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# SOLVABILITY OF SECOND ORDER DELTA-NABLA $p$-LAPLACIAN $m$-POINT EIGENVALUE PROBLEM ON TIME SCALES

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Abstract. In this paper, we are concerned with the following eigenvalue problem of $m$-point boundary value problem for $p$-Laplacian dynamic equation on time scales,

$$\Big(\phi_p(u^\Delta(t))\Big)^\nabla + \lambda h(t) f\big(u(t)\big) = 0,\ t \in [a,b]_{\mathbb{T}},$$
$$u(a) - u^\Delta(a) = \sum_{i=1}^{m-2} u^\Delta(\xi_i),\ u^\Delta(b) = 0,\ m \geq 3,$$

where $\phi_p(u) = |u|^{p-2}u, p > 1$ and $\lambda > 0$ is a real parameter. Under certain assumptions, some new results on existence of one or two positive solutions and nonexistence are obtained for $\lambda$ evaluated in different intervals by using Guo-Krasnosel'skii fixed point theorem.

Keywords: eigenvalue, time scale, $p$-Laplacian, positive solution, fixed point, cone.

AMS Subject Classification: 34B18, 34A34

## 1. Introduction

The theory of dynamic equations on time scales was introduced firstly by Stefan Hilger in 1988. Since then, more and more scholars are interested in this area. The main reason is that the time scale theory can not only unite continuous and discrete dynamic equations but also have important applications, for example, in the study of insect population models, neural networks, heat transfer, economic, stock market, and epidemic models. Throughout this work we assume a working knowledge of time scales and time scales natation, where any nonempty closed subset of $\mathbb{R}$ can serve as a time scale $\mathbb{T}$, see Hilger [18], Bohner and Peterson [8]

Very recently, there is an increasing attention paid to question of positive solution for second order boundary value problem on time scales [4, 5, 9, 10, 11, 22]. But very little work has been done to the existence of positive solutions for $p$-Laplacian boundary value problem on time scales [6, 24, 17]. In particular, we would like to mention some results of Agarwal, Lü and O'Regan [3], Anderson, Avery and Henderson [6], Fan and Li [12], Guo and Sun [14], Goodrich [15, 16], Nageswararao [21], Prasad, Nageswararao and Murali [22], Sun and Li [23, 24], Sun, Tang and Wang [25], which are motivate us to consider our problem.

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In this paper we consider the eigenvalue problem of $m$-point boundary value problem for the one-dimensional $p$-Laplacian dynamic equation on time scales

$$\Big(\phi_p(u^{\Delta}(t))\Big)^{\nabla} + \lambda h(t) f(u(t)) = 0,\ t \in [a,b]_{\mathbb{T}} \tag{1}$$

$$u(a) - u^{\Delta}(a) = \sum_{i=1}^{m-2} u^{\Delta}(\xi_i),\ u^{\Delta}(b) = 0,\ m \geq 3. \tag{2}$$

Under certain assumptions, results on existence of one or two positive solutions and nonexistence are obtained for $\lambda$ evaluated in different intervals.
For convenience, throughout this paper, we denote $\phi_p(u)$ as the $p$-Laplacian operator, i.e., $\phi_p(u) = |u|^{p-2}u$ for $p > 1$ with $(\phi_p)^{-1} = \phi_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.
We make the following assumptions throughout:

(A1) $f \in C([0,\infty),[0,\infty))$ does not vanish identically on any closed subinterval of $[a,b]_{\mathbb{T}}$.

(A2) $\lambda > 0$ is a parameter, the function $h : (a,b) \to [0,\infty)$ is left-dense continuous such that $h(t_0) > 0$ for at least one $t_0 \in [a,b)$ and $0 < \int_a^b h(\tau)\nabla\tau < \infty$.

We define the positive extended real numbers $f^0, f_0, f^{\infty}$ and $f_{\infty}$ by

$$f^0 = \lim\sup_{u\to 0^+} \frac{f(u)}{\phi_p(u)},\ f_0 = \lim\inf_{u\to 0^+} \frac{f(u)}{\phi_p(u)},$$
$$f^{\infty} = \lim\sup_{u\to \infty} \frac{f(u)}{\phi_p(u)} \text{ and } f_{\infty} = \lim\inf_{u\to \infty} \frac{f(u)}{\phi_p(u)}$$

assume that they will exist.

