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ON TRIGONOMETRIC APPROXIMATION IN THE SPACE $L^{p(x)}$

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Abstract. In this paper we have introduced two new class of numerical sequences, named almost monotone decreasing (increasing) upper second mean sequences. Moreover, we have presented some results on trigonometric approximation of functions by means of a special transformation related to the partial sums of a Fourier series.

Keywords: Numerical sequences, classes $Lip(\alpha, p(x))$, trigonometric approximation, $L^{p(x)}$-norm.

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1. Introduction

Let $f \in L$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{1}$$

with its $n-$th partial sums at the point x

$$S_n(f;x) = \sum_{k=0}^{n} U_k(f;x),$$

where

$$U_0(f;x) := \frac{a_0}{2}; \quad U_k(f;x) := a_k \cos kx + b_k \sin kx, \quad k = 1, 2, \dots \ .$$

Let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. We consider the so-called Nörlund means) of the sums $S_n(f;x)$ defined by

$$N_n(f;x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m} S_m(f;x),$$

where $P_n := \sum_{m=0}^{n} p_m$, $p_{-1} := P_{-1} := 0$. In the case $p_m = 1$ for all $m \geq 0$, the means $N_n(f;x)$ reduced to the Cesàro mean given by equality

$$\sigma_n(f;x) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f;x).$$

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The approximation properties of the mean $\sigma_n(f;x)$ in classes $Lip(\alpha,p)$, $1 \le p < +\infty$, $0 < \alpha \le 1$ were established first by E. S. Quade [11]. His results are generalized by R. N. Mohapatra and D. C. Russell [10], P. Chandra [2]-[5] and L. Leindler [9].

Let $p : \mathbb{R} \to [1,\infty)$ be a measurable 2π periodic function. Denote by $L^{p(x)} = L^{p(x)}([0,2\pi])$ the set of all measurable 2π periodic functions f such that $m_p(\lambda f) < \infty$ for $\lambda = \lambda(f) > 0$, where

$$m_p(f) := \int_0^{2\pi} |f(x)|^{p(x)} dx.$$

$L^{p(x)}$ becomes a Banach space with respect to the norm

$$\|f\|_{p(x)} := \inf\left\{\lambda > 0 : m_p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

If the function $p(x) = p$ is a constant one $(1 \le p < \infty)$, then the space $L^{p(x)}$ is isometrically isomorphic to the Lebesgue space L^p.

If the function p satisfies

$$1 < p_- := \operatorname{ess}\inf_{x\in[0,2\pi]} p(x), \quad p_+ := \operatorname{ess}\sup_{x\in[0,2\pi]} p(x) < \infty, \tag{2}$$

then the function

$$p'(x) := \frac{p(x)}{p(x)-1}$$

is well defined and satisfies (2) itself.

The space $L^{p(x)}$ consists of all measurable 2π periodic functions f such that

$$\int_0^{2\pi} |f(x)g(x)| dx < \infty$$

for all measurable functions g with $m_{p'}(g) \le 1$.

Denote by $M(f)$ the Hardy-Littlewood maximal operator, defined for $f \in L^1$ by

$$M(f)(x) = \sup_I \frac{1}{|I|}\int_I |f(t)| dt, \quad x \in [0,2\pi],$$

where the supremum is taken over all intervals with $x \in I$.

It was proved in [6] that if the function $p(x)$ satisfies (2) and the condition

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \quad 0 < |x-y| \le \frac{1}{2}, \tag{3}$$

then the maximal operator $M(f)$ is bounded on $L^{p(x)}$, that is,

$$\|M(f)\|_{p(x)} \le A\|f\|_{p(x)} \tag{4}$$

for all $f \in L^{p(x)}$, where A is a constant depending only on p.

The set of all measurable 2π periodic functions $p : \mathbb{R} \to [0,\infty)$ satisfies the conditions (2) and (3) will be denoted by $\mathcal{M}$.

