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BEST COAPPROXIMATION IN $L^\infty(\mu, X)$

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ABSTRACT. Let X be a real Banach space and let G be a closed subset of X. The set G is called coproximinal in X if for each $x \in X$, there exists $y_0 \in G$ such that $\|y - y_0\| \leq \|x - y\|$, for all $y \in G$. In this paper, we study coproximinality of $L^\infty(\mu, G)$ in $L^\infty(\mu, X)$, when G is either separable or reflexive coproximinal subspace of X.

Keywords: Best Coapproximation, coproximinal set, essentially bounded functions.

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1. Introduction and Preliminaries

The theory of best coapproximation in normed linear spaces was developed as a counterpart to the theory of best approximation. It was initially introduced by Franchetti and Furi in 1972, [2], in order to study some characteristic properties of real Hilbert spaces. Many researches have been done since then, see [9-12]. Let X be a Banach space and G a bounded subset of X. For an element $x \in X$, the element $y_0 \in G$ is called a best coapproximation point of G to x, if

$$\|y_0 - y\| \leq \|x - y\|, \forall y \in G.$$

Consider the set-valued map $R_G : X \to 2^G$ defined by

$$R_G(x) = \{y_0 \in G : \|y_0 - y\| \leq \|x - y\|, \forall y \in G\},$$

namely, $R_G(x)$ is the set of all best coapproximation points to x from G. Notice that $R_G(x)$ is closed and bounded for each x, see [10], [9]. G is called coproximinal in X, if for each $x \in X$, there exists at least one point of best coapproximation to x in G. In other words, G is coproximinal in X iff $R(G) = X$, where $R(G) = \{x \in X : R_G(x) \neq \phi\}$. Clearly, $G \subset R(G)$. If $R(G)$ is dense in X then G is called densely coproximinal in X. On the other hand, G is called co-Chebyshev in X, if for each $x \in X$, $R_G(x)$ is singelton. Notice that, see Theorem 2 in [12], if G is convex in X, then $R_G(x)$ is a convex subset of G, for any $x \in X$ such that $R_G(x) \neq \phi$. Now, let G be a coproximinal subspace of X and denote by $\breve{G}$ the following set

$$\breve{G} = \{x \in X : \|y\| \leq \|y - x\|, \forall y \in G\}.$$

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Then, $X = G + \breve{G}$, [10]. The set $\breve{G}$ is sometimes written as $ker(R_G)$ and it is called the cometric complement of G, whereas R_G above is called the cometric projection onto G.

Clearly, when G is coproximinal subspace of X then for each $x \in X$, $R_G(x) = \{y_0 \in G : x - y_0 \in \breve{G}\}$.

Let X be a Banach space, (T, Σ, μ) a σ-finite complete measure space and let $L^p(\mu, X)$, $1 \leq p < \infty$, be the Banach spaces of all equivalence classes of strongly measurable, Bochner p-integrable functions on T i.e,

$$\int_T \|f(t)\|^p dt < \infty.$$

Usually, $L^p(\mu, X)$, $1 \leq p < \infty$ are called Bochner p-integrable function spaces, with norm defined as follows,

$$\|f\|_p = \{\int_T \|f(t)\|^p dt\}^{1/p}.$$

Let $L\infty(\mu, X)$ be the Banach space of all equivalence classes of strongly measurable, X-valued, essentially bounded functions on T (i.e bounded except on a set of measure zero). For $f \in L^{\infty}(\mu, X)$ the norm of f, namely $\|f\|_{\infty}$ is given by

$$\|f\|_{\infty} = ess\ sup_{t \in T} \|f(t)\|$$

For more on the theory of $L^p(\mu, X)$, $1 \leq p \leq \infty$, see [1] or [7].

The theory of best coapproximation has been studied for $L^p(\mu, X)$, $1 \leq p < \infty$, by [3] and [8], where several properties have been obtained. In [4], some results were generalized to Köthe Bochner function spaces. In this paper, we will study best coapproximation in $L^{\infty}(\mu, X)$ by elements in $L^{\infty}(\mu, G)$, where G is a closed subspace of X. Main results concerning coproximinality of $L^{\infty}(\mu, G)$, when G is either separable or reflexive coproximinal subspace of X, are presented in section 3.

