TWMS J. App. Eng. Math. V.8, N.2, 2018, pp. 483-494

SINGLE-VALUED NEUTROSOPHIC LINE GRAPHS

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Abstract. In this paper, the concept of a single-valued neutrosophic line graph (SVNLG) of a single-valued neutrosophic graph (SVNG) is introduced and its properties are investigated. We state a necessary and sufficient condition for a SVNG to be isomorphic to its corresponding SVNLG. Moreover, a necessary and sufficient condition for a SVNG to be the SVNLG of some SVNG is given. The notion of a single-valued neutrosophic clique (SVNC) is introduced. A complete characterization of the structure of the SVNC is presented.

Keywords: Neutrosophic set, single-valued neutrosophic set, single-valued neutrosophic graph, single-valued neutrosophic line graph, single-valued neutrosophic clique.

AMS Subject Classification: 05C99

1. Introduction

In 1965, Zadeh [25] originally introduced the concept of fuzzy set, characterized by a membership function in $[0, 1]$, which shows great advantages in expressing vague information when accurate judgments to the things cannot be given. But sometimes, to explain the characters of things, membership function of the fuzzy set is not adequate. To overcome this shortcoming of the fuzzy set, Atanassov [5] proposed an extension of fuzzy set by introducing a non-membership and a hesitancy function, and defined intuitionistic fuzzy set (IFS) which has been applied in many areas, such as pattern recognition, decision making, medical diagnosis and cluster analysis. Atanassov and Gargov [6] introduced interval-valued intuitionistic fuzzy set (IVIFS) which consists of a membership function and a non-membership function, whose values are intervals rather than exact numbers. The IVIFS comprehensively depicts the characters of things. However IFSs and IVIFSs cannot deal with all types of uncertainty, such as indeterminate information and inconsistent information, which exist commonly in different real-world problems.

To effectively accommodate such situations, Smarandache [21] firstly proposed neutrosophic set theory from philosophical point of view. Its prominent characteristic is that a truth-membership degree, an indeterminacy-membership degree and a falsity-membership degree, in non-standard unit interval $]0^-, 1^+[$, are independently assigned to each element in the set. Gradually, it has been discovered that without a specific description, neutrosophic sets are difficult to apply in the real applications. After analyzing this difficulty, Wang et al. [23] initiated the concept of a single-valued neutrosophic set (SVNS) from

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§ Manuscript received: December 18, 2016; accepted: January 04, 2017.
TWMS Journal of Applied and Engineering Mathematics Vol.8, No.2

scientific or engineering point of view, as an instance of the neutrosophic set and an extension of IFS, and provided its various properties. SVNSs represent uncertainty, incomplete, inconsistent, and imprecise information which exist in real world.

The concept of fuzzy graphs was initiated by Kaufmann [11], based on Zadeh's fuzzy relations. Later, another elaborated definition of fuzzy graph with fuzzy vertex and fuzzy edges was introduced by Rosenfeld [18] and obtaining analogs of several graph theoretical concepts such as paths, cycles and connectedness etc, he developed the structure of fuzzy graphs. Some remarks on fuzzy graphs were given by Bhattacharya [7]. Fuzzy line graphs were studied in [12] by Mordeson. Nair and Cheng [13] defined the concept of a fuzzy clique consistent with the definition of fuzzy cycles in fuzzy graphs. Intuitionistic fuzzy graphs with vertex sets and edge sets as IFS were introduced by Akram and Davvaz [1]. Sahoo and Pal [19, 20] introduced some new concepts of intuitionistic fuzzy graphs. Naz et al. [3, 16, 17] put forward many new concepts concerning the extended structures of fuzzy graphs. Kandasamy et al. [22] put forward the notion of neutrosophic graphs. Neutrosophic graphs, particularly SVNGs [2, 4, 8, 9, 14, 15] have attracted significant interest from researchers in recent years. In literature, the study of SVNLGs and SVNCs is still blank. To fill this vacancy, we shall focus on the study of SVNLGs and SVNCs, in this paper.