The rest of this paper is organized as follows. In Section 2, we shall provide some preliminaries. For convenience, we also state the Krasnosel'skii's fixed point theorem in a cone. In Section 3, we establish a criteria to determine eigenvalue intervals for which there exist at least one or two positive solutions. In the last Section, we will consider the conditions of the nonexistence of the positive solution. An example is also given to illustrate the main results.

## 2. Preliminaries

Let the Banach space $\mathcal{B} = C_{ld}[a,b]$ be endowed with the norm

$$\|u\| = \sup_{t\in[a,b]_{\mathbb{T}}} |u(t)|,$$

and choose the cone $\mathcal{P} \subset \mathcal{B}$ defined by

$$\mathcal{P} = \Big\{ u \in \mathcal{B} : u(t) \geq 0, \text{ on } [a,b]_{\mathbb{T}} \text{ and } u^{\Delta\nabla}(t) \leq 0,\ u^{\Delta}(t) \geq 0,$$
$$\text{for } t \in [a,b]_{\mathbb{T}},\ u(a) - u^{\Delta}(a) = \sum_{i=1}^{m-2} u^{\Delta}(\xi_i),\ m \geq 3 \Big\}$$

clearly, we can obtain $\| u \| = u(b)$ for $u \in \mathcal{P}$.
A function $u : \mathbb{T} \to \mathbb{R}$ is said to be a solution of the BVP (1)-(2) provided that $u$ is delta differential, $u^{\Delta}$ and $\big(\phi_p(u^{\Delta}(t))\big)^{\nabla}$ are continuous on $[a,b]_{\mathbb{T}}$, and $u$ satisfies the BVP (1)-(2).
To obtain our main results, we make use of the following lemmas.

**Lemma 2.1.** *Assume that* $(A1)$ *and* $(A2)$ *are satisfied. Then* $u(t)$ *is the solution of the* $BVP$ (1)-(2) *if and only if*

$$u(t) = \phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \int_a^t \phi_q\Big(\int_s^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\Delta s.$$

*Proof.* Firstly, we prove the necessity. Let $u(t)$ be the solution of the BVP (1)-(2). $\nabla$-integrating on (1) from t to b, we have $\phi_p(u^\Delta(b)) - \phi_p(u^\Delta(t)) = -\int_t^b \lambda h(\tau)f(u(\tau))\nabla\tau$. By using second boundary condition, we get $u^\Delta(t) = \phi_q\Big(\int_t^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)$ $\Delta$-integrating from a to t, we get $u(t) - u(a) = \int_a^t \phi_q\Big(\int_s^b \lambda h(\tau)f(u(\tau))\Big)\Delta\tau$.
Therefore,

$$u(t) = u(a) + \int_a^t \phi_q\Big(\int_s^b \lambda h(\tau)f(u(\tau))\Big)\Delta\tau.$$

We have $u^\Delta(a) = \phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)$, $u^\Delta(\xi_i) = \phi_q\Big(\int_{\xi_i}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)$. Now we submitting $u^\Delta(a), u^\Delta(\xi_i)$ into the first boundary condition of the (2), we obtain

$$u(a) = \phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big).$$

Thus, we have

$$u(t) = \phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \int_a^t \phi_q\Big(\int_s^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\Delta s.$$

Secondly, we can prove the sufficiency easily. □

Define the operator $T : \mathcal{P} \to \mathcal{B}$, for $u \in \mathcal{P}$, by

$$(Tu)(t) = \phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \int_a^t \phi_q\Big(\int_s^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\Delta s. \tag{3}$$

By using Lemma 2.1, $u(t)$ is the solution of the BVP (1)-(2) if and only if $u(t) = (Tu)(t)$.

