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AN ADJACENT EDGE GRACEFUL LABELING OF PENTAGONAL SNAKE GRAPH

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Abstract. Let G be a graph with p vertices and q edges. The graph G is said to be an adjacent edge graceful graph if there exists bijection mapping $f : E(G) \to \{1, 2, 3, ..., q\}$ such that the induced vertex mapping $f^* : V(G) \to N$, where N is a natural number by $f^*(r) = \sum_k f(e_k)$ taken over all edges e_k incident to adjacent vertices of r is an injection. In this article we will go to prove the Pentagonal snake, Alternative Pentagonal Snake, Double Pentagonal snake and Alternative Double Pentagonal Snake are Adjacent Edge Graceful graph.

Keywords: Graph labeling, Adjacent edge graceful graphs, Pentagonal snake graphs, Alternative Pentagonal snake graphs

AMS Subject Classification: 05C78.

1. Introduction

In this article, we shall examine basic finite and undirected graphs. Assume that the vertex set and edge set of the graph G is represent by $V(G)$ and $E(G)$. For all terminologies and comments we follow Harary[7] . Graph theory encompasses many different topics. Labeling of graphs is one of the topics in Graph Theory. Graph labeling is applied in many areas like communication network address, radar, error-correcting codes, cryptography, etc. "Alex Rosa initialy conceptualized graph labeling in 1967[1]. He called a function f is a β-valuation of a graph G with q edges if f is an injecton from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that when each edge rs is assigned the label $|f(r) - f(s)|$, the resulting edge labels are distinct values". Also he named it β-labeling. In 1972, S.W.Golomb[5] renamed β-labeling by graceful labeling. "Harmonius labeling was initialy conceptualized by Graham and Slone in 1980[6]. A graph G is known as Harmonious labeling if there is an injection f from the vertices of G to the group of integers mod(q)

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such that, when each edge rs is assinged the label $(f(r)+f(s))(mod\ q)$, the resulting edge labels have distinct values". Following these labeling many kinds of labeling have been introduced in the field of graph labeling such as cordial labeling, Mean labeling, Prime labeling, Square labeling, etc. A detailed survey can be found in Gallian[3]. "In 1985, Lo [9] was introduced the idea of edge-graceful labeling that is, a graph G is described as edge-graceful if there exists a bijection $f : E \to \{1, 2, ..., q\}$ such that the induced mapping $f^* : V \to \{0, 1, 2, ..., p-1\}$ defined by $f^*(r) = \sum_{e=(r,s)\in E} f(e)\ (mod\ p)$ is also a bijection". In 2009, Solairaju and Chithra [17]introduced a labeling of a graph known as edge odd graceful labeling, which is a bijective function $f : E(G) \to \{1, 3, ..., 2q-1\}$ such that the induced map $f^* : V(G) \to \{0, 1, 2, ..., 2q-1\}$ given by $f^*(r) = (\sum_{rs\in E(G)} f(rs))mod(2q)$ is an injective.

"In 2014, a new labeling was introduced which is called as Adjacent Edge Graceful(AEG) labeling of a graph in[20], that is, the graph G is described as an AEG-graph if there exists bijection mapping $f : E(G) \to \{1, 2, 3, ..., q\}$ such that the induced vertex mapping $f^* : V(G) \to N$, where N is a natural number by $f^*(r) = \sum_k f(e_k)$ taken over all edges e_k incident to adjacent vertices of r is an injection. The functon f is called an AEG-labeling of G ". They showed that some graphs such as path P_n, cycle C_n, triangular snake T_n, Helm H_n, fan f_n, etc., are an AEG-graphs[20]. "From this labeling they introduced Strongly Adjacent Edge Graceful (SAEG) labeling in 2014 , that is a (p, q) graph $G(V, E)$ is described to be a SAEG-graph if there exists a bijection $f : E(G) \to \{1, 2, ..., q\}$ such that the induced mapping $f^* : V(G) \to \{0, 1, 2, ..., p-1\}$ by, $f^*(r) = (\sum_k f(e_k))(mod\ p)$ taken overall edges e_k incident to adjacent vertices of r is a bijection and the function f is called a SAEG-Labeling of G [22] ". They proved that every cycle C_n for odd positive integer, the graph mK_3 for odd positive m, the graph mC_5 for odd positive integer and the path P_n for $5 \le n \le 7$ are SAEG-graphs.