The paper is structured as follows: Section 2 contains a brief background about SVNSs and SVNGs. Section 3 introduces the concept of SVNLG of a SVNG and, investigates their properties. In Section 4, the notion of SVNC consistent with single-valued neutrosophic cycles in SVNGs is proposed and a complete characterization of the structure of the SVNC is presented, and finally we draw conclusions in Section 5.

2. Preliminaries

In the following, some basic concepts on SVNSs and SVNGs are reviewed to facilitate next sections.

A graph is a pair of sets $G = (V, E)$, satisfying $E \subseteq V \times V$. The elements of V and E are the vertices and edges of the graph G, respectively. The line graph $L(G)$ of G is the graph whose vertices are the edges of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent. A connected graph is isomorphic to its line graph if and only if it is a cycle [10]. The idea of line graph of any graph is so natural that many authors have discovered it independently.

Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$. Let $S_i = \{v_i, x_{i1}, \ldots, x_{ip_i}\}$ where $x_{ij} \in E$ has v_i as a vertex, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, p_i$. Let $S = \{S_1, S_2, \ldots, S_n\}$ and $T = \{S_iS_j \mid S_i, S_j \in S, S_i \cap S_j \neq \emptyset, i \neq j\}$. Then $\Omega(S) = (S, T)$ is an intersection graph and $G \cong \Omega(S)$.

For a graph G, the line graph $L(G)$ is the intersection graph of the set of lines of G. That is, $L(G) = (Z, W)$ where $Z = \{\{x\} \cup \{u_x, v_x\} \mid x \in E, u_x, v_x \in V, x = u_xv_x\}$ and $W = \{S_xS_y \mid S_x \cap S_y \neq \emptyset, x, y \in E, x \neq y\}$, with $S_x = \{x\} \cup \{u_x, v_x\}, x \in E$.

A clique in a graph G is a complete subgraph of G. A subgraph H of a graph G is a disjoint union of cliques if $V(H)$ can be partitioned into $H_1, H_2, \ldots, H_k$ such that $xy \in E(H)$ for all $x, y \in V(H)$ if and only if $\{x, y\} \subseteq H_i$, for some $i,\ i = 1, 2, \ldots, k$ [13].

Definition 2.1. [18, 25] *A fuzzy subset η of a set V is a function $\eta : V \to [0, 1]$. A fuzzy (binary) relation on a set V is a mapping $\mu : V \times V \to [0, 1]$ such that $\mu(x, y) \leq \min\{\eta(x), \eta(y)\}$ for all $x, y \in V$. A fuzzy relation μ is symmetric if $\mu(x, y) = \mu(y, x)$ for* *all $x, y \in V$. A fuzzy graph is a pair $\mathcal{G} = (\eta, \mu)$, where η is a fuzzy subset of a set V and μ is a (symmetric) fuzzy relation on η.*

Definition 2.2. [5] *An IFS X in V is an object having the form*

$$X = \{\langle x, \mu_X(x), \nu_X(x)\rangle \mid x \in V\},$$

where the functions $\mu_X : V \to [0,1]$ and $\nu_X : V \to [0,1]$ define the degree of membership and degree of non-membership of the element $x \in V$, respectively, such that $0 \leq \mu_X(x) + \nu_X(x) \leq 1$ for all $x \in V$.

For each IFS X in V, $\pi_X(x) = 1 - \mu_X(x) - \nu_X(x)$ is called a hesitancy degree of x in X. If $\pi_X(x) = 0$ for all $x \in V$, then IFS reduces to Zadeh's fuzzy set.

Definition 2.3. [21] *Let V be a space of points (objects), with a generic element in V denoted by x. A neutrosophic set X in V is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X. $T_X(x), I_X(x)$ and $F_X(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $T_X : V \to]0^-, 1^+[, I_X : V \to]0^-, 1^+[$ and $F_X : V \to]0^-, 1^+[$.*

There is no restriction on the sum of $T_X(x), I_X(x)$ and $F_X(x)$, therefore $0^- \leq \sup T_X(x) + \sup I_X(x) + \sup F_X(x) \leq 3^+$.