2. Coproximinality in $L^{\infty}(\mu, X)$

Throughout this section, (T, Σ, μ) is a finite measure space, X is a real Banach space and G a closed subspace of X. $L^{\infty}(\mu, X)$ is the Banach space defined as above. The folllowing theorem is the first to start with,

Theorem 2.1. *For f in $L^{\infty}(\mu, X)$ and g in $L^{\infty}(\mu, G)$ such that $g(t)$ is a best coapproximation point in G to $f(t)$ in X, a.e $t \in T$, then g is a best coapproximation to f.*

Proof. Let $g(t)$ be a best coapproximation element in G to $f(t) \in X$, a.e $t \in T$. Then

$$\|g(t) - y\| \leq \|f(t) - y\|, \forall y \in G, a.e\ t \in T.$$

Hence, in particular, for any function h in $L^{\infty}(\mu, G)$, we have

$$\|g(t) - h(t)\| \leq \|f(t) - h(t)\|, \ a.e\ t \in T.$$

This implies for all $h \in L^{\infty}(\mu, G)$,

$$ess\ sup_{t \in T} \|g(t) - h(t)\| \leq ess\ sup_{t \in T} \|f(t) - h(t)\|.$$

So, we get

$$\|g - h\|_{\infty} \leq \|f - h\|_{\infty}, \forall h \in L^{\infty}(\mu, G).$$

Hence, g is a best coapproximation to f. □

On the other hand, consider the following theorem,

Theorem 2.2. *Let G be closed subspace of X. If $L^\infty(\mu, G)$ is coproximinal in $L^\infty(\mu, X)$ then G is coproximinal in X.*

Proof. Let $x \in X$. Define the function f_x as follows: $f_x(t) = x,\ a.e\ t \in T$. Then, it is clear that $f_x \in L^\infty(\mu, X)$.
Now, from the given there exists $w \in L^\infty(\mu, G)$ such that

$$\|w - h\|_\infty \leq \|f_x - h\|_\infty\,,\ \forall h \in L^\infty(\mu, G).$$

In particular, for $h = f_y$, where $y \in G$. Hence,

$$\begin{aligned} \|w - f_y\|_\infty &\leq \|f_x - f_y\|_\infty\,,\ \forall y \in G. \\ &= ess\ sup_{t\in T} \|f_x(t) - f_y(t)\|\,,\ \forall y \in G. \\ &= \|x - y\|\,,\ \forall y \in G. \end{aligned}$$

So, for some t_0 in T, then

$$\|w(t_0) - y\| \leq \|w - f_y\|_\infty \leq \|x - y\|\,,\ \forall y \in G.$$

This implies that $w(t_0) \in G$ is a best coapproximation of $x \in X$, where $w \in L^\infty(\mu, G)$ is a best coapproximation of the constant function $f_x \in L^\infty(\mu, X)$. Hence, G is coproximinal in X. □

Next, let us consider the set of countably-valued functions which is dense in $L^\infty(\mu, X)$. For a countable collection $A_1, ..., A_n, ...$ of mutually disjoint measurable subsets of T, such that $\cup_{i=1}^\infty A_i = T$ and a sequence $x_1, ..., x_n, ...$ of elements in X, a function with countable range (countably-valued function) $f : T \to X$ is defined as follows,

$$f(t) = \sum_{i=1}^\infty x_i \chi_{A_i}(t),\ t \in T,$$

where for each i, χ_{A_i} is the characteristic function on A_i. Clearly, simple functions are included.

Theorem 2.3. *Let G be a coproximinal subspace in X. Then every countably-valued function in $L^\infty(\mu, X)$ has a best coapproximation in $L^\infty(\mu, G)$.*

Proof. Let $f = \sum\limits_{i=1}^\infty x_i \chi_{A_i}$ be a countably-valued function in $L^\infty(\mu, X)$. For each $g \in L^\infty(\mu, G)$ and a given $\epsilon > 0$, there exists a countably-valued function $\varphi_g = \sum\limits_{i=1}^\infty y_i \chi_{A_i}$ in $L^\infty(\mu, G)$, with $y_i \in G$, such that $\|g - \varphi_g\|_\infty < \epsilon/2$.