Let $p \in \mathcal{M}$ and $f \in L^{p(x)}$. The modulus of continuity of the function f is defined by equality

$$\Omega_{p(x)}(f,\delta) = \sup_{|h|\le\delta} \|T_h(f)\|_{p(x)}, \quad \delta > 0,$$

where

$$T_h(f;x) := \frac{1}{h}\int_0^h |f(x+t) - f(x)| dt.$$

The modulus of continuity $\Omega_{p(x)}(f,\delta)$ and the classical integral modulus of continuity $\omega_p(f,\delta)$ in the Lebesgue space L^p are equivalent (for details see [8]).

Let $p \in \mathcal{M}$ and $0 < \alpha \le 1$. Very recently, A. Guven and D. Israfilov [7] defined the Lipschitz class $Lip(\alpha, p(x))$ as

$$Lip(\alpha, p(x)) = \left\{ f \in L^{p(x)} : \Omega_{p(x)}(f, \delta) = \mathcal{O}(\delta^{\alpha}),\ \delta > 0 \right\},$$

and gave $L^{p(x)}$ counterparts of the results obtained by L. Leindler [9] and P. Chandra [5].

Before we write their results we need first to recall some known notions.

A sequence of positive real numbers $(p_n)_0^{\infty}$ is called almost monotone decreasing (increasing) if there exists a constant K, depending only on $(p_n)_0^{\infty}$ such that for all $n \ge m$ the inequality

$$p_n \le K p_m \quad (p_n \ge K p_m)$$

holds. Such sequences will be denoted by $(p_n)_0^{\infty} \in AMDS$ ($(p_n)_0^{\infty} \in AMIS$).

Among others they have proved the following.

Theorem 1.1 ([7]). *Let $p \in \mathcal{M}$, $0 < \alpha < 1$, $f \in Lip(\alpha, p(x))$ and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. If*

$$(p_n)_{n=0}^{\infty} \in AMDS$$

or

$$(p_n)_{n=0}^{\infty} \in AMIS \quad \textit{and} \quad (n+1)p_n = \mathcal{O}(P_n),$$

then

$$\|f - N_n(f)\|_{p(x)} = \mathcal{O}(n^{-\alpha}).$$

holds.

Let $(a_{n,k})$ be a lower triangular infinite matrix of real numbers such that

$$a_{n,k} \ge 0, \quad k \le n; \quad a_{n,k} = 0, \quad k > n, \quad \text{and} \quad \sum_{k=0}^{n} a_{n,k} = 1, \quad (k, n = 0, 1, \dots).$$

Let $A_{n,k} = \frac{1}{k+1} \sum_{i=n-k}^{n} a_{n,i}$. The following class of numerical sequences was introduced in [13]:

If $(A_{n,k}) \in AMDS$ ($(A_{n,k}) \in AMIS$), then it is said that $(a_{n,k})$ is an almost monotone decreasing (increasing) upper mean sequence, briefly $(a_{n,k}) \in AMDUMS$ ($(a_{n,k}) \in AMIUMS$).

Now denoting $A_{n,k}^{(2)} = \frac{2}{(k+1)(k+2)} \sum_{i=n-k}^{n} a_{n,i}$, we introduce two new classes of numerical sequences as follows:

If $(A_{n,k}^{(2)}) \in AMDS$ ($(A_{n,k}^{(2)}) \in AMIS$), then we shall say that $(a_{n,k})$ is an *almost monotone decreasing (increasing) upper second mean sequence*, briefly $(a_{n,k}) \in AMDUSMS$ ($(a_{n,k}) \in AMIUSMS$).

The main object of this paper is to prove the Theorem 1.1 under assumptions that

$$(p_n)_{n=0}^{\infty} \in AMDUSMS$$

or

$$(p_n)_{n=0}^{\infty} \in AMIUSMS \quad \text{and} \quad (n+1)^2 p_n = \mathcal{O}(P_n),$$

instead of

$$(p_n)_{n=0}^{\infty} \in AMDS$$

or

$$(p_n)_{n=0}^{\infty} \in AMIS \quad \text{and} \quad (n+1)p_n = \mathcal{O}(P_n),$$

respectively, and also we will consider the special case $\alpha = 1$.