**Lemma 2.2.** *Assume that* $(A1)$ *and* $(A2)$ *hold.* $T : \mathcal{P} \to \mathcal{P}$ *is a completely continuous operator.*

*Proof.* Firstly we prove $T : \mathcal{P} \to \mathcal{P}$. For $u \in \mathcal{P}$, by Lemma 2.1, $(A1)$, and $(A2)$, we obtain $Tu(t) \geq 0,\ t \in [a,b]_{\mathbb{T}}$,

$$(Tu)^\Delta(t) = \phi_q\Big(\int_t^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big),\ (Tu)^{\Delta\nabla}(t) \leq 0,$$

$$(Tu)(a) - (Tu)^\Delta(a) = \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) = \sum_{i=1}^{m-2}(Tu)^\Delta(\xi_i),$$

which implies $Tu \in \mathcal{P}$.
Secondly, $T$ maps a bounded set into a bounded set. Assume $c > 0$ is a constant and $u \in \overline{P_c} = \{x \in P : \| x \| \leq c\}$. Note that the continuity of $f(u)$ is continuous, there exists a $C > 0$ such that $f(u) \leq \phi_p(C)$. Hence, for $t \in [a,b]_{\mathbb{T}}, u \in \overline{P_c}$, we have

$$
\begin{aligned}
|(Tu)^{\Delta}(t)| &= \Big|\phi_q\Big(\int_t^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big)\Big| < C\phi_q(\lambda)\phi_q\Big(\int_a^b h(\tau)\nabla\tau\Big) < \infty,\\
|(Tu)(t)| &= \Big|\phi_q\Big(\int_a^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big) + \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big)\\
&\quad + \int_a^t \phi_q\Big(\int_s^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big)\Delta s\Big|\\
&\leq C(m-1+b)\phi_q(\lambda)\phi_q\Big(\int_a^b h(\tau)\nabla\tau\Big) < \infty.
\end{aligned}
$$

Consequently, $T\overline{P_c}$ is bounded.
For $t_1, t_2 \in [a,b]_{\mathbb{T}}, u \in \overline{P_c}$, we get

$$
\begin{aligned}
|(Tu)(t_1) - (Tu)(t_2)| &\leq \Big|\int_{t_1}^{t_2} \phi_q\Big(\int_s^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big)\Delta s\Big|\\
&\leq \Big|\int_{t_1}^{t_2} \phi_q\Big(\int_s^b \lambda h(\tau)\nabla\tau\Big)\Delta s\Big|\\
&\leq |t_1 - t_2| C\phi_q(\lambda)\phi_q\Big(\int_a^b h(\tau)\nabla\tau\Big) \to 0,\ t_1 \to t_2.
\end{aligned}
$$

So, by applying Arzela-Ascoli theorem on time scales [2], we obtain that $T\overline{P_c}$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [7], it is easy to prove that $T$ is continuous. Hence, $T$ is completely continuous. □

**Theorem 2.1.** [**Krasnosel'skii**] [13, 20] *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that $\Omega_1$ and $\Omega_2$ are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

*be a completely continuous operator such that either*

(i) $\| Tu \| \leq \| u \|,\ u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\| Tu \| \geq \| u \|,\ u \in \mathcal{P} \cap \partial\Omega_2$, *or*
(ii) $\| Tu \|\ geq \| u \|,\ u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\| Tu \| \leq \| u \|,\ u \in \mathcal{P} \cap \partial\Omega_2$.

*Then, $T$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

## 3. Eigenvalue intervals

In this section, we shall apply Theorem 2.1 to derive explicit eigenvalue intervals, and also give sufficient conditions that the BVP (1)-(2) has at least one or two positive solutions.
To begin, we shall define some important constants

$$
\begin{aligned}
L_1 &= (m+b-a-1)\phi_q\Big(\int_a^b h(\tau)\nabla\tau\Big)\\
L_2 &= \Big(\frac{\xi_{m-2}-a}{b-a}\Big)(m+\xi_{m-2}-a-1)\phi_q\Big(\int_{\xi_{m-2}}^b h(\tau)\nabla\tau\Big)
\end{aligned}
$$

**Theorem 3.1.** *Suppose* $(A1)-(A2)$ *hold. Then for each*

$$(f_\infty)^{-1}\phi_q\Big(\frac{(\xi_{m-2}-a)L_2}{b-a}\Big) < \lambda < (f^0)^{-1}\phi_q(L_1), \tag{4}$$

*the BVP* (1)-(2) *has at least one positive solution. Here we impose* $(f_\infty)^{-1}\phi_q\Big(\frac{(\xi_{m-2}-a)L_2}{b-a}\Big) = 0$ *if* $f_\infty = \infty$ *and* $(f^0)^{-1}\phi_q(L_1) = \infty$ *if* $f^0 = 0$.

