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EXTENSIONS OF CONSTANT CURVATURE

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Abstract. In this paper, we discuss some examples of two dimensional spaces of constant curvature and their Riemannian extensions.

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1. Introduction

Patterson and Walker[2] have defined Riemann extensions and showed how a Riemannian structure can be given to the $2n$ dimensional tangent bundle of an $n-$ dimensional manifold with given non-Riemannian structure.

A Riemann extension provides a solution of the general problem of embedding a manifold M carrying a given structure in a manifold M' carrying another structure, the embedding being carried out in such a way that the structure on M' induces in a natural way the given structure on M.

The Riemann extension of Riemannian or non-Riemannian spaces can be constructed with the help of the Christoffel coefficients Γ^i_{jk} of corresponding Riemann space or with connection coefficients Π^i_{jk} in the case of the space of affine connection. Here we discuss examples on flat spaces, spaces of constant positive and negative curvatures.

In each case it may be noted that the Christoffel symbols and hence the geodesic equations are extended. We give examples where extension of flat metric is flat and that of extension of a metric of non zero constant curvature is not a metric of constant curvature. The theory of Riemann extensions has been extensively studied by Afifi [1], Dryuma,[3], [4], [5], [6].

Preliminaries

Let (M, g) be a Riemannian manifold and let g_{ij} be components of the metric tensor g. The metric g is of constant curvature if

$$R_{ijkl} = \lambda(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{1}$$

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where R_{ijkl} are components of the Riemannian curvature tensor on M and λ is a constant. For a metric

$$ds^2 = g_{ij}dx^i dx^j, \tag{2}$$

the system of geodesic equations is given by

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0, \tag{3}$$

where Γ^k_{ij} are the Christophel symbols constructed from g.

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}(g_{mi,j} + g_{mj,i} - g_{ij,m}). \tag{4}$$

The Riemann extension of the metric (2) is given by

$$ds^2 = -2\Gamma^k_{ij}(x^l)\psi_k dx^i dx^j + 2d\psi_k dx^k, \tag{5}$$

where ψ_k are the coordinates of additional space. The extended metric(5) has the following property. The geodesic equations of (5) consists of two parts.

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{6}$$

and

$$\frac{\delta^2\psi_k}{ds^2} + R^l_{kji}\frac{dx^j}{ds}\frac{dx^i}{ds}\psi_l, \tag{7}$$

where

$$\frac{\delta^2\psi_k}{ds^2} = \frac{d\psi_k}{ds} - \Gamma^l_{jk}\psi_l\frac{dx^j}{ds}. \tag{8}$$

The system of equations (6) is the system of geodesic equations for geodesics of basic space with local coordinates x^i and it does not contain the coordinates ψ_k.
The system of equations (7) is a $n \times n$ linear matrix system of second order of the form given by

$$\frac{d^2\psi}{ds^2} + A(s)\frac{d\psi}{ds} + B(s)\psi = 0, \tag{9}$$

where $A(s)$ and $B(s)$ are matrices. The geometric properties of the basic space are preserved by Riemann extension. For example, Riemann extension of flat space is flat. We give an example for this below. This gives us the possibility of the basic space to use the linear system of equations for studying the properties of basic space.
In section 1 we give an example of a flat space where geodesic parallel to an axis is transformed in to a curve with 4-components in the extended space. In section 2 we study the metric similar to the metric of the 2-sphere embedded in three dimensional Euclidean space, with constant radius. We study the properties of geodesics in the extended space. It will be shown that stationary points are transformed into geodesics with extended components changing at constant rate. It can further be noted that stationary extended components are not solutions to this system of equations except for the components being zero. In section 3 we see that a line parallel to an axis can be transformed in to a curve with the extended components related parabolically. Throughout this paper we use the same notation for Christoffel symbols of base and extended spaces.