2. Helpful Lemmas

To achieve the aim, which we mentioned above, we need some helpful statements given below.

Lemma 2.1 ([7])**.** *Let $p \in \mathcal{M}$. Then the estimate*

$$\|\sigma_n(f) - S_n(f)\|_{p(x)} = \mathcal{O}\left(n^{-1}\right), \ n = 1, 2, \ldots,$$

holds for every $f \in Lip(1, p(x))$.

Lemma 2.2 ([7])**.** *Let $p \in \mathcal{M}$ and $0 < \alpha \leq 1$. Then the estimate*

$$\|f - S_n(f)\|_{p(x)} = \mathcal{O}\left(n^{-\alpha}\right), \ n = 1, 2, \ldots,$$

holds for every $f \in Lip(\alpha, p(x))$.

Lemma 2.3. *Let (p_n) be a positive sequence so that*
  (i) $(p_n) \in AMDUSMS$ *or,*
  (ii) $(p_n) \in AMIUSMS$*, and* $(n+1)^2 p_n = \mathcal{O}(P_n)$
are satisfied. Then

$$\Sigma := \sum_{k=0}^{n} \frac{p_{n-k}}{(k+1)^\alpha} = \mathcal{O}\left(\frac{P_n}{(n+1)^\alpha}\right)$$

holds for all $0 < \alpha < 1$.

Proof. Let $r = [n/2]$ be the integer part of $n/2$ and $A_{n,k}^{(N,2)} = \frac{2}{(k+1)(k+2)P_n} \sum_{i=n-k}^{n} p_i$. Then under assumptions of the lemma, applying the summation by parts and Lagrangue's mean value theorem, we have

$$\begin{aligned}
\Sigma &\leq \sum_{k=0}^{r} \frac{p_{n-k}}{(k+1)^\alpha} + \frac{1}{(r+1)^\alpha} \sum_{k=r+1}^{n} p_{n-k} \\
&= \sum_{k=0}^{r-1} \left[\frac{1}{(k+1)^\alpha} - \frac{1}{(k+2)^\alpha}\right] \sum_{i=0}^{k} p_{n-i} + \frac{1}{(r+1)^\alpha} \sum_{k=0}^{r} p_{n-k} + \frac{P_n}{(r+1)^\alpha} \\
&\leq \frac{\alpha P_n}{2} \sum_{k=0}^{r-1} \frac{(k+1)^{\alpha-1}}{[(k+1)(k+2)]^{\alpha-1}} A_{n,k}^{(N,2)} + \frac{P_r}{(r+1)^\alpha} + \frac{P_n}{(r+1)^\alpha} \\
&= P_n \left[\frac{\alpha}{2} \sum_{k=0}^{r-1} \frac{A_{n,k}^{(N,2)}}{(k+2)^{\alpha-1}} + \frac{2}{(r+1)^\alpha}\right].
\end{aligned}$$

If $(p_n) \in AMDUSMS$, then

$$\begin{aligned}
\Sigma &\leq P_n \left[\frac{\alpha}{2} A_{n,r}^{(N,2)} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha-1}} + \frac{2}{(r+1)^\alpha}\right] \\
&\ll P_n \left[\frac{1}{(r+1)(r+2)P_n} \sum_{i=n-r}^{n} p_i \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha-1}} + \frac{1}{(r+1)^\alpha}\right] \\
&\ll P_n \left[\frac{1}{(r+1)(r+2)} (r+1)^{1-(\alpha-1)} + \frac{1}{(r+1)^\alpha}\right] \\
&\ll \frac{P_n}{(r+1)^\alpha} \ll \frac{P_n}{(n+1)^\alpha}.
\end{aligned}$$