*Proof.* Let $\lambda$ satisfy (4) and $\epsilon > 0$ be such that

$$(f_\infty-\epsilon)^{-1}\phi_q\Big(\frac{(\xi_{m-2}-a)L_2}{b-a}\Big) \leq \lambda \leq (f^0+\epsilon)^{-1}\phi_q(L_1). \tag{5}$$

By the definition of $f^0$, we see that there exists $r_1 > 0$ such that

$$f(u) \leq (f^0+\epsilon)\phi_p(u),\ 0 < u < r_1. \tag{6}$$

So, if $u \in \partial P_{r_1}$, then by (5) and (6) we have

$$\begin{aligned}
\parallel Tu \parallel = (Tu)(b) &\leq \phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\\
&\quad + (m-2)\phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + (b-a)\phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\\
&\leq (m+b-a-1)\ u\ \phi_q(\lambda(f^0+\epsilon))\phi_q\Big(\int_a^b h(\tau)\nabla\tau\Big)\\
&\leq \phi_q(\lambda(f^0+\epsilon))L_1 \parallel u \parallel \leq \parallel u \parallel .
\end{aligned}$$

Hence if we let $\Omega_1 = \{u \in \mathcal{B} : \parallel u \parallel < r_1\}$, then

$$\parallel Tu \parallel \leq \parallel y \parallel,\ \text{for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{7}$$

Let $r_3 > 0$ be such that

$$f(u) \geq (f_\infty-\epsilon)\phi_p(u),\ u \geq r_3. \tag{8}$$

If $u \in \mathcal{B}$ with $\parallel u \parallel = r_2 = \max\Big\{2r_1, \frac{(b-a)r_3}{\xi_{m-2}-a}\Big\}$. Then in view of (8)we have

$$\begin{aligned}
\parallel Tu \parallel &\geq (Tu)(\xi_{m-2}) \geq \Big(\frac{\xi_{m-2}-a}{b-a}\Big)(Tu)(b)\\
&\geq \Big(\frac{\xi_{m-2}-a}{b-a}\Big)\Big[\phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\\
&\quad + \int_a^b \phi_q\Big(\int_s^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\Delta s\Big]\\
&\geq \Big(\frac{\xi_{m-2}-a}{b-a}\Big)\Big[\phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + (m-2)\phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\\
&\quad + \int_a^{\xi_{m-2}} \phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big)\Delta s\Big]\\
&\geq \Big(\frac{\xi_{m-2}-a}{b-a}\Big)(m+\xi_{m-2}-a-1)\phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau)(f_\infty-\epsilon)\phi_p(u(\tau))\nabla\tau\Big)\\
&\geq L_2\phi_q\big(\lambda(f_\infty-\epsilon)\big)\Big(\frac{\xi_{m-2}-a}{b-a}\Big) \parallel u \parallel \geq \parallel u \parallel .
\end{aligned}$$

Thus if we set $\Omega_2 = \{u \in \mathcal{B} : \| u \| < r_2\}$, then

$$\| Tu \| \leq \| u \|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{9}$$

Now, (7),(9) and Theorem 2.1 guarantee that $T$ has a fixed point $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$ with $r_1 \leq \| u \| \leq r_2$, and clearly $u$ is a positive solution of (1)-(2). □

**Theorem 3.2.** *Suppose* $(A1) - (A2)$ *hold. Then for each*

$$(f_0)^{-1}\phi_q\Big(\frac{(\xi_{m-2} - a)L_2}{b - a}\Big) < \lambda < (f^\infty)^{-1}\phi_q(L_1), \tag{10}$$

*the* $BVP$ (1)-(2) *has at least one positive solution.*

*Proof.* Let $\lambda$ satisfy (10) and $\epsilon > 0$ be such that

$$(f_0 - \epsilon)^{-1}\phi_q\Big(\frac{(\xi_{m-2} - a)L_2}{b - a}\Big) \leq \lambda \leq (f^\infty + \epsilon)^{-1}\phi_q(L_1). \tag{11}$$