Motivated by these ideas we continue to examine the graphs such as pentagonal snake PS_n, alternative pentagonal snake $A(PS_n)$, double pentagonal snake DPS_n, alternative double pentagonal snake $A(DPS_n)$ are AEG-graph. Moreover A graph which have more than one vertex which are adjacent to same vertex is not an AEG-graphs. Several results have been shown to be not an AEG-graphs in [20, 21]. Our aim is to explore and detect some new types of graphs that exhibit AEG-graphs.

2. PRELIMINARIES

Definition 2.1. [20, 21] *"The graph G is described as an* ***Adjacent edge graceful graph (AEG-graph)*** *if there exists bijection mapping $f : E(G) \to \{1, 2, 3, ..., q\}$ such that the induced vertex mapping $f^* : V(G) \to N$, where N is a natural number by $f^*(r) = \sum_k f(e_k)$ taken over all edges e_k incident to adjacent vertices of r is an injection.The functon f is called an AEG-labeling of G ".*

Definition 2.2. [11, 15, 19, 24] *"The* ***Pentagonal snake*** *PS_n is obtained from a path $r_1, r_2, ..., r_n$ by joining r_k and r_{k+1} ; $1 \le k \le n-1$, to two new vertices s_k, t_k and by joining s_k and t_k to a new vertex u_k respectively that is every path's edge is changed out by a cycle C_5."*

Definition 2.3. [19, 24] *"An* ***Alternative pentagonal snake*** *$A(PS_n)$ is obtained from a path $r_1, r_2, ..., r_n$ by joining r_k and r_{k+1} to two new vertices s_k, t_k and by joining s_k and t_k to a new vertex u_k respectively that is every path's alternate edge is changed out by a cycle C_5."*

Definition 2.4. [11] *The* ***Double pentagonal snake*** *DPS_n is obtained by two pentagonal snakes that shared a common path.*

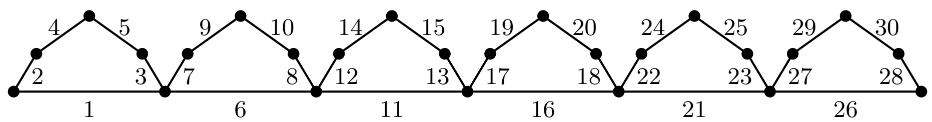
Definition 2.5. *An **Alternative double pentagonal snake** $A(DPS_n)$ is obtained by two alternative pentagonal snakes that follow a common path.*

3. Main results

Theorem 3.1. *The Pentagonal Snake $PS_n,\ n \geq 3$ is an AEG-graph*

Proof. Let $PS_n, n \geq 3$ be the Pentagonal Snake with $4n-3$ vertices and $5(n-1)$ edges.
Let $V(PS_n) = \begin{cases} r_k : 1 \leq k \leq n \\ s_k, t_k, u_k : 1 \leq k \leq n-1 \end{cases}$
and $E(PS_n) = \Big\{ r_k r_{k+1}, r_k s_k, s_k t_k, t_k u_k, u_k r_{k+1} : 1 \leq k \leq n-1$
Thus $|V(PS_n)| = 4n-3$ *and* $|E(PS_n)| = 5(n-1)$.
We now define the labeling of edges $f : E(PS_n) \to \{1, 2, ..., 5(n-1)\}$ by
$f(r_k r_{k+1}) = 5k-4,\ 1 \leq k \leq n-1$
$f(r_k s_k) = 5k-3,\ 1 \leq k \leq n-1$
$f(s_k t_k) = 5k-1,\ 1 \leq k \leq n-1$
$f(t_k u_k) = 5k,\ 1 \leq k \leq n-1$
$f(u_k r_{k+1}) = 5k-2,\ 1 \leq k \leq n-1.$
Consequently, the following are the induced vertex labels
for $n = 3$,
$f^*(r_1) = 23;\ f^*(r_2) = 41;\ f^*(r_3) = 35;$
$f^*(s_k) = 12(2k-1),\ 1 \leq k \leq n-1$
$f^*(t_k) = 2(10k-3),\ 1 \leq k \leq n-1$
$f^*(u_k) = 7k+19,\ 1 \leq k \leq n-1.$
for $n \geq 4$,
$f^*(r_1) = 23,\ f^*(r_2) = 64,$
$f^*(r_{k+2}) = 2(30k-29); 1 \leq k \leq n-4,$
$f^*(r_{n-1}) = 5(10n-23),\ f^*(r_n) = 5(6n-11),$
$f^*(s_1) = 12,$
$f^*(u_k) = 30k-4; 1 \leq k \leq n-1,$
$f^*(s_{k+1}) = 6(5k+1); 1 \leq k \leq n-1,$
$f^*(u_n) = 20n-27,$
$f^*(t_k) = 2(10k-3); 1 \leq k \leq n-1$
As a result, each vertex label is unique. Hence the Pentagonal Snake $PS_n,\ n \geq 3$ is an AEG-graph.