Definition 2.4. [23] *Let V be a space of points (objects), with a generic element in V denoted by x. A SVNS X in V is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X. For each point $x \in X$, $T_X(x), I_X(x), F_X(x) \in [0,1]$. Therefore, a SVNS X in V can be written as*

$$X = \{\langle x, T_X(x), I_X(x), F_X(x)\rangle \mid x \in V\},$$

Definition 2.5. [24] *A SVNS Y in $V \times V$ is said to be a single-valued neutrosophic relation in V, denoted by*

$$Y = \{\langle xy, T_Y(xy), I_Y(xy), F_Y(xy)\rangle \mid xy \in V \times V\}$$

where $T_Y : V \times V \to [0,1]$, $I_Y : V \times V \to [0,1]$ and $F_Y : V \times V \to [0,1]$ represent the truth-membership, indeterminacy-membership and falsity-membership function of Y, respectively.

Definition 2.6. [8, 14] *A single-valued neutrosophic graph (SVNG) on a non-empty set V is a pair $\mathcal{G} = (X, Y)$, where X is a SVNS in V and Y is a single-valued neutrosophic relation on V such that*

$$T_Y(xy) \leq \min\{T_X(x), T_X(y)\},$$

$$I_Y(xy) \geq \max\{I_X(x), I_X(y)\},$$

$$F_Y(xy) \geq \max\{F_X(x), F_X(y)\}$$

for all $x, y \in V$. X and Y are called the single-valued neutrosophic vertex set and the single-valued neutrosophic edge set of $\mathcal{G}$, respectively. Here Y is a symmetric single-valued neutrosophic relation on X.

Definition 2.7. [8] *A SVNG $\mathcal{G} = (X, Y)$ is called strong if $T_Y(xy) = \min\{T_X(x), T_X(y)\}$, $I_Y(xy) = \max\{I_X(x), I_X(y)\}$ and $F_Y(xy) = \max\{F_X(x), F_X(y)\}$ for all $xy \in E$.*

3. Single-valued neutrosophic line graphs

This section proposes the concept of a SVNLG of a SVNG based on the extension of the concept of a fuzzy line graph of a fuzzy graph which was proposed by Mordeson [12], and investigates its properties.

Definition 3.1. *Let $\Omega(S) = (S, T)$ be an intersection graph of a graph $G = (V, E)$ and let $\mathcal{G} = (X_1, Y_1)$ be a SVNG with underlying set V. A SVNG of $\Omega(S)$ is a pair (X_2, Y_2), where $X_2 = \langle T_{X_2}, I_{X_2}, F_{X_2}\rangle$ and $Y_2 = \langle T_{Y_2}, I_{Y_2}, F_{Y_2}\rangle$ are SVNSs of S and T, respectively, such that*

- **(i):** $T_{X_2}(S_i) = T_{X_1}(v_i)$, $I_{X_2}(S_i) = I_{X_1}(v_i)$ *and* $F_{X_2}(S_i) = F_{X_1}(v_i)$ *for all* $S_i, S_j \in S$,
- **(ii):** $T_{Y_2}(S_iS_j) = T_{Y_1}(v_iv_j)$, $I_{Y_2}(S_iS_j) = I_{Y_1}(v_iv_j)$ *and* $F_{Y_2}(S_iS_j) = F_{Y_1}(v_iv_j)$ *for all* $S_iS_j \in T$.

That is, any SVNG of intersection graph $\Omega(S)$ is a single valued neutrosophic intersection graph of $\mathcal{G}$.