By the definition of $f_0$, we see that there exists $r_1 > 0$ such that

$$f(u) \leq (f_0 - \epsilon)\phi_p(u), \ 0 < u \leq r_1.$$

Further, if $u \in \mathcal{P}$ with $\| u \| = r_1$, then $u(t) \geq \left(\frac{\xi_{m-2}-a}{b-a}\right) \| u \|, \ t \in [\xi_{m-2}, b]_{\mathbb{T}}$, and similar to the second part of Theorem 3.1, we can obtain that $\| Tu \| \geq \| u \|$ . Thus, if we let $\Omega_1 = \{u \in \mathcal{B} : \| u \| < r_1\}$, then

$$\| Tu \| \geq \| u \|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{12}$$

Next, we may choose $R_2 > 0$ such that

$$f(u) \leq (f^\infty + \epsilon)\phi_p(u), u \geq R_2. \tag{13}$$

Here there are two cases to consider, namely, where $f$ is bounded and $f$ is unbounded.
**Case (i)**. Suppose $f$ is bounded, then there exists some $M > 0$, such that $f(u) \leq M, \ u \in (0, \infty)$. We define $r_3 = \max\Big\{2r_1, \phi_q(\lambda M)L_1, \Big\}$ and $u \in \mathcal{P}$ be such that $\| u \| = r_3$, then

$$\begin{aligned}
&\| Tu \| = (Tu)(b) \\
&\leq \phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) + (m-2)\phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) \\
&\ + (b-a)\phi_q\Big(\int_a^b \lambda h(\tau)f(u(\tau))\nabla\tau\Big) \\
&\leq \phi_q(\lambda M)L_1 \leq r_3 = \| u \| .
\end{aligned}$$

Hence,

$$\| Tu \| \leq \| u \|, \text{ for } u \in \partial\mathcal{P}_{r_3}. \tag{14}$$

**Case (ii)**. Suppose $f$ is unbounded, then there exists $r_4 > \max\{2r_1, R_2\}$ such that

$$f(u) \leq f(r_4), 0 < u < r_4. \tag{15}$$

Let $u \in \mathcal{P}$ be such that $\| u \| = r_4$, then by (11), we have

$$\begin{aligned}
&\| Tu \| = (Tu)(b) \\
&\leq \phi_q\Big(\int_a^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big) + (m-2)\phi_q\Big(\int_a^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big) \\
&\ + (b-a)\phi_q\Big(\int_a^b \lambda h(\tau) f(u(\tau))\nabla\tau\Big) \\
&\leq \phi_q(\lambda(f^\infty + \epsilon))L_1 \| u \| \leq \| u \| .
\end{aligned}$$

Thus, (14) is also true. In both Case (i) and Case (ii), if we set

$$\Omega_2 = \Big\{u \in \mathcal{B} : \| u \| < r_2 = \max\{r_3, r_4\}\Big\},$$

then (13) hold for $u \in \mathcal{P} \cap \partial\Omega_2$. Now that we have obtained (12) and (14), it follows from Theorem 2.1 that $T$ has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$, and $r_1 \leq \| u \| \leq r_2$. It is clear that $u$ is a positive solution of (1)-(2). □

In the rest of this section, we consider the existence of two positive solutions of (1)-(2). First, we give lemma.

**Lemma 3.1.** *Assume that* $(A1)$ *and* $(A2)$ *hold. In addition, assume there exist* $r_2 > r_1 > 0$, *such that*

$$\max_{0\leq u\leq r_1} f(u) \leq \frac{\phi_p(\frac{r_1}{L_1})}{\lambda} \tag{16}$$

$$\min_{\frac{(\xi_{m-2}-a)r_2}{b-a}} \geq f(u) \leq r_2\frac{\phi_p(\frac{r_2}{L_2})}{\lambda}. \tag{17}$$

*Then,* (1)-(2) *has a solution* $u \in \mathcal{P}$ *with* $r_1 \leq \| u \| \leq r_2$.