The labeling pattern of AEG-labeling of the Pentagonal Snake is under below, □



Theorem 3.2. *Every Alternative pentagonal snake $A(PS_n)$ is an AEG-graph for odd n if the pentagonal is start from the first vertex.*

Proof. let$A(PS_n); n \geq 5$ be an Alternative pentagonal snake graph and the pentagonal snake starts from the first vertex.

Let $V(A(PS_n)) = \begin{cases} r_k : 1 \leq k \leq n \\ s_k, t_k, u_k : 1 \leq k \leq \frac{n-1}{2} \end{cases}$ and

$E(A(PS_n)) = \Big\{ r_k r_{k+1}, r_{2k-1} s_k, s_k t_k, t_k u_k, u_k r_{2k} : 1 \leq k \leq \frac{n-1}{2}$

Thus $|V(A(PS_n))| = \frac{5n-3}{2}$ and $|E(A(PS_n))| = 3(n-1)$.

We now define the labeling of edges $f : E(A(PS_n)) \to \{1, 2, ..., 3n-3\}$ by

$f(r_{2k-1} r_{2k}) = 6k - 1;\ 1 \leq k \leq \frac{n-1}{2}$,

$f(r_{2k} r 2k + 1) = 6k;\ 1 \leq k \leq \frac{n-1}{2}$,

$f(r_{2k-1} s_k) = 6k - 5;\ 1 \leq k \leq \frac{n-1}{2}$,

$f(s_k t_k) = 2(3k - 2);\ 1 \leq k \leq \frac{n-1}{2}$,

$f(t_k u_k) = 3(2k - 1);\ 1 \leq k \leq \frac{n-1}{2}$,

$f(u_k r_{2k}) = 2(3k - 1);\ 1 \leq k \leq \frac{n-1}{2}$

Consequently, the following are the induced vertex labels

$f^*(r_1) = 18,\ f^*(r_{2k}) = 48k - 11;\ 1 \leq k \leq \frac{n-3}{2}$

$f^*(r_{2k+1}) = 3(16k + 5);\ 1 \leq k \leq \frac{n-3}{2}$,

$f^*(r_{n-1}) = 18n - 35,\ f^*(r_n) = 3(3n - 4)$,

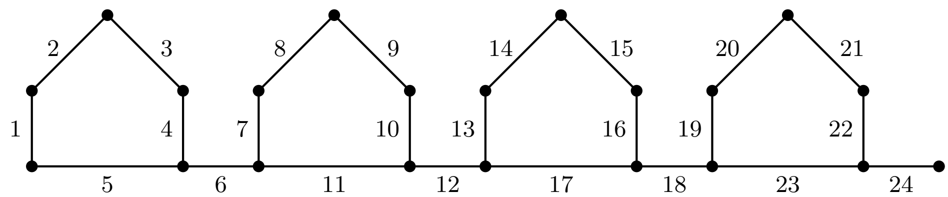
$f^*(s_k) = 30k - 19;\ 1 \leq k \leq \frac{n-1}{2}$,

$f^*(t_k) = 2(12k - 7);\ 1 \leq k \leq \frac{n-1}{2}$,

$f^*(u_k) = 10(3k - 1);\ 1 \leq k \leq \frac{n-1}{2}$.

As a result, each vertex label is unique. Hence Every Alternative pentagonal snake $A(PS_n)$ is an AEG-graph for odd n if the pentagonal is start from the first vertex.

The labeling pattern of AEG-labeling of the Alternative Pentagonal snake starting fron the first vertex is under below,



□

Theorem 3.3. *Every Alternative Pentagonal snake is an AEG-graph if the pentagonal starts from the second vertex.*

Proof. Let $A(PS_n)$ be the Alternative Pentagonal snake graph.
In this theorem we prove this result by two cases for odd n and for even n.

case(i) n **is odd and** $n \geq 5$

In this case, let $V(A(PS_n)) = \begin{cases} r_k : 1 \leq k \leq n \\ s_k, t_k, u_k : 1 \leq k \leq \frac{n-1}{2} \end{cases}$ and