2. Two dimensional flat metric

Consider the metric of the form,

$$ds^2 = dx^2 + x^2 dy^2. \tag{10}$$

For this metric we have $\Gamma^1_{22} = x$, $\Gamma^2_{12} = \frac{1}{x}$ rest being zero.
The geodesic equations are

$$\frac{d^2x}{ds^2} - x\left(\frac{dy}{ds}\right)^2 = 0 \tag{11}$$

and

$$\frac{d^2y}{ds^2} + \frac{1}{x}\frac{dx}{ds}\frac{dy}{ds} = 0. \tag{12}$$

Simplifying we get,

$$\left(\frac{dx}{ds}\right)^2 = klogx \tag{13}$$

and

$$\frac{dy}{ds} = \frac{c}{x}. \tag{14}$$

On applying (5) we obtain the extended metric given by

$$ds^2 = 4xPdy^2 - \frac{4Q}{x}dxdy + 2dPdx + 2dQdy, \tag{15}$$

where P and Q are the extended coordinates.
For this extended metric, the Christoffel symbols are given by,

$$\Gamma^1_{22} = x, \quad \Gamma^2_{12} = \frac{1}{x}, \quad \Gamma^3_{12} = \frac{2Q}{x^2}, \quad \Gamma^3_{22} = -P, \quad \Gamma^3_{24} = -\frac{1}{x},$$
$$\Gamma^4_{11} = \frac{2Q}{x^2}, \quad \Gamma^4_{12} = -P, \quad \Gamma^4_{14} = -\frac{1}{x}, \quad \Gamma^4_{22} = -2Q, \quad \Gamma^4_{23} = x.$$

The geodesic equations of the extended metric are

$$\left(\frac{dx}{ds}\right)^2 = klogx, \tag{16}$$

$$\frac{dy}{ds} = \frac{c}{x}, \tag{17}$$

$$\frac{d^2P}{ds^2} + \frac{4Q}{x^2}\frac{dx}{ds}\frac{dy}{ds} - P\left(\frac{dy}{ds}\right)^2 - \frac{2}{x}\frac{dy}{ds}\frac{dQ}{ds} = 0 \tag{18}$$

and

$$\frac{d^2Q}{ds^2} + \frac{2Q}{x^2}\left(\frac{dx}{ds}\right)^2 - 2p\frac{dx}{ds}\frac{dy}{ds} - \frac{2}{x}\frac{dx}{ds}\frac{dQ}{ds} - 2Q\left(\frac{dy}{ds}\right)^2 + 2x\frac{dy}{ds}\frac{dp}{ds} = 0. \tag{19}$$

Making $k = 0$, the system of equations reduces to

$$x = m, \tag{20}$$

$$\frac{dy}{ds} = \frac{c}{m}, \tag{21}$$

$$\frac{d^2P}{ds^2} - \frac{c^2P}{m^2} - \frac{2c}{m}\frac{dQ}{ds} = 0, \tag{22}$$

$$\frac{d^2Q}{ds^2} - 2Q(\frac{c^2}{m^2}) + 2c\frac{dp}{ds} = 0, \tag{23}$$

where m and c are constants. This is a second order linear system and can be solved. The solution is lengthy though. The point of discussing such a direction is that straight line motions parallel to y-axis with constant velocity are transformed into motion with exponential characteristics in the extended components. Clearly, this is geodesical extension and equation (15) is a flat metric which can be easily verified. If a metric is flat its extension is also flat. But for a non zero constant metric this is not so. In fact there can be no Riemann extension of constant curvature.

