.*Therefore* $\gamma_{sm}(G) = 9$.*If* $n = 6$ *then* $G = K_{1,3} \boxdot P_6$,$S_m = \{f_1, f_4, f_6, f_8, f_{11}, f_{13}, f_{15}, f_{18}, f_{20}, f_{22}\}$ *is a minimum SMD set of* G.*Therefore* $\gamma_{sm}(G) = 10$. *Hence* $\gamma_{sm}(G) = \begin{cases} n+2 & if \quad n = 2,3 \\ n+3 & if \quad n = 4 \\ n+4 & if \quad n = 5,6 \end{cases}$ *Let* $n \geq 7$. *Consider the following cases.*

Case a Subcase(i): $n \equiv 0(mod7)$

Consider $G = K_{1,3} \boxdot P_7$. *Let* $S_m = \{f_2, f_4, f_6, ..., f_{4n-1}\}$. *For each* $g \in V \backslash S_m$ *there exists* $f \in S_m$ *such that* g *is adjacent to* f *and* $\{S_m \backslash f\} \bigcup \{g\}$ *is a monophonic dominating set. In general,* $S_m = \{\bigcup_{j=0}^{4k-1} f_{7j+2}, f_{7j+4}, f_{7j+6}\}$ *is a minimum SMD set of* G.*Therefore* $\gamma_{sm}(G) = \lceil \frac{12n}{7} \rceil$.

Subcase(ii): $n \equiv 1(mod7)$

Consider $G = K_{1,3} \boxdot P_8$. *Let* $S_m = \{f_1, f_4, f_6, ..., f_{4n-1}\}$. *Proceeding the same argument as in Subcase(i), we get* $S_m = \{\bigcup_{j=0}^{4k-1} f_{7j+4}, f_{7j+6}, f_{7j+8}\} \cup \{f_1, f_{4n-1}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{12n}{7} \rceil$.

Subcase(iii): $n \equiv 2(mod7)$

Consider $G = K_{1,3} \boxdot P_9$.*Let* $S_m = \{f_1, f_4, f_6, ..., f_{4n}\}$. *Proceeding the same argument as in Subcase(i), we get* $S_m = \{f_1\} \cup \{\bigcup_{j=0}^{4k} f_{7j+4}, f_{7j+6}, f_{7j+8}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{12n}{7} \rceil$.

Subcase(iv): $n \equiv 4(mod7)$

Consider $G = K_{1,3} \boxdot P_{11}$. *Let* $S_m = \{f_1, f_4, ..., f_{4n-1}\}$. *Proceeding the same argument as in Subcase(i), we get* $S_m = \{f_1\} \cup \{\bigcup_{j=0}^{4k+1} f_{7j+4}, f_{7j+6}, f_{7j+8}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{12n}{7} \rceil$.

Case b Subcase(i): $n \equiv 3(mod7)$

Consider $G = K_{1,3} \boxdot P_{10}$.*Let* $S_m = \{f_1, f_3, ..., f_{4n-2}\}$. *Proceeding the same argument as in Case a Subcase(i), we get* $S_m = \{\bigcup_{j=0}^{4k} f_{7j+1}, f_{7j+3}, f_{7j+5}\} \cup \{f_{4n-4}, f_{4n-2}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{12n}{7} \rceil - 1$.

Subcase(ii) : $n \equiv 5(mod7)$

Consider $G = K_{1,3} \boxdot P_{12}$. *Let* $S_m = \{f_2, f_5, f_7, ..., f_{4n-1}\}$. *Proceeding the same argument as in Case a Subcase(i), we get* $S_m = \{\bigcup_{j=0}^{4k+1} f_{7j+5}, f_{7j+7}, f_{7j+9}\} \cup \{f_2, f_{4n-1}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{12n}{7} \rceil - 1$.