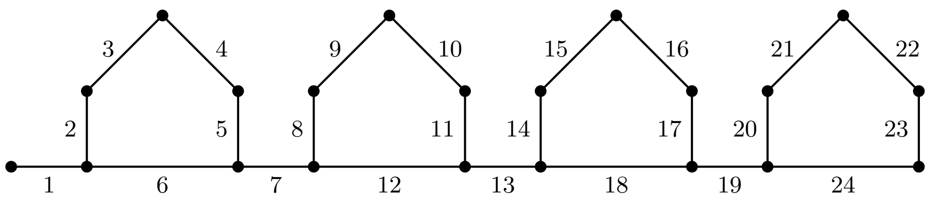
$E(A(PS_n)) = \begin{cases} r_k r_{k+1} : 1 \leq k \leq n-1 \\ r_{2k} s_k, s_k t_k, t_k u_k, u_k r_{2k+1} : 1 \leq k \leq \frac{n-1}{2} \end{cases}$

Thus $|V(A(PS_n))| = \frac{5n-3}{2}$ and $|E(A(PS_n))| = 3(n-1)$.

We now define the labeling of edges $f : E(A(PS_n)) \to \{1, 2, ..., \frac{5n-3}{2}\}$ by

$f(r_{2k-1}r_{2k}) = 6k-5;\ 1 \le k \le \frac{n-1}{2}$,
$f(r_{2k}r_{2k+1}) = 6k;\ 1 \le k \le \frac{n-1}{2}$,
$f(r_{2k}s_k) = 2(3k-2);\ 1 \le k \le \frac{n-1}{2}$,
$f(s_kt_k) = 3(2k-1);\ 1 \le k \le \frac{n-1}{2}$,
$f(t_ku_k) = 2(3k-1);\ 1 \le k \le \frac{n-1}{2}$,
$f(u_kr_{2k+1}) = 6k-1;\ 1 \le k \le \frac{n-1}{2}$
Consequently, the following are the induced vertex labels
$f^*(r_1) = 9,\ f^*(r_2) = 24$,
$f^*(r_{2k+1}) = 3(16k-1); 1 \le k \le \frac{n-3}{2}$
$f^*(r_{2(k+1)}) = 48k+23; 1 \le k \le \frac{n-5}{2}$,
$f^*(r_{n-1}) = 21n-47, f^*(r_n) = 3(5n-9)$,
$f^*(s_k) = 2(15k-7); 1 \le k \le \frac{n-1}{2}$,
$f^*(t_k) = 2(12k-5); 1 \le k \le \frac{n-1}{2}$
$f^*(u_k) = 5(6k-1); 1 \le k \le \frac{n-3}{2}$,
$f^*(u_n) = 6(2n-3)$.
As a result, each vertex label is unique. Hence in this case the Alternative Pentagonal snake is an AEG-graph.

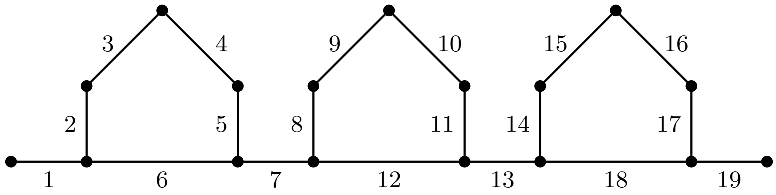
The labeling pattern of AEG-graph of $A(PS_9)$ is under below,



Case(ii) n **is even and** $n \ge 6$
In this case, let $V(A(PS_n)) = \begin{cases} r_k : 1 \le k \le n \\ s_k, t_k, u_k : 1 \le k \le \frac{n-1}{2} \end{cases}$ and
$E(A(PS_n)) = \begin{cases} r_kr_{k+1} : 1 \le k \le n-1 \\ r_{2k}s_k, s_kt_k, t_ku_k, u_kr_{2k+1} : 1 \le k \le \frac{n-2}{2} \end{cases}$
Thus $|V(A(PS_n))| = \frac{5n-6}{2}$ *and* $|E(A(PS_n))| = 3n-2$.
We now define the labeling of edges $f : E(A(PS_n)) \to \{1, 2, ..., \frac{5n-3}{2}\}$ by
$f(r_{2k-1}r_{2k}) = 6k-5;\ 1 \le k \le \frac{n}{2}$,
$f(r_{2k}r_{2k+1}) = 6k;\ 1 \le k \le \frac{n-2}{2}$
$f(r_{2k}s_k) = 2(3k-2);\ 1 \le k \le \frac{n-2}{2}$,
$f(s_kt_k) = 3(2k-1);\ 1 \le k \le \frac{n-2}{2}$
$f(t_ku_k) = 2(3k-1);\ 1 \le k \le \frac{n-2}{2}$,
$f(u_kr_{2k+1}) = 6k-1;\ 1 \le k \le \frac{n-2}{2}$
Consequently, the following are the induced vertex labels
$f^*(r_1) = 9,\ f^*(r_2) = 24$,
$f^*(r_{2k+1}) = 3(16k-1);\ 1 \le k \le \frac{n-4}{2}$
$f^*(r_{2(k+1)}) = 48k+23;\ 1 \le k \le \frac{n-4}{2}$,
$f^*(r_{n-1}) = 18n-47,\ f^*(r_n) = 9(n-2)$,
$f^*(s_k) = 2(15k-7);\ 1 \le k \le \frac{n-2}{2}$,