Definition 3.2. *Let $L(G) = (Z, W)$ be a line graph of a graph $G = (V, E)$. A SVNLG of a SVNG $\mathcal{G} = (X_1, Y_1)$ (with underlying set V) is a pair $L(\mathcal{G}) = (X_2, Y_2)$, where $X_2 = \langle T_{X_2}, I_{X_2}, F_{X_2}\rangle$ and $Y_2 = \langle T_{Y_2}, I_{Y_2}, F_{Y_2}\rangle$ are SVNSs of Z and W, respectively, such that*

$$T_{X_2}(S_x) = T_{Y_1}(x) = T_{Y_1}(u_xv_x), I_{X_2}(S_x) = I_{Y_1}(x) = I_{Y_1}(u_xv_x)$$
$$\text{and } F_{X_2}(S_x) = F_{Y_1}(x) = F_{Y_1}(u_xv_x) \text{ for all } S_x, S_y \in Z,$$

$$T_{Y_2}(S_xS_y) = \min\{T_{Y_1}(x), T_{Y_1}(y)\}, I_{Y_2}(S_xS_y) = \max\{I_{Y_1}(x), I_{Y_1}(y)\}$$
$$\text{and } F_{Y_2}(S_xS_y) = \max\{F_{Y_1}(x), F_{Y_1}(y)\} \text{ for all } S_xS_y \in W.$$

Example 3.1. *Consider a graph $G = (V, E)$, where $V = \{a_1, a_2, a_3\}$ and $E = \{x_1 = a_1a_2, x_2 = a_2a_3, x_3 = a_1a_3\}$. Let $\mathcal{G} = (X_1, Y_1)$ be a SVNG of G, as shown in Figure 1, defined by*

	a_1	a_2	a_3
T_{X_1}	0.5	0.4	0.6
I_{X_1}	0.2	0.3	0.1
F_{X_1}	0.4	0.1	0.2

	a_1a_2	a_2a_3	a_1a_3
T_{Y_1}	0.2	0.1	0.1
I_{Y_1}	0.6	0.4	0.3
F_{Y_1}	0.4	0.3	0.5

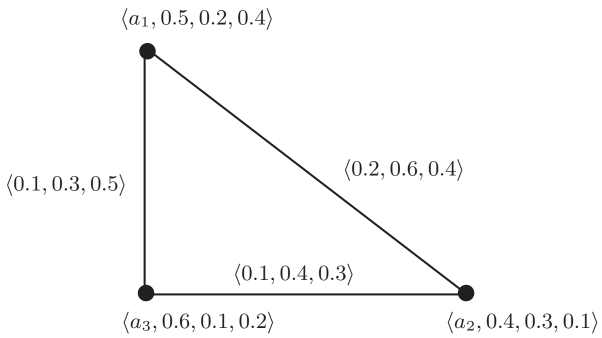

Figure 1. Single-valued neutrosophic graph $\mathcal{G} = (X_1, Y_1)$.

Consider a line graph $L(G) = (Z, W)$, where $Z = \{S_{x_1}, S_{x_2}, S_{x_3}\}$ and $W = \{S_{x_1}S_{x_2}, S_{x_2}S_{x_3},$

$S_{x_3}S_{x_1}\}$. *Let* $L(\mathcal{G})$ *be the SVNLG, as shown in Figure 2. Then, we have*

$$T_{X_2}(S_{x_1}) = 0.2, I_{X_2}(S_{x_1}) = 0.6, F_{X_2}(S_{x_1}) = 0.4,$$
$$T_{X_2}(S_{x_2}) = 0.1, I_{X_2}(S_{x_2}) = 0.4, F_{X_2}(S_{x_2}) = 0.3,$$
$$T_{X_2}(S_{x_3}) = 0.1, I_{X_2}(S_{x_3}) = 0.3, F_{X_2}(S_{x_3}) = 0.5.$$

$$T_{Y_2}(S_{x_1}S_{x_2}) = 0.1, I_{Y_2}(S_{x_1}S_{x_2}) = 0.6, F_{Y_2}(S_{x_1}S_{x_2}) = 0.4,$$
$$T_{Y_2}(S_{x_2}S_{x_3}) = 0.1, I_{Y_2}(S_{x_2}S_{x_3}) = 0.4, F_{Y_2}(S_{x_2}S_{x_3}) = 0.5,$$
$$T_{Y_2}(S_{x_1}S_{x_3}) = 0.1, I_{Y_2}(S_{x_1}S_{x_3}) = 0.6, F_{Y_2}(S_{x_1}S_{x_3}) = 0.5.$$