3. Two dimensional space of constant positive curvature

Consider two dimensional metric

$$ds^2 = \frac{1}{1-kx^2}dx^2 + x^2dy^2. \tag{24}$$

For this metric, we have

$$\Gamma^1_{11} = \frac{kx}{1-kx^2}, \qquad \Gamma^1_{22} = -x(1-kx^2), \qquad \Gamma^2_{12} = \frac{1}{x}.$$

It is easy to check that metric (24) is a metric of constant curvature. On applying (3), the geodesic equations of this metric are

$$\frac{d^2x}{ds^2} + \frac{kx}{1-kx^2}\left(\frac{dx}{ds}\right)^2 - x(1-kx^2)\left(\frac{dy}{ds}\right)^2 = 0 \tag{25}$$

and

$$\frac{d^2y}{ds^2} + \frac{2}{x}\frac{dx}{ds}\frac{dy}{ds} = 0. \tag{26}$$

On applying (5) to the metric (24), we get the extended metric as

$$ds^2 = \frac{-4kxp}{1-kx^2}dx^2 + 4px(1-kx^2)dy^2 - \frac{4Q}{x}dxdy + 2dxdp + 2dQdy, \tag{27}$$

$$\Gamma^1_{11} = \frac{kx}{1-kx^2}, \quad \Gamma^1_{22} = -x(1-kx^2), \quad \Gamma^2_{12} = \frac{1}{x}, \quad \Gamma^3_{11} = -\frac{kP}{(1-kx^2)^2},$$

$$\Gamma^3_{12} = \frac{2Q}{x^2}, \quad \Gamma^3_{13} = -\frac{kx}{1-kx^2}, \quad \Gamma^3_{22} = 2kx^2P - P, \quad \Gamma^3_{24} = -\frac{1}{x},$$

$$\Gamma^4_{11} = \frac{2Q}{x^2(1-kx^2)}, \quad \Gamma^4_{12} = -2kx^2P, \quad \Gamma^4_{14} = -\frac{1}{x}, \quad \Gamma^4_{22} = -2(1-kx^2)Q,$$

$$\Gamma^4_{23} = x(1-kx^2).$$

It may be noted that it is not a metric of constant curvature. For instance $R_{1113} = \frac{2k^2x^2}{(1-kx^2)^2}$ as obtained from usual calculations. But the right side of equation(1) is zero. The system of geodesic equations are

$$\frac{d^2x}{ds^2} + \frac{kx}{1-kx^2}\left(\frac{dx}{ds}\right)^2 - x(1-kx^2)\left(\frac{dy}{ds}\right)^2 = 0, \tag{28}$$

$$\frac{d^2y}{ds^2} + \frac{2}{x}\frac{dx}{ds}\frac{dy}{ds} = 0, \tag{29}$$

$$\begin{aligned}\frac{d^2P}{ds^2} - \frac{kP}{(1-kx^2)^2}\left(\frac{dx}{ds}\right)^2 + \frac{4Q}{x^2}\frac{dx}{ds}\frac{dy}{ds} - \frac{2kx}{1-kx^2}\frac{dx}{ds}\frac{dP}{ds}\\ + (5kx^2P - 4P)\left(\frac{dy}{ds}\right)^2 - \frac{2}{x}\frac{dy}{ds}\frac{dQ}{ds} = 0\end{aligned} \tag{30}$$

and

$$\frac{d^2Q}{ds^2} + \frac{2Q}{x^2(1-kx^2)}\left(\frac{dx}{ds}\right)^2 - (P + 4kx^2P)\frac{dx}{ds}\frac{dy}{ds} - \frac{2}{x}\frac{dx}{ds}\frac{dy}{ds}$$
$$- 2Q(1-kx^2)\left(\frac{dy}{ds}\right)^2 + 2x(1-kx^2)\frac{dy}{ds}\frac{dP}{ds} = 0. \tag{31}$$

Clearly, $\frac{dx}{ds} = 0$ and $\frac{dy}{ds} = 0$ form a solution to the systems of equations (28) and (29). Then (30) and (31) reduce to

$$\frac{d^2P}{ds^2} = 0, \tag{32}$$

$$\frac{d^2Q}{ds^2} = 0. \tag{33}$$

Solving (32) and (33), we get $P = As + B$, $Q = Cs + D$.
This shows that the extended velocity components are constants. Further we note that stationary points are transformed in to either stationary points or points moving with constant extended velocity components.