Subcase (iii) : $n \equiv 6(mod7)$

Consider $G = K_{1,3} \boxdot P_{13}$. *Let* $S_m = \{f_1, f_4, f_6, ..., f_{4n-2}\}$. *Proceeding the same argument as in Case a Subcase(i), we get* $S_m = \{f_1\} \cup \{\bigcup_{j=0}^{4k+2} f_{7j+4}, f_{7j+6}, f_{7j+8}\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil \frac{12n}{7} \rceil - 1$. *Finally we conclude that,*

$$\gamma_{sm}(G) = \begin{cases} \lceil \frac{12n}{7} \rceil & if \quad n \equiv 0, 1, 2, 4(mod7)(n \geq 7) \\ \lceil \frac{12n}{7} \rceil - 1 & if \quad n \equiv 3, 5, 6(mod7) \end{cases}$$

$$\text{Hence } \gamma_{sm}(G) = \begin{cases} n+2 & if \quad n = 2, 3 \\ n+3 & if \quad n = 4 \\ n+4 & if \quad n = 5, 6 \\ \lceil \frac{12n}{7} \rceil & if \quad n \equiv 0, 1, 2, 4(mod7)(n \geq 7) \\ \lceil \frac{12n}{7} \rceil - 1 & if \quad n \equiv 3, 5, 6(mod7) \end{cases}$$

Theorem 3.4. *If* $G = P_n \odot \overline{K_m}(m \geq 1, n \geq 3)$, *then*
$\gamma_{sm}(G) = \{ \lceil (\frac{3m+1}{3})n \rceil \quad if \quad n \equiv 0, 1, 2(mod3)$

Proof: *Let* G *be the corona product of Path graph* P_n *and complement of complete graph* $\overline{K_m}$ *whose vertex set is* $V(G) = \{f_i, g_j / 1 \leq i \leq n, 1 \leq j \leq mn\}$ *and edge set is* $E(G) = \{f_i f_{i+1} / 1 \leq i \leq n-1\} \cup \{f_i g_{m(i-1)+j} / 1 \leq j \leq m, 1 \leq i \leq n\}$ *such that* $|V(G)| = n(m+1)$ *and* $|E(G)| = n(m+1) - 1$. *Consider the following cases.*

Case(i): $n \equiv 0(mod3)$

Consider $G = P_3 \odot \overline{K_m}$. *Consider* $V(G) = \{f_i, g_j / 1 \leq i \leq 3, 1 \leq j \leq 3m\}$ *and* $S_m = \{f_2, g_j / 1 \leq j \leq 3m\} \subset V(G)$. *For each* $g \in V \backslash S_m$, *then there exists* $f \in S_m$

such that g is adjacent to f and $\{S_m \backslash f\} \cup \{g\}$ *is a monophonic dominating set of* G. *Therefore* S_m *is a minimum SMD set of* G. *In general* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq mn\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil(\frac{3m+1}{3})n\rceil$.

Case(ii): $n \equiv 1(mod3)$

Consider $G = P_4 \odot \overline{K_m}$. *Consider* $V(G) = \{f_i, g_j/1 \leq i \leq 4, 1 \leq j \leq 4m\}$ *and* $S_m = \{f_2, f_4, g_j/1 \leq j \leq 4m\} \subset V(G)$. *Proceeding the same argument as in Case(i), we get* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq mn\} \cup \{f_n\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil(\frac{3m+1}{3})n\rceil$.

Case(iii): $n \equiv 2(mod3)$

Consider $G = P_5 \odot \overline{K_m}$. *Consider* $V(G) = \{f_i, g_j/1 \leq i \leq 5, 1 \leq j \leq 5m\}$ *and* $S_m = \{f_2, f_5, g_j/1 \leq j \leq 5m\} \subset V(G)$. *Proceeding the same argument as in Case(i), we get* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq mn\} \cup \{f_n\}$ *is a minimum SMD set of* G. *Therefore* $\gamma_{sm}(G) = \lceil(\frac{3m+1}{3})n\rceil$. *Finally,* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq mn\} \bigcup \begin{cases} \phi & if \quad r = 0 \\ f_n & if \quad r = 1, 2 \end{cases}$ *is a minimum SMD set of* G. *Hence* $\gamma_{sm}(G) = \{ \lceil(\frac{3m+1}{3})n\rceil \quad if \quad n \equiv 0, 1, 2(mod3) \quad , m \geq 1$.