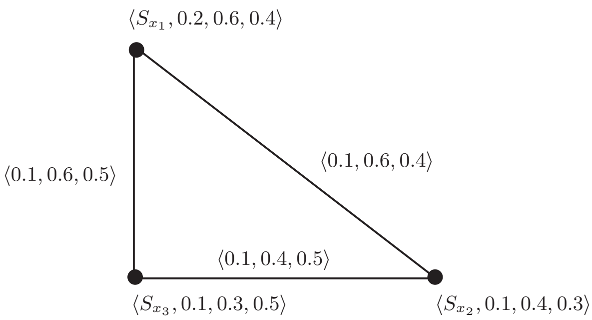

FIGURE 2. Single-valued neutrosophic line graph $L(\mathcal{G})$.

Proposition 3.1. *A SVNLG is always a strong SVNG.*

Proof. It is obvious, therefore omitted. □

Proposition 3.2. *If* $L(\mathcal{G})$ *is a SVNLG of SVNG* $\mathcal{G}$. *Then* $L(G)$ *is the line graph of* G.

Proof. Since $\mathcal{G} = (X_1, Y_1)$ is a SVNG of G and $L(\mathcal{G}) = (X_2, Y_2)$ is a SVNG of $L(G)$,

$$T_{X_2}(S_x) = T_{Y_1}(x), I_{X_2}(S_x) = I_{Y_1}(x), F_{X_2}(S_x) = F_{Y_1}(x) \text{ for all } x \in E$$

and so $S_x \in Z$ if and only if $x \in E$. Also

$$T_{Y_2}(S_xS_y) = \min\{T_{Y_1}(x), T_{Y_1}(y)\}, I_{Y_2}(S_xS_y) = \max\{I_{Y_1}(x), I_{Y_1}(y)\},$$
$$F_{Y_2}(S_xS_y) = \max\{F_{Y_1}(x), F_{Y_1}(y)\} \text{ for all } S_xS_y \in W$$

and so $W = \{S_xS_y \mid S_x \cap S_y \neq \emptyset, x, y \in E, x \neq y\}$. Hence proved. □

Proposition 3.3. *Let* $L(\mathcal{G}) = (X_2, Y_2)$ *be a SVNG of* $L(G)$. *Then* $L(\mathcal{G})$ *is a SVNLG of some SVNG of* G *if and only if*

$$T_{Y_2}(S_xS_y) = \min\{T_{X_2}(S_x), T_{X_2}(S_y)\}, I_{Y_2}(S_xS_y) = \max\{I_{X_2}(S_x), I_{X_2}(S_y)\},$$
$$F_{Y_2}(S_xS_y) = \max\{F_{X_2}(S_x), F_{X_2}(S_y)\} \text{ for all } S_xS_y \in W.$$

Proof. Suppose that $T_{Y_2}(S_xS_y) = \min\{T_{X_2}(S_x), T_{X_2}(S_y)\}, I_{Y_2}(S_xS_y) = \max\{I_{X_2}(S_x), I_{X_2}(S_y)\}$, $F_{Y_2}(S_xS_y) = \max\{F_{X_2}(S_x), F_{X_2}(S_y)\}$ for all $S_xS_y \in W$. Define

$$T_{X_2}(S_x) = T_{Y_1}(x), I_{X_2}(S_x) = I_{Y_1}(x), F_{X_2}(S_x) = F_{Y_1}(x) \text{ for all } x \in E.$$

Then

$$T_{Y_2}(S_xS_y) = \min\{T_{X_2}(S_x), T_{X_2}(S_y)\} = \min\{T_{Y_1}(x), T_{Y_1}(y)\},$$
$$I_{Y_2}(S_xS_y) = \max\{I_{X_2}(S_x), I_{X_2}(S_y)\} = \max\{I_{Y_1}(x), I_{Y_1}(y)\},$$
$$F_{Y_2}(S_xS_y) = \max\{F_{X_2}(S_x), F_{X_2}(S_y)\} = \max\{F_{Y_1}(x), F_{Y_1}(y)\} \text{ for all } S_xS_y \in W.$$

Any SVNS X_1 that yields the property $T_{Y_1}(uv) \leq \min\{T_{X_1}(u), T_{X_1}(v)\}$, $I_{Y_1}(uv) \geq \max\{I_{X_1}(u), I_{X_1}(v)\}$ and $F_{Y_1}(uv) \geq \max\{F_{X_1}(u), F_{X_1}(v)\}$ will suffice. The converse is immediate. Hence proved. □

The following theorem shows that when a SVNG is a SVNLG of some SVNG.