If $(p_n) \in AMIUSMS$ and $(n+1)^2 p_n = \mathcal{O}(P_n)$, we obtain

$$\begin{aligned} \Sigma &\leq P_n \left[\frac{\alpha}{2} A_{n,0}^{(N,2)} \sum_{k=0}^{r-1} \frac{1}{(k+2)^{\alpha-1}} + \frac{2}{(r+1)^{\alpha}} \right] \\ &\ll P_n \left[\frac{p_n}{P_n} \sum_{k=0}^{n-1} \frac{1}{(k+2)^{\alpha-1}} + \frac{1}{(r+1)^{\alpha}} \right] \\ &\ll P_n \left[\frac{p_n}{P_n} (n+1)^{2-\alpha} + \frac{1}{(r+1)^{\alpha}} \right] \ll \frac{P_n}{(n+1)^{\alpha}}. \end{aligned}$$

□

Next section will be devoted to the main results.

3. Main Results

Theorem 3.1. *Let $p \in \mathcal{M}$, $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$, and $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. Let*

$$(p_n)_{n=0}^{\infty} \in AMDUSMS \quad or$$

$$(p_n)_{n=0}^{\infty} \in AMIUSMS \quad and \quad (n+1)^2 p_n = \mathcal{O}(P_n), \tag{5}$$

then

$$\|f - N_n(f)\|_{p(x)} = \mathcal{O}\left(\frac{1}{(n+1)^{\alpha}}\right)$$

holds for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Since

$$f(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} f(x),$$

then we can write

$$f(x) - N_n(f;x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \{f(x) - S_m(f;x)\}.$$

Whence, using Lemma 2.2, Lemma 2.3, and conditions (5) we get

$$\begin{aligned} \|f - N_n(f)\|_{p(x)} &\leq \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \|f(x) - S_m(f;x)\|_{p(x)} \\ &= \frac{1}{P_n} \sum_{m=1}^{n} p_{n-m} \|f(x) - S_m(f;x)\|_{p(x)} + \frac{p_n}{P_n} \|f(x) - S_0(f;x)\|_{p(x)} \\ &= \frac{1}{P_n} \sum_{m=1}^{n} p_{n-m} \mathcal{O}\left(m^{-\alpha}\right) + \mathcal{O}\left(\frac{1}{(n+1)^2}\right) \\ &= \frac{1}{P_n} \mathcal{O}\left(\sum_{m=1}^{n} p_{n-m} (m+1)^{-\alpha}\right) + \mathcal{O}\left(\frac{1}{n+1}\right) \\ &= \mathcal{O}\left(\frac{1}{(n+1)^{\alpha}}\right). \end{aligned}$$

□

Next theorem gives the same degree of approximation with different conditions from those of Theorem 3.1, considering the case $\alpha = 1$.

Theorem 3.2. *Let $p \in \mathcal{M}$, $f \in Lip(1, p(x))$ and let $(p_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. If*

$$\sum_{m=0}^{n-2} \left| A_{n,m}^{(N,1)} - A_{n,m+1}^{(N,1)} \right| = \mathcal{O}\left(n^{-1}\right),$$

then for $n = 1, 2, \ldots$ the estimate

$$\|f - N_n(f)\|_{p(x)} = \mathcal{O}\left(n^{-1}\right)$$

holds.

Proof. According to the definition of $N_n(f;x)$ the following equality is true

$$D_n(f;x) := N_n(f;x) - f(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m}\{S_m(f;x) - f(x)\}.$$