$f^*(t_k) = 2(12k-5);\ 1 \leq k \leq \frac{n-2}{2}$
$f^*(u_k) = 5(6k-1);\ 1 \leq k \leq \frac{n-2}{2}$.
As a result, each vertex label is unique. Hence in this case the Alternative Pentagonal snake is an AEG-graph.

The labeling pattern of AEG-graph of $A(PS_8)$ is under below,



Hence every Alternative Pentagonal snake is an AEG-graph if the pentagonal starts from the second vertex. □

Theorem 3.4. *The Double Pentagonal Snake is an AEG-graph.*

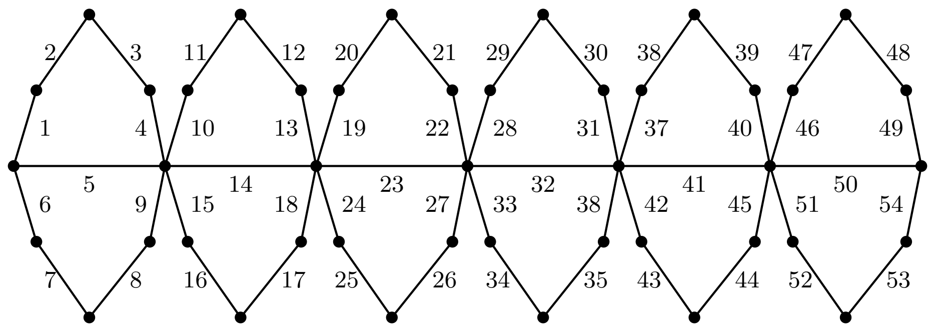
Proof. Let $DPS_n, n \geq 3$ be the Double pentagonal snake.
let $V(DPS_n) = \begin{cases} r_k : 1 \leq k \leq n \\ s_k, t_k, u_k, s'_k, t'_k, u'_k : 1 \leq k \leq n-1 \end{cases}$ and
$E(DPS_n) = \Big\{ r_k r_{k+1}, r_k s_k, s_k t_k, t_k u_k, u_k r_{2k}, r_k s'_k, s'_k t'_k, t'_k u'_k, u'_k r_{2k} : 1 \leq k \leq n-1$

Thus $|V(DPS_n)| = 7n-6$ and $|E(DPS_n)| = 9(n-1)$.
We now define the labeling of edges $f : E(DPS_n) \to \{1, 2, ..., 9(n-1)\}$ by
$f(r_k r_{k+1}) = 9k-4;\ 1 \leq k \leq n-1$,
$f(r_k s_k) = 9k-8;\ 1 \leq k \leq n-1$
$f(s_k t_k) = 9k-7;\ 1 \leq k \leq n-1$
$f(t_k u_k) = 3(3k-2);\ 1 \leq k \leq n-1$
$f(t_k r_{2k}) = 9k-5;\ 1 \leq k \leq n-1$
$f(r_k s'_k) = 3(3k-1);\ 1 \leq k \leq n-1$
$f(s'_k t'_k) = 9k-2;\ 1 \leq k \leq n-1$
$f(t'_k u'_k) = 9k-1;\ 1 \leq k \leq n-1$
$f(u'_k r_{2k}) = 9k;\ 1 \leq k \leq n-1$.
Consequently, the following are the induced vertex labels

$for\ n = 3$,
$f^*(r_1) = 73,\ f^*(r_2) = 133,\ f^*(r_n) = 9(10n-17)$,
$f^*(s_1) = 17,\ f^*(s_2) = 80$,
$f^*(t_k) = 2(18k-13);\ 1 \leq k \leq n-1$,
$f^*(u_k) = 2(36k-5); 1 \leq k \leq n-2$,
$f^*(u_n) = 45n-67,\ f^*(s'_1) = 27,\ f^*(s'_2) = 90$,
$f^*(t'_k) = 6(6k-1); 1 \leq k \leq n-1$,
$f^*(u'_k) = 72k; 1 \leq k \leq n-2,\ f^*(u'_n) = 78$.