*Proof.* The proof of Lemma 3.1 is similar to that of Theorem 3.2, we omit it here. □

For the remainder of the paper, we will need the following condition:
$(A3)$

$$\sup_{r>0} \min_{u\in\big(\frac{(\xi_{m-2}-a)r}{b-a},\ r\big)} f(u) > 0.$$

$$\lambda_1 = \sup_{r>0} \frac{\phi_p(\frac{r}{L_1})}{\max_{0\leq u\leq r} f(u)}, \tag{18}$$

$$\lambda_2 = \inf_{r>0} \frac{\phi_p(\frac{r}{L_2})}{\min_{\big(\frac{(\xi_{m-2}-a)r}{b-a}\big)\leq u\leq r} f(u)}, \tag{19}$$

In view of $(A1)$ and $(A3)$, we can easily obtain that $0 < \lambda_1 \leq \infty$ and $0 \leq \lambda_2 < \infty$.

**Theorem 3.3.** *Suppose* $(A1) - (A3)$ *hold, if* $f_0 = \infty$ *and* $f_\infty = \infty$, *then the BVP* (1)-(2) *has at least two positive solutions for each* $\lambda \in (0, \lambda_1)$.

*Proof.* Define

$$a(r) = \frac{\phi_p(\frac{r}{L_1})}{\max_{0\leq u\leq r} f(u)},\ r > 0,$$

then by $(A1)$, $f_0 = \infty$ and $f_\infty = \infty$, we have that $a(r) : (0, \infty) \to (0, \infty)$ is continuous and $\lim_{r\to 0} a(r) = \lim_{r\to\infty} a(r) = 0$. By (17) there exists $r_0 \in (0, \infty)$, such that

$a(r_0) = \sup_{r>0} a(r) = \lambda_1$, then for $\lambda \in (0, \lambda_1)$, there exist constants $c_1, c_2$ $(0 < c_1 < r_0 < c_2 < \infty)$ with $a(c_1) = a(c_2) = \lambda$. Thus

$$f(u) \leq \frac{\phi_p(\frac{c_1}{L_1})}{\lambda} \text{ for } u \in [0, c_1], \tag{20}$$

$$f(u) \leq \frac{\phi_p(\frac{c_2}{L_1})}{\lambda} \text{ for } u \in [0, c_2]. \tag{21}$$

On the other hand, applying the condition $f_0 = \infty$ and $f_\infty = \infty$, there exist constants $d_1, d_2$ $(0 < d_1 < c_1 < r_0 < c_2 < d_2 < \infty)$, with

$$\frac{f(u)}{\phi_p(u)} \geq \lambda^{-1}\phi_q\Big(\frac{\xi_{m-2} - a)L_2}{b-a}\Big) \text{ for } u \in (0, d_1) \cup \Big(\frac{(\xi_{m-2} - a)d_2}{b-a}, \infty\Big) \tag{22}$$

then

$$\min_{\frac{(\xi_{m-2}-a)d_1}{b-a} \leq u \leq d_1} f(u) \geq \lambda^{-1}\phi_p\Big(\frac{d_1}{L_2}\Big) \tag{23}$$

$$\min_{\frac{(\xi_{m-2}-a)d_2}{b-a} \leq u \leq d_2} f(u) \geq \lambda^{-1}\phi_p\Big(\frac{d_2}{L_2}\Big). \tag{24}$$

By (20) and (23), (21) and (24), Lemma 3.1, we can complete the proof. □

**Theorem 3.4.** *Suppose* $(A1)-(A3)$ *hold, if* $f_0 = 0$ *and* $f_\infty = 0$, *then for each* $\lambda \in (\lambda_2, \infty)$ *the BVP* (1)-(2) *has at least two positive solutions.*

*Proof.* Define

$$b(r) = \frac{\phi_p(\frac{r}{L_2})}{\min_{\frac{(\xi_{m-2}-a)r}{b-a} \leq u \leq r} f(u)}, \ r \in (0, \infty).$$