Applying the summation by parts twice and puting $A_{n,k}^{(N,1)} = \frac{1}{(k+1)P_n} \sum_{i=n-k}^{n} p_i$ we get (with the same technique as in [13] page 584)

$$\begin{aligned}
D_n(f;x) &= \sum_{m=0}^{n-1} \left(S_m(f;x) - S_{m+1}(f;x)\right) \frac{1}{P_n} \sum_{i=0}^{m} p_{n-i} + S_n(f;x) - f(x) \\
&= -\sum_{m=0}^{n-1} (m+1)\, U_{m+1}(f;x) A_{n,m}^{(N,1)} + S_n(f;x) - f(x) \\
&= -\sum_{m=0}^{n-2} \left(A_{n,m}^{(N,1)} - A_{n,m+1}^{(N,1)}\right) \sum_{j=0}^{m} (j+1) U_{j+1}(f;x) \\
&\quad -\frac{1}{nP_n} \sum_{j=1}^{n} p_i \sum_{j=0}^{n-1} (j+1) U_{j+1}(f;x) + S_n(f;x) - f(x).
\end{aligned}$$

Subsequently,

$$\begin{aligned}
\|D_n(f)\|_{p(x)} &\leq \sum_{m=0}^{n-2} \left| A_{n,m}^{(N,1)} - A_{n,m+1}^{(N,1)} \right| \left\| \sum_{j=1}^{m+1} jU_j(f) \right\|_{p(x)} \\
&\quad + \frac{1}{n} \left\| \sum_{j=1}^{n} jU_j(f) \right\|_{p(x)} + \|S_n(f) - f\|_{p(x)}.
\end{aligned}$$

Based on Lemma 2.1 and the equality

$$\sum_{j=1}^{n} jU_j(f;x) = (n+1)(S_n(f;x) - \sigma_n(f;x)),$$

we have

$$\left\| \sum_{j=1}^{n} jU_j(f) \right\|_{p(x)} = \mathcal{O}(1).$$

Hence, using Lemma 2.2 and the latter estimation we get

$$\|D_n(f)\|_{p(x)} = \mathcal{O}\left(\sum_{m=0}^{n-2}\left|A_{n,m}^{(N,1)} - A_{n,m+1}^{(N,1)}\right|\right) + \mathcal{O}\left(\frac{1}{n}\right).$$

Finally, if the condition $\sum_{m=0}^{n-2}\left|A_{n,m}^{(N,1)} - A_{n,m+1}^{(N,1)}\right| = \mathcal{O}\left(\frac{1}{n}\right)$ is satisfied, then we obtain

$$\|N_n(f) - f(x)\|_{p(x)} = \mathcal{O}\left(\frac{1}{n}\right).$$

The proof of theorem is completed. □

Note that in the special case, when $p_n = A_{n-m}^{\nu-1}A_m^{\beta} = O(n^{\nu+\beta})$ with $\nu+\beta > -1$, where $A_0^{\nu+\beta} = 1$, then the mean $N_n(f;x)$ reduces to the $n-$th Cesàro mean of order (ν,β) (see [1]):

$$N_n(f;x) \equiv \sigma_n^{\nu,\beta}(f;x) = \frac{1}{A_n^{\nu+\beta}}\sum_{m=0}^{n} A_{n-m}^{\nu-1}A_m^{\beta}S_m(f;x).$$

Therefore, the degree of approximation of the function $f \in Lip(\alpha, p(x))$ with Cesàro mean of order (ν,β), is an immediate result of Theorem 3.1.

Corollary 3.1. *Let $p \in \mathcal{M}$ and $\nu+\beta > -1$. Under assumptions of theorem 3.1 the estimate*

$$\|f - \sigma_n^{\nu,\beta}(f)\|_{p(x)} = \mathcal{O}\left(\frac{1}{(n+1)^{\alpha}}\right)$$

holds for every $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$ and all $n = 0, 1, 2, \dots$.

If we put $\beta = 0$ in the above corollary, then we immediately obtain the following.

Corollary 3.2. *Let $p \in \mathcal{M}$ and $\nu > -1$. Under assumptions of theorem 3.1 the estimate*

$$\|f - \sigma_n^{\nu}(f)\|_{p(x)} = \mathcal{O}\left(\frac{1}{(n+1)^{\alpha}}\right)$$

holds for every $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$ and all $n = 0, 1, 2, \dots$.

Remark 3.1. *Similar corollaries can be derived from theorem 3.2 when $\alpha = 1$.*

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