4. Two dimensional space of constant negative curvature

Consider two dimensional metric as

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{34}$$

Here

$$\Gamma^1_{12} = -\frac{1}{y}, \qquad \Gamma^2_{11} = \frac{1}{y}, \qquad \Gamma^2_{22} = -\frac{1}{y}.$$

It can be easily verified that this is a metric constant negative curvature -1.
Its geodesic equations are

$$\frac{d^2x}{ds^2} - \frac{2}{y}\left(\frac{dx}{ds}\right)^2 = 0 \tag{35}$$

and

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}(\frac{dy}{ds})^2 = 0. \tag{36}$$

On applying (5) to (34), the four dimensional Riemann extension is given by

$$ds^2 = -\frac{4Q}{y}dx^2 + \frac{4P}{y}dxdy + \frac{4Q}{y}dy^2 + 2dxdp + 2dydQ, \tag{37}$$

where p and q are coordinates in the extended space. We have

$$\Gamma^1_{12} = -\frac{1}{y}, \qquad \Gamma^2_{11} = \frac{1}{y}, \qquad \Gamma^2_{22} = -\frac{1}{y}, \qquad \Gamma^3_{11} = -\frac{2P}{y^2},$$
$$\Gamma^3_{12} = -\frac{Q}{y^2}, \qquad \Gamma^3_{14} = -\frac{1}{y}, \qquad \Gamma^3_{23} = \frac{1}{y}, \qquad \Gamma^4_{11} = -\frac{3Q}{y^2},$$
$$\Gamma^4_{12} = \frac{2P}{y^2}, \qquad \Gamma^4_{13} = \frac{1}{y}, \qquad \Gamma^4_{22} = \frac{Q}{y^2}, \qquad \Gamma^4_{24} = \frac{1}{y}.$$

The system of geodesic equations of (37) are

$$\frac{d^2x}{ds^2} - \frac{2}{y}\left(\frac{dx}{ds}\right)^2 = 0, \tag{38}$$

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}\left(\frac{dy}{ds}\right)^2 = 0, \tag{39}$$

$$\frac{d^2P}{ds^2} - \frac{2P}{y^2}(\frac{dx}{ds})^2 - \frac{2Q}{y^2}\frac{dx}{ds}\frac{dy}{ds} - \frac{2}{y}\frac{dx}{ds}\frac{dQ}{ds} + \frac{2}{y}\frac{dy}{ds}\frac{dP}{ds} = 0 \tag{40}$$

and

$$\frac{d^2Q}{ds^2} - \frac{3Q}{y^2}\left(\frac{dx}{ds}\right)^2 + \frac{4P}{y^2}\frac{dx}{ds}\frac{dy}{ds} + \frac{2}{y}\frac{dx}{ds}\frac{dP}{ds} + \frac{Q}{y^2}\left(\frac{dy}{ds}\right)^2 + \frac{2}{y}\frac{dy}{ds}\frac{dQ}{ds} = 0. \tag{41}$$

Clearly, $x = c$, a constant is a solution of (38) and the system reduces to the form

$$x = c, \tag{42}$$

$$\frac{dy}{ds} = cy, \tag{43}$$

$$\frac{d^2P}{ds^2} + 2c\frac{dP}{ds} = 0 \tag{44}$$

and

$$\frac{d^2Q}{ds^2} + 2c\frac{dQ}{ds} + c^2Q = 0. \tag{45}$$

Solving above system of equations, we get

$$x = c, \tag{46}$$

$$y = Ae^{cs}, \tag{47}$$

$$P = \frac{-B}{2c}e^{-2cs} + D \tag{48}$$

$$\text{and} \quad Q = (E + Fs)e^{-cs}. \tag{49}$$

Here again, lines parallel to y-axis are transformed into curves in 4-dimensions. For $D = F = 0$, we have $P = kQ^2$, where $k = -\frac{B}{2cE^2}$. Thus in all three cases we note that the geodesics in the extended space generally give very drastic behaviour though preserving the properties of the base space.

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