Theorem 3.1. *$L(\mathcal{G})$ is a SVNLG if and only if $L(G)$ is a line graph and*

$$T_{Y_2}(uv) = \min\{T_{X_2}(u), T_{X_2}(v)\}, I_{Y_2}(uv) = \max\{I_{X_2}(u), I_{X_2}(v)\},$$
$$F_{Y_2}(uv) = \max\{F_{X_2}(u), F_{X_2}(v)\} \text{ for all } uv \in W.$$

Proof. Straightforward using Propositions 3.2 and 3.3. □

Definition 3.3. *A homomorphism $\chi : \mathcal{G}_1 \to \mathcal{G}_2$ of two SVNGs $\mathcal{G}_1 = (X_1, Y_1)$ and $\mathcal{G}_2 = (X_2, Y_2)$ is a mapping $\chi : V_1 \to V_2$ such that*

(a): $T_{X_1}(x_1) \leq T_{X_2}(\chi(x_1))$,$I_{X_1}(x_1) \geq I_{X_2}(\chi(x_1))$,$F_{X_1}(x_1) \geq F_{X_2}(\chi(x_1))$ *for all* $x_1 \in V_1$,

(b): $T_{Y_1}(x_1y_1) \leq T_{Y_2}(\chi(x_1)\chi(y_1))$,$I_{Y_1}(x_1y_1) \geq I_{Y_2}(\chi(x_1)\chi(y_1))$, $F_{Y_1}(x_1y_1) \geq F_{Y_2}(\chi(x_1)\chi(y_1))$ *for all* $x_1y_1 \in E_1$.

Definition 3.4. *A (weak) vertex-isomorphism is a bijective homomorphism $\chi : \mathcal{G}_1 \to \mathcal{G}_2$ such that $T_{X_1}(x_1) = T_{X_2}(\chi(x_1))$,$I_{X_1}(x_1) = I_{X_2}(\chi(x_1))$,$F_{X_1}(x_1) = F_{X_2}(\chi(x_1))$ for all $x_1 \in V_1$.*

A (weak) line-isomorphism is a bijective homomorphism $\chi : \mathcal{G}_1 \to \mathcal{G}_2$ such that $T_{Y_1}(x_1y_1) = T_{Y_2}(\chi(x_1)\chi(y_1))$,$I_{Y_1}(x_1y_1) = I_{Y_2}(\chi(x_1)\chi(y_1))$,$F_{Y_1}(x_1y_1) = F_{Y_2}(\chi(x_1)\chi(y_1))$ for all $x_1y_1 \in E_1$.

If $\chi : \mathcal{G}_1 \to \mathcal{G}_2$ is a (weak) vertex-isomorphism and a (weak) line-isomorphism, then χ is called a (weak) isomorphism.

The following proposition shows that any SVNG is isomorphic to a single-valued neutrosophic intersection graph.

Proposition 3.4. *Let $\mathcal{G} = (X_1, Y_1)$ be a SVNG with underlying set V. Then*

(i): *(X_2, Y_2) is a SVNG of $\Omega(S)$,*

(ii): *$(X_1, Y_1) \cong (X_2, Y_2)$.*

Proposition 3.5. *Let $\mathcal{G}$ and $\mathcal{G}'$ be SVNGs of G and G', respectively. If $\chi : \mathcal{G} \to \mathcal{G}'$ is a weak isomorphism, then $\chi : G \to G'$ is an isomorphism.*

Proof. Let $\chi : \mathcal{G} \to \mathcal{G}'$ be a weak isomorphism, then $v \in V$ if and only if $\chi(v) \in V'$ and $uv \in E$ if and only if $\chi(u)\chi(v) \in E'$.
Hence proved. □

Now we provide a necessary and sufficient condition for a SVNG to be isomorphic to its corresponding SVNLG.