Theorem 3.5. *If* $G = C_n \odot \overline{K_m}$, *then*
$\gamma_{sm}(G) = \{ \lceil(\frac{3m+1}{3})n\rceil \quad if \quad n \equiv 0, 1, 2(mod3) \quad , m \geq 1, n \geq 3$.

Proof: *This follows from Theorem* 3.4

Theorem 3.6. *If* $G = Y_{n+1} \odot \overline{K_m}$, *then*
$\gamma_{sm}(G) = \left\{ \lceil\frac{(3m+1)n+3m}{3}\rceil \quad if \quad n \equiv 0, 1, 2(mod3) \right. \quad , m \geq 1, n \geq 3$

Proof: *Let* G *be the corona product of* Y*-tree graph* Y_{n+1} *and complement of complete graph* $\overline{K_m}$ *whose vertex set is* $V(G) = \{f_i, g_j/1 \leq i \leq n+1, 1 \leq j \leq (n+1)m\}$ *and edge set is* $E(G) = \{f_i f_{i+1}/1 \leq i \leq n-2\} \cup \{f_i f_{i+1}, f_i f_{i+2}/i = n-1\} \cup \{ f_i g_{m(i-1)+j}/1 \leq j \leq m, 1 \leq i \leq n+1\}$ *such that* $|V(G)| = (m+1)(n+1)$ *and* $|E(G)| = (m+1)(n+1) - 1$. *Now we consider the following cases.*

Case(i): $n \equiv 0(mod3)$

Consider $G = Y_{3+1} \odot \overline{K_m}$. *Consider* $V(G) = \{f_i, g_j/1 \leq i \leq 4, 1 \leq j \leq 4m\}$ *and* $S_m = \{f_2, g_j/1 \leq j \leq 4m\} \subset V(G)$. *For each* $g \in V \backslash S_m$, *then there exists* $f \in S_m$ *such that* g *is adjacent to* f *and* $\{S_m \backslash f\} \cup \{g\}$ *is a monophonic dominating set of* G. *Therefore* S_m *is a minimum SMD set of* G. *In general* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq (n+1)m\}$ *is a minimum SMD set of* G. *Therefore* $|S_m| = \lceil\frac{(3m+1)n+3m}{3}\rceil$.

Case(ii): $n \equiv 1(mod3)$

Consider $G = Y_{4+1} \odot \overline{K_m}$. *Consider* $V(G) = \{f_i, g_j/1 \leq i \leq 5, 1 \leq j \leq 5m\}$ *and* $S_m = \{f_2, f_3, g_j/1 \leq j \leq 5m\} \subset V(G)$. *Proceeding the same argument as in Case(i), we get* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq (n+1)m\} \cup \{f_{n-1}\}$ *is a minimum SMD set of* G. *Therefore* $|S_m| = \lceil\frac{(3m+1)n+3m}{3}\rceil$.

Case(iii): $n \equiv 2(mod3)$

Consider $G = Y_{5+1} \odot \overline{K_m}$. *Consider* $V(G) = \{f_i, g_j/1 \leq i \leq 6, 1 \leq j \leq 6m\}$ *and* $S_m = \{f_2, f_4, g_j/1 \leq j \leq 6m\} \subset V(G)$. *Proceeding the same argument as in Case(i),* *we get* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq (n+1)m\} \cup \{f_{n-1}\}$ *is a minimum SMD set of* G*. Therefore* $|S_m| = \lceil\frac{(3m+1)n+3m}{3}\rceil$*. Finally we conclude that,* $S_m = \{\bigcup_{i=0}^{k-1} f_{3i+2}, g_j/1 \leq j \leq (n+1)m\} \bigcup \left\{\begin{array}{ll} \phi & if \quad r=0 \\ f_{n-1} & if \quad r=1,2 \end{array}\right.$ *is a minimum SMD set of* G*. Hence* $\gamma_{sm}(G) = \left\{ \lceil\frac{(3m+1)n+3m}{3}\rceil \quad if \quad n \equiv 0,1,2(mod3) \right., m \geq 1, n \geq 3.$