Now, for all $g \in L^\infty(\mu, G)$, we can write

$$\begin{aligned} \|f - g\|_\infty &\geq \|f - \varphi_g\|_\infty - \|\varphi_g - g\|_\infty \\ &> \|f - \varphi_g\|_\infty - \epsilon/2 \end{aligned}$$

And since G is coproximinal, let z_i, for each i, be the best coapproximation in G to $x_i \in X$. Thus, for each i, we have

$$\|x_i - y_i\|_X \geq \|z_i - y_i\|_X\,.$$

Hence,

$$ess\ sup_{t\in T}\{\sum_{i=1}^{\infty}\|x_i - y_i\|_X\,\chi_{A_i}(t)\} \geq ess\ sup_{t\in T}\{\sum_{i=1}^{\infty}\|z_i - y_i\|_X\,\chi_{A_i}(t)\}$$

So, by taking $g_0 = \sum_{i=1}^{\infty} z_i\chi_{A_i}$, we get

$$\|f - \varphi_g\|_{\infty} \geq \|g_0 - \varphi_g\|_{\infty}.$$

But again write

$$\|g_0 - \varphi_g\|_{\infty} \geq \|g_0 - g\|_{\infty} - \|g - \varphi_g\|_{\infty}$$

which implies

$$\|f - g\|_{\infty} > \|g_0 - g\|_{\infty} - \epsilon.$$

And since ϵ arbitrary, we get

$$\|f - g\|_{\infty} \geq \|g_0 - g\|_{\infty},$$

for all $g \in L^{\infty}(\mu, G)$. □

Corollary 2.1. *Let G be a coproximinal subspace in X. Then $L^{\infty}(\mu, G)$ is densely coproximinal in $L^{\infty}(\mu, X)$.*

3. Main Results

In this section, we will give two main results concerning coproximinality of $L^{\infty}(\mu, G)$ in $L^{\infty}(\mu, X)$ when G is either separable or reflexive coproximinal subspace of the Banach space X. First, we deal with G being separable. Let us recall, see [6], pp.133, that a set-valued map on a measure space (T, Σ, μ), $F : T \to 2^X$ is said to be weakly measurable if for any open set U of X, the set $\{t \in T : F(t) \bigcap U \neq \phi\}$ is measurable (i.e belongs to Σ). A measurable selection of F is a measurable function $h : T \to X$ such that $h(t) \in F(t)$, for all $t \in T$. The following Lemma, known as Kuratowski-Ryll-Nardzewski Measurable Selection Theorem [5], can also be found in [6].

Lemma 3.1. *Let $F : T \to 2^X$ be a weakly measurable set-valued map carrying each $t \in T$ to a nonempty closed and bounded subset of X. If X is a separable Banach space then F has a measurable selection.*

Now, let G be a coproximinal subspace in the Banach space X and $\breve{G}$ the cometric complement of G in X. For each $f \in L^{\infty}(\mu, X)$, define the map $\pi_f : T \to 2^G$ as

$$\pi_f(t) = \{z_t \in G : f(t) - z_t \in \breve{G}\},\ t \in T.$$

Then π_f is a set-valued map, taking each element $t \in T$ into a subset of G, precisely the set of best coapproximation points to $f(t) : R_G(f(t))$.

Theorem 3.1. *Let G be a separable subspace of X such that π_f as defined above is weakly measurable. Then $L^{\infty}(\mu, G)$ is coproximinal in $L^{\infty}(\mu, X)$ if G is coproximinal in X.*

Proof. Suppose that G is coproximinal in X and let f be in $L^{\infty}(\mu, X)$. Let $\pi_f : T \to 2^G$ be the set-valued map defined as above. Hence, we can write,

$$\pi_f(t) = \{z_t \in G : \|z_t - y\| \leq \|f(t) - y\|,\ \text{for all } y \in G\}.$$