By $f_0 = 0$ and $f_\infty = 0$ we easily see that $b(r) : (0, \infty) \to (0, \infty)$ is continuous and

$$\lim_{r\to 0} b(r) = \lim_{r\to\infty} b(r) = \infty.$$

Thus there exists $r_0 \in (0, \infty)$, such that $b(r_0) = \inf_{r>0} b(r) = \lambda_2$. For $\lambda \in (\lambda_2, \infty)$, there exist constants $d_1, d_2 (0 < d_1 < r_0 < d_2 < \infty)$ with $b(d_1) = b(d_2) = \lambda$. Therefore

$$f(u) \geq \frac{\phi_p(\frac{d_1}{L_1})}{\lambda} \text{ for } u \in \Big[\frac{(\xi_{m-2} - a)d_1}{b-a}, d_1\Big],$$

$$f(u) \geq \frac{\phi_p(\frac{d_2}{L_2})}{\lambda} \text{ for } u \in \Big[\frac{(\xi_{m-2} - a)d_2}{b-a}, d_2\Big],$$

On the other hand, using $f_0 = 0$, we know that there is a constant $c_1(0 < c_1 < d_1)$ with

$$\frac{f(u)}{\phi_p(u)} \leq \lambda^{-1}\phi_q(L_1) \text{ for } u \in (0, c_1)$$

$$\max_{0\leq u\leq c_1} f(u) \leq \frac{\phi_p\Big(\frac{c_1}{L_1}\Big)}{\lambda}. \tag{25}$$

In view of $f_\infty = 0$, there exits a constant $c_2 \in (d_2, \infty)$ such that

$$\frac{f(u)}{\phi_p(u)} \leq \phi_q(L_1\lambda) \text{ for } u \in (c_2, \infty).$$

Let $M = \sup_{u\in[0,c_2]} f(u)$ and $c_2 \geq L_1\phi_q(\lambda M)$. It is easily seen that

$$\max_{0\leq u\leq c_2} f(u) \leq \frac{\phi_p(\frac{c_2}{L_1})}{\lambda}. \tag{26}$$

By (25), (26) and Lemma 3.1, we can complete the proof. □

## 4. Nonexistence

In this section, we give some sufficient conditions for the nonexistence of positive solution to the BVP (1)-(2).

**Theorem 4.1.** *Suppose* $(A1)-(A3)$ *hold. If* $f^0<\infty, f^\infty<\infty$, *then there exists a* $\lambda_0>0$ *such that for all* $0<\lambda<\lambda_0$, (1)-(2) *has no positive solution.*

*Proof.* Since $f^0<\infty$ and $f^\infty<\infty$, there exist positive numbers $l_1, l_2, r_1$ *and* $r_2$ such that $r_1<r_2$ and

$$f(u)\le l_1\phi_p(u)\ for\ u\in[0,r_1],$$

$$f(u)\le l_2\phi_p(u)\ for\ u\in[r_1,\infty],$$

Let $L=\max\Big\{l_1,l_2,\max_{r_1\le u\le r_2}\{f(u)\phi_q(u)\}\Big\}$, then we have

$$f(u)\le L\phi_p(u) \text{ for } u\in[0,\infty)$$

Assume that $v(t)$ is a positive solution of BVP (1)-(2). We will show that this leads to a contradiction for $0<\lambda<\lambda_0=L^{-1}\phi_q(L_1)$. Since $Tv(t)=v(t)$ for $t\in[a,b]_{\mathbb{T}}$, then

$$\begin{aligned}\parallel Tv\parallel=(Tv)(b)&\le\phi_q\Big(\int_a^b\lambda h(\tau)f(v(\tau))\nabla\tau\Big)\\&+(m-2)\phi_q\Big(\int_a^b\lambda h(\tau)f(v(\tau))\nabla\tau\Big)+(b-a)\phi_q\Big(\int_a^b\lambda h(\tau)f(v(\tau))\nabla\tau\Big)\\&\le\phi_q(\lambda L)\parallel v\parallel L_1<\parallel v\parallel.\end{aligned}$$

which is a contradiction. Therefore, (1)-(2) has no positive solutions. □

**Theorem 4.2.** *Suppose* $(A1)-(A3)$ *hold, if* $f_0>0, f_\infty>0$, *then there exist a* $\lambda_0>0$ *such that for all* $\lambda>\lambda_0$, (1)-(2) *has no positive solution.*

*Proof.* By $f_0>0, f_\infty>0$, we know that there exist $m_1, m_2, r_1$ and $r_2$ such that $r_1<r_2$ and

$$f(u)\ge m_1\phi_p(u) \text{ for } u\in[0,r_1],\ f(u)\ge m_2\phi_p(u) \text{ for } u\in[r_2,\infty).$$