Theorem 3.7. *If* $G = K_{1,n-1} \odot \overline{K_m}$*, then* $\gamma_{sm}(G) = m(n-1)+1, n \geq 3, m \geq 1.$

Proof: *Let* G *be the corona product of Star graph* $K_{1,n-1}$ *and complement of complete graph* $\overline{K_m}$ *whose vertex set is* $V(G) = \{f_i, g_j/1 \leq i \leq n-1, 1 \leq j \leq m(n-1)\}$ *and edge set is* $E(G) = \{\{\bigcup_{i=1}^{n-1} f_i f_{i+1}\} \cup \{f_i g_{m(i-1)+j}/1 \leq j \leq m, 1 \leq i \leq n-1\}\}$ *such that* $|V(G)| = (n-1)(m+1)$ *and* $|E(G)| = (n-1)(m+1)-1$*. Let* $P = \{g_i/1 \leq i \leq m(n-1)\}$ *be the set of end vertices of* G*. By observation(2.2),* P *is a subset of every SMD set of* G*. For each* $g \in V\backslash P$*, there exists* $f \in P$ *such that* g *is adjacent to* f *and* $\{P\backslash f\} \cup \{g\}$*is not a SMD set of* G*. Therefore* $\gamma_{sm}(G) \geq m(n-1)+1$*. Let* $P' = P \cup \{f_1\}$*. For each* $g \in V\backslash P'$*, then there exists* $f \in P'$ *such that* g *is adjacent to* f *and* $\{P'\backslash f\} \cup \{g\}$ *is a monophonic dominating set of* G*. Then* P' *is a minimum SMD set of* G*, so that* $\gamma_{sm}(G) = m(n-1)+1$*.*

Theorem 3.8. *If* $G = Hd_n \odot \overline{K_m}$ *,* $\gamma_{sm}(G) = (2n-2)(m+1)-n, m \geq 1, n \geq 3.$

Proof: *Let* G *be the corona product of Hurdle graph* Hd_n *and complement of complete graph* $\overline{K_m}$ *whose vertex set is* $V(G) = \{f_i, g_j, h_k, d_l/1 \leq i \leq n, 1 \leq j \leq (2n-2)m, 1 \leq k \leq n-2, 1 \leq l \leq (n-2)m\}$ *and edge set is* $E(G) = \{f_i f_{i+1}, h_j f_{j+1}/1 \leq i \leq n-1, 1 \leq j \leq n-2\} \cup \{f_i g_{m(i-1)+j}/1 \leq j \leq m, 1 \leq i \leq n\} \cup \{h_i d_{m(i-1)+j}/1 \leq j \leq m, 1 \leq i \leq n-2\}$ *such that* $|V(G)| = (2n-2)(m+1)$ *and* $|E(G)| = (2n-2)(m+1)-1$*. Let* $P = \{g_i, d_j/1 \leq i \leq mn, 1 \leq j \leq (n-2)m\}$ *be the set of end vertices of* G*. By observation(2.2),* P *is a subset of every SMD set of* G*. For each* $g \in V\backslash P$*, there exists* $f \in P$ *such that* g *is adjacent to* f *and* $\{P\backslash f\} \cup \{g\}$*is not a SMD set of* G*. Therefore* $\gamma_{sm}(G) \geq (2n-2)(m+1)-n$*. Let* $P' = P \cup \{f_i/2 \leq i \leq n-1\}$*. Then for each* $g \in V\backslash P'$*, then there exists* $f \in P'$ *such that* g *is adjacent to* f *and* $\{P'\backslash f\} \cup \{g\}$ *is a monophonic dominating set of* G*. Therefore* P' *is a minimum SMD set of* G*, so that* $\gamma_{sm}(G) = (2n-2)(m+1)-n$*.*

4. Conclusion

The study on domination of different graph structures is one of the potential areas of research. In this article, we introduced the new concept namely 'SMD number of cartesian and corona product of graphs'. To find new theorems on Secure Monophonic Domination Number for other graph families is an open area of research.

5. Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

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