$for\ n \geq 4$,
$f^*(r_1) = 73,\ f^*(r_2) = 199$,
$f^*(r_{k+2}) = 10(18k+19);\ 1 \leq k \leq n-4$

$f^*(r_{n-1}) = 153n - 335,\ f^*(r_n) = 9(10n - 17),$
$f^*(s_1) = 17,$
$f^*(s_{k+1}) = 8(9k + 1);\ 1 \le k \le n - 2$
$f^*(t_k) = 2(18k - 13);\ 1 \le k \le n - 1$
$f^*(u_k) = 2(36k - 5); 1 \le k \le n - 2$
$f^*(u_n) = 45n - 67,\ f^*(s'_1) = 27,$
$f^*(s'_{k+1}) = 18(4k + 1);\ 1 \le k \le n - 1$
$f^*(t'_k) = 6(6k - 1); 1 \le k \le n - 1$
$f^*(u'_k) = 72k; 1 \le k \le n - 2$
$f^*(u'_n) = 3(15n - 19).$
As a result, each vertex label is unique. Hence the Double pentagonal snake is an AEG-graph.
The labeling pattern of AEG-graph of DPS_7 is under below,



□

Theorem 3.5. *The Alternative Double Pendagonal Snake is an AEG-graph for odd n if the double pentagonal start from the first vertex.*

Proof. Let $A(DPS_n)$, be the alternative Double Pendagonal Snake and n is odd.

The Alternative Double Pendagonal Snake start from the first vertex and end with the $(n-1)^{th}$ vertex.

Let $V(A(DPS_n)) = \begin{cases} r_k : 1 \le k \le n \\ s_k, t_k, u_k, s'_k, t'_k, u'_k : 1 \le k \le \frac{n-1}{2} \end{cases}$ and

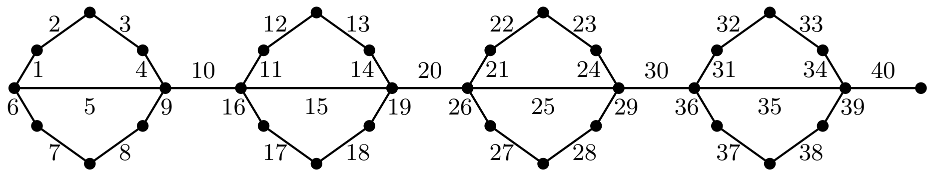
$$E(A(DPS_n)) = \begin{cases} r_k r_{k+1} : 1 \le k \le n-1 \\ r_{2k-1}s_k, s_k t_k, t_k u_k, u_k r_{2k}, \\ r_{2k-1}s'_k, s'_k t'_k, t'_k u'_k, u'_k r_{2k} : 1 \le k \le \frac{n-1}{2} \end{cases}$$

Thus $|V(A(DPS_n))| = 4n - 3$ and $|E(A(DPS_n))| = 5(n-1)$.
We now define the labeling of edges $f : E(A(DPS_n)) \to \{1, 2, ..., 5(n-1)\}$ by
$f(r_k r_{k+1}) = 5k;\ 1 \le k \le n - 1,$
$f(r_{2k-1}s_k) = 10k - 9;\ 1 \le k \le \frac{n-1}{2}$
$f(s_k t_k) = 2(5k - 4);\ 1 \le k \le \frac{n-1}{2},$
$f(t_k u_k) = 10n - 7;\ 1 \le k \le \frac{n-1}{2}$
$f(u_k r_{2k}) = 2(5k - 3);\ 1 \le k \le \frac{n-1}{2},$
$f(r_{2k-1}s'_k) = 2(5k - 2);\ 1 \le k \le \frac{n-1}{2}$