Let $m_3=\min\Big\{m_1,m_2,\min_{r_1\le u\le r_2}\{f(u)\phi_q(u)\}\Big\}>0$, then we get

$$f(u)\ge m_3\phi_p(u) \text{ for } u\in[0,\infty).$$

Assume $v(t)$ is a positive solution of (1)-(2). We will show that this leads to a contradiction for $\lambda > \lambda_0 = (m_3)^{-1}\phi_p\Big(\frac{(\xi_{m-2}-a)L_2}{b-a}\Big)$. Since $Tv(t) = v(t)$ for $t \in [a,b]_{\mathbb{T}}$, then

$$\begin{aligned}
\| v \| &\geq v(\xi_{m-2}) = (Tv)(\xi_{m-2}) \geq \Big(\frac{\xi_{m-2}-a}{b-a}\Big)(Tv)(b) \\
&\geq \Big(\frac{\xi_{m-2}-a}{b-a}\Big)\Bigg[\phi_q\Big(\int_a^b \lambda h(\tau) f(v(\tau))\nabla\tau\Big) + \sum_{i=1}^{m-2}\phi_q\Big(\int_{\xi_i}^b \lambda h(\tau) f(v(\tau))\nabla\tau\Big) \\
&\quad + \int_a^b \phi_q\Big(\int_s^b \lambda h(\tau) f(v(\tau))\nabla\tau\Big)\Delta s\Bigg] \\
&\geq \Big(\frac{\xi_{m-2}-a}{b-a}\Big)\Bigg[\phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau) f(v(\tau))\nabla\tau\Big) + (m-2)\phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau) f(v(\tau))\nabla\tau\Big) \\
&\quad + \int_a^{\xi_{m-2}} \phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau) f(v(\tau))\nabla\tau\Big)\Delta s\Bigg] \\
&= \Big(\frac{\xi_{m-2}-a}{b-a}\Big)(m+\xi_{m-2}-a-1)\phi_q\Big(\int_{\xi_{m-2}}^b \lambda h(\tau) f(v(\tau))\nabla\tau\Big) \\
&\geq \phi_q(\lambda m_3)\Big(\frac{\xi_{m-2}-a}{b-a}\Big) \| v \| L_2 = \| v \|
\end{aligned}$$

which is a contradiction. Thus, (1)-(2) has no positive solution. □

**Example 4.1.** *Let* $\mathbb{T} = \{1-(\frac{1}{2})^{\mathbb{N}_0}\} \cup \{1\}$, *where* $\mathbb{N}_0$ *denotes the set of all nonnegative integers. Taking* $a = 0,\ b = 1,\ p = 2,\ m = 3,\ \xi_1 = \frac{1}{2}$, *if we let*
$h(s) = 1$, *then* $L_1 = 2,\ L_2 = \frac{3}{2}$. *Suppose* $f(u) = \frac{1+49u}{1+u}\Big(\frac{3}{2} + \sin u\Big),\ u \geq 0,$ *and*
$f^0 = f_0 = \frac{3}{2}, f^\infty = 235, f_\infty = 45.$

*By direct calculation, it is easy to get that*

$$(f_\infty)^{-1}\phi_q\Big(\frac{(\xi_1-a)L_2}{b-a}\Big) = 0.029,\ \ (f^0)^{-1}\phi_q(L_1) = 0.334,$$
$$L^{-1}\phi_q(L_1) = 0.0122 \text{ and } (m_3)^{-1}\phi_p\Big(\frac{(\xi_1-a)L_2}{b-a}\Big) = 4.$$

*Thus the boundary value problem*

$$u^{\Delta\nabla}(t) + \lambda f(u(t)) = 0,\ t \in [0,1]_{\mathbb{T}},$$
$$u(0) - u^\Delta(0) = u^\Delta\Big(\frac{1}{2}\Big),\ u^\Delta(1) = 0$$

*has at least one positive solutions for* $0.029 < \lambda < 0.334$, *has no solution on* $\mathcal{P}$ *for* $0 < \lambda < 0.0122$ or $\lambda > 4$ *by Theorem 3.1, Theorem 4.1 and Theorem 4.2 respectively.*

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