Hence, for each $t \in T$, $\pi_f(t)$ is closed, bounded and nonempty subset in G, since it takes $t \in T$ to the set of best coapproximation points in G to $f(t)$. The assumption that the map π_f is weakly measurable implies (by Lemma 3.1) that it has a measurable selection, say $w : T \to G$ such that $w(t) \in \pi_f(t)$, a.e $t \in T$. But since G separable, w is strongly measurable by Lemma 10.3 in [6]. Hence the result follows, from Theorem 2.1, if we show that $w \in L^\infty(\mu, G)$. Indeed, since $w : T \to G$ satisfies

$$\|w(t) - y\| \leq \|f(t) - y\|, \text{ for all } y \in G.$$

So, in particular

$$\|w(t)\| \leq \|f(t)\|, \text{ a.e } t \in T,$$

which implies $\|w\|_\infty \leq \|f\|_\infty$. Hence, $w \in L^\infty(\mu, G)$. □

For the next main result, Theorem 3.2, we need the following Lemma which has been proved in [4], (see Theorem 7 in [4])

Lemma 3.2. *Let (I, μ) be a finite measure space, G be a separable coproximinal subspace of X and $f : I \to X$ be measurable function. Then there is a measurable function $g : I \to G$ such that $g(t)$ is a point of coapproximation to $f(t)$ in G, a.e $t \in I$.*

Theorem 3.2. *Let G be a seperable subspace of X. G is coproximinal in X iff $L^\infty(\mu, G)$ is coproximinal in $L^\infty(\mu, X)$.*

Proof. Suppose that G is separable and coproximinal in X and let $f \in L^\infty(\mu, X)$. Lemma 3.2 above guarantees that there exists a measurable function g defined on T with values in G (hence g is strongly measurable since G separable) such that $g(t)$ is a point of best coapproximation to $f(t)$, a.e $t \in T$. Thus, we have $f(t) - g(t) \in \breve{G}$, a.e $t \in T$, which implies that

$$\|y\| \leq \|y - (f(t) - g(t))\| \leq \|f(t) - g(t) + y\|, \forall y \in G.$$

In particular, taking $y = g(t)$, we get a.e $t \in T$,

$$\|g(t)\| \leq \|f(t) - g(t) + g(t)\| = \|f(t)\|.$$

Hence, $g \in L^\infty(\mu, G)$ and $g(t)$ is a best coapproximation point to $f(t)$. It follows from Theorem 2.1 that g is a point of best coapproximation to f in $L^\infty(\mu, G)$. The other direction follows from Theorem 2.2. □

In the remaining part of this section, we will deal with coproximinality of $L^\infty(\mu, G)$ in $L^\infty(\mu, X)$, when G is reflexive coproximinal subspace in X. We assume that (T, μ) is a finite measure space.

Theorem 3.3. *If $L^1(\mu, G)$ is coproximinal in $L^1(\mu, X)$, then $L^\infty(\mu, G)$ is coproximinal in $L^\infty(\mu, X)$.*

Proof. Let $f \in L^\infty(\mu, X)$. Since the measure space (T, μ) is finite then $f \in L^1(\mu, X)$. Hence, by the given there exists $g_0 \in L^1(\mu, G)$ such that

$$\|g_0 - g\|_1 \leq \|f - g\|_1, \text{ for all } g \in L^1(\mu, G).$$

By Lemma 2.2 in [3], we have

$$\|g_0(t) - g(t)\| \leq \|f(t) - g(t)\|, \ \mu\text{-a.e } t \in T.$$

Hence, in particular, for all $g(t) = w(t) \in G$, where $w \in L^\infty(\mu, G)$. But, since $0 \in G$, then $\|g_0(t)\| \leq \|f(t)\|$, μ-a.e $t \in T$. This implies $\|g_0\|_\infty \leq \|f\|_\infty$. Hence, $g_0 \in L^\infty(\mu, G)$.
Now,

$$\|g_0(t) - w(t)\| \leq \|f(t) - w(t)\|, \ \mu\text{-a.e } t \in T,$$

which implies $\|g_0 - w\|_\infty \leq \|f - w\|_\infty, \forall w \in L^\infty(\mu, G)$. □

Theorem 3.4. *Let G be a reflexive coproximinal subspace of X. Then $L^\infty(\mu, G)$ is coproximinal in $L^\infty(\mu, X)$.*

Proof. Let G be a reflexive coproximinal subspace in X. It has been proved in [3] (see Theorem 3.6 in [3]) that $L^1(\mu, G)$ is coproximinal in $L^1(\mu, X)$. Hence the result follows from Theorem 3.3. □

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