$f(s'_k t'_k) = 10k - 3;\ 1 \le k \le \frac{n-1}{2}$,
$f(t'_k u'_k) = 2(5k - 1);\ 1 \le k \le \frac{n-1}{2}$
$f(u'_k r_{2k}) = 10k - 1;\ 1 \le k \le \frac{n-1}{2}$.
Consequently, the following are the induced vertex labels
$f^*(r_1) = 44$,
$f^*(r_{2k}) = 8(15k - 4);\ 1 \le k \le \frac{n-3}{2}$,
$f^*(r_{2k+1}) = 8(15k + 4); 1 \le k \le \frac{n-3}{2}$,
$f^*(r_{n-1}) = 45n - 89,\ f^*(r_n) = 4(5n - 8)$,
$f^*(s_k) = 60k - 43;\ 1 \le k \le \frac{n-1}{2}$,
$f^*(t_k) = 10(4k - 3);\ 1 \le k \le \frac{n-1}{2}$,
$f^*(u_k) = 3(20k - 9);\ 1 \le k \le \frac{n-1}{2}$,
$f^*(s'_k) = 3(20k - 11);\ 1 \le k \le \frac{n-1}{2}$,
$f^*(t'_k) = 10(4k - 1);\ 1 \le k \le \frac{n-1}{2}$,
$f^*(u'_k) = 60k - 17;\ 1 \le k \le \frac{n-1}{2}$, .

As a result, each vertex label is unique. Hence the Alternative Double Pentagonal Snake is an AEG-graph for odd n if the double pentagonal is start from the first vertex.

The labeling pattern of AEG-labeling of the Alternative Double Pentagonal Snake is under below.



□

Theorem 3.6. *Every Alternative double pentagonal snake is an AEG-graph when the pentagonal starts from the second vertex.*

Proof. let $A(DPS_n)$ be the Alternative double pentagonal snake starting the double pentagonal is an second vertex.

Case(i) n is odd

Let $V(A(DPS_n)) = \begin{cases} r_k : 1 \le k \le n \\ s_k, t_k, u_k, s'_k, t'_k, u'_k : 1 \le k \le \frac{n-1}{2} \end{cases}$ and

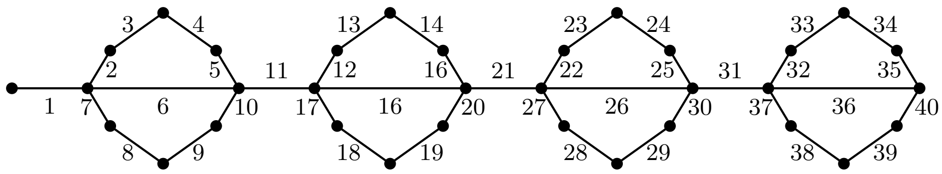
$E(A|(DPS_n)) = \begin{cases} r_k r_{k+1} : 1 \le k \le (n-1) \\ r_{2k}s_k, s_k t_k, t_k u_k, u_k r_{2k+1}, r_{2k}s'_k, s'_k t'_k, t'_k u'_k, u'_k r_{2k+1} : 1 \le k \le \frac{n-1}{2} \end{cases}$

Thus $|V(A(DPS_n))| = 4n - 3$ and $|E(A(DPS_n))| = 5n - 6$

We now define the labeling of edges $f : E(A(DPS_n)) \to \{1, 2, ..., 5n - 6\}$ by
$f(r_k r_{k+1}) = 5k - 4;\ 1 \le k \le n - 1$,
$f(r_{2k}s_k) = 2(5k - 4);\ 1 \le k \le \frac{n-1}{2}$
$f(s_k t_k) = 10k - 7;\ 1 \le k \le \frac{n-1}{2}$,
$f(t_k u_k) = 2(5k - 3);\ 1 \le k \le \frac{n-1}{2}$

$f(u_k r_{2k+1}) = 5(2k-1);\ 1 \le k \le \frac{n-1}{2}$,
$f(r_{2k}s'_k) = 10k-3;\ 1 \le k \le \frac{n-1}{2}$
$f(s'_k t'_k) = 2(5k-1);\ 1 \le k \le \frac{n-1}{2}$,
$f(t'_k u'_k) = 10k-1;\ 1 \le k \le \frac{n-1}{2}$
$f(u'_k r_{2k+1}) = 10k;\ 1 \le k \le \frac{n-1}{2}$
Consequently, the following are the induced vertex labels
$f^*(r_1) = 16,\ f^*(r_2) = 53$,
$f^*(r_{2k+1}) = 20(6k-1);\ 1 \le k \le \frac{n-3}{2}$,
$f^*(r_{2(k+1)}) = 4(30k+11);\ 1 \le k \le \frac{n-5}{2}$,
$f^*(r_{n-1}) = 55n-132,\ f^*(r_n) = 40n-76$,
$f^*(s_k) = 60k-37;\ 1 \le k \le \frac{n-1}{2}$,
$f^*(t_k) = 2(20k-13);\ 1 \le k \le \frac{n-1}{2}$,
$f^*(u_k) = 3(20k-7);\ 1 \le k \le \frac{n-3}{2}$,
$f^*(u_n) = 25n-47$,
$f^*(s'_k) = 3(20k-9);\ 1 \le k \le \frac{n-1}{2}$,
$f^*(t'_k) = 2(20k-3);\ 1 \le k \le \frac{n-1}{2}$,
$f^*(u'_k) = 60k-11;\ 1 \le k \le \frac{n-3}{2}$,
$f^*(u'_n) = 25n-37$.
As a result, each vertex label is unique. Hence in this case the alternative double pentagonal snake is an AEG-graph.

The labeling pattern of AEG-graph of $A(DPS_9)$ is under below,



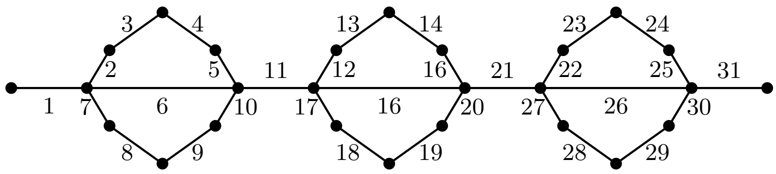
Case(ii) n **is even**

let $V(A(DPS_n)) = \begin{cases} r_k : 1 \le k \le n \\ s_k, t_k, u_k, s'_k, t'_k, u'_k : 1 \le k \le \frac{n-2}{2} \end{cases}$ and

$$E(A(DPS_n)) = \begin{cases} r_k r_{k+1} : 1 \le k \le (n-1) \\ r_{2k}s_k, s_k t_k, t_k u_k, u_k r_{2k+1}, r_{2k}s'_k, s'_k t'_k, t'_k u'_k, u'_k r_{2k+1} : 1 \le k \le \frac{n-2}{2} \end{cases}$$

Thus $|V(A(DPS_n))| = 4n-6\ and |E(A(DPS_n))| = 5n-9$
We now define the labeling of edges $f : E(A(DPS_n)) \to \{1, 2, ..., 5n-9\}$ by
$f(r_k r_{k+1}) = 5k-4;\ 1 \le k \le n-1$,
$f(r_{2k}s_k) = 2(5k-4);\ 1 \le k \le \frac{n-2}{2}$
$f(s_k t_k) = 10k-7;\ 1 \le k \le \frac{n-2}{2}$,
$f(t_k u_k) = 2(5k-3);\ 1 \le k \le \frac{n-2}{2}$
$f(u_k r_{2k+1}) = 5(2k-1);\ 1 \le k \le \frac{n-2}{2}$,
$f(r_{2k}s'_k) = 10k-3;\ 1 \le k \le \frac{n-2}{2}$
$f(s'_k t'_k) = 2(5k-1);\ 1 \le k \le \frac{n-2}{2}$,
$f(t'_k u'_k) = 10k-1;\ 1 \le k \le \frac{n-2}{2}$
$f(u'_k r_{2k+1}) = 10k;\ 1 \le k \le \frac{n-2}{2}$

Consequently, the following are the induced vertex labels
$f^*(r_1) = 16,\ f^*(r_2) = 53,$
$f^*(r_{2k+1}) = 20(6k-1); 1 \le k \le \frac{n-4}{2}$
$f^*(r_{2(k+1)}) = 4(30k+11); 1 \le k \le \frac{n-5}{2},$
$f^*(r_{n-1}) = 45n - 125,$
$f^*(r_n) = 4(5n-12),$
$f^*(s_k) = 60k - 37; 1 \le k \le \frac{n-2}{2},$
$f^*(t_k) = 2(20k-13); 1 \le k \le \frac{n-2}{2},$
$f^*(u_k) = 3(20k-7); 1 \le k \le \frac{n-2}{2},$
$f^*(s'_k) = 3(20k-9); 1 \le k \le \frac{n-2}{2},$
$f^*(t'_k) = 2(20k-3); 1 \le k \le \frac{n-2}{2},$
$f^*(u'_k) = 60k - 11; 1 \le k \le \frac{n-2}{2}.$
As a result, each vertex label is unique. Hence in this case the alternative double pentagonal snake is an AEG-graph.
The labeling pattern of AEG-graph of $A(DPS_8)$ is under below,



Hence every Alternative double pentagonal snake is an AEG-graph when the pentagonal starts from the second vertex. □

4. Conclusions

In this article we have proved the Pendagonal snake, Alternative Pendagonal snake, Double pentagonal snake, Alternative Double Pendagonal snake are an adjacent edge graceful graph. We havealso found various graphs that accepts AEG-graphs. It can be further investigated to graph operations such as union, intersection, etc., using this concept.

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