Proposition 3.6. *Let $G = (V, E)$ be a connected graph. Suppose that $L(\mathcal{G}) = (X_2, Y_2)$ is a SVNLG corresponding to a SVNG $\mathcal{G} = (X_1, Y_1)$. Then*

(i): *there exists a weak isomorphism of $\mathcal{G}$ onto $L(\mathcal{G})$ if and only if G is a cycle and for all $v \in V, x \in E, T_{X_1}(v) = T_{Y_1}(x), I_{X_1}(v) = I_{Y_1}(x), F_{X_1}(v) = F_{Y_1}(x)$, that is, $X_1 = \langle T_{X_1}, I_{X_1}, F_{X_1}\rangle$ and $Y_1 = \langle T_{Y_1}, I_{Y_1}, F_{Y_1}\rangle$ are constant functions on V and E, respectively, taking on the same value.*

(ii): *If $\chi : \mathcal{G} \to L(\mathcal{G})$ is a weak isomorphism, then χ is an isomorphism.*

Proof. Suppose that $\chi : \mathcal{G} \to L(\mathcal{G})$ is a weak isomorphism. From Proposition 3.5, it follows that $G = (V, E)$ is a cycle. Consider $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2 = x_1, v_2v_3 = x_2, \ldots, v_nv_1 = x_n\}$, where $v_1v_2, \ldots, v_nv_1$ is a cycle. Let

$$T_{X_1}(v_i) = s_i, I_{X_1}(v_i) = s_i^{'}, F_{X_1}(v_i) = s_i^{''},$$
$$T_{Y_1}(x_i) = r_i, I_{Y_1}(x_i) = r_i^{'}, F_{Y_1}(x_i) = r_i^{''}$$

$i = 1, 2, \ldots, n, v_{n+1} = v_1$. Then for $s_1 = s_{n+1}, s_1^{'} = s_{n+1}^{'}, s_1^{''} = s_{n+1}^{''}$,

$$r_i \leq \min\{s_i, s_{i+1}\}, r_i^{'} \geq \max\{s_i^{'}, s_{i+1}^{'}\}, r_i^{''} \geq \max\{s_i^{''}, s_{i+1}^{''}\} \; i = 1, 2, \ldots, n. \tag{1}$$

Now $Z = \{S_{x_i} \mid i = 1, 2, \ldots, n\}$ and $W = \{S_{x_i}S_{x_{i+1}} \mid i = 1, 2, \ldots, n-1\}$.
Also for $r_1 = r_{n+1}, r_1^{'} = r_{n+1}^{'}, r_1^{''} = r_{n+1}^{''}$,

$$\begin{aligned}
&T_{X_2}(S_{x_i}) = T_{Y_1}(x_i) = r_i,\\
&I_{X_2}(S_{x_i}) = I_{Y_1}(x_i) = r_i^{'},\\
&F_{X_2}(S_{x_i}) = F_{Y_1}(x_i) = r_i^{''},\\
&T_{Y_2}(S_{x_i}S_{x_{i+1}}) = \min\{T_{Y_1}(x_i), T_{Y_1}(x_{i+1})\} = \min\{r_i, r_{i+1}\},\\
&I_{Y_2}(S_{x_i}S_{x_{i+1}}) = \max\{I_{Y_1}(x_i), I_{Y_1}(x_{i+1})\} = \max\{r_i^{'}, r_{i+1}^{'}\},\\
&F_{Y_2}(S_{x_i}S_{x_{i+1}}) = \max\{F_{Y_1}(x_i), F_{Y_1}(x_{i+1})\} = \max\{r_i^{''}, r_{i+1}^{''}\}
\end{aligned}$$

for all $i = 1, 2, \ldots, n$, where $v_{n+2} = v_2$.
Since χ is an isomorphism of G onto $L(G)$, χ maps V one-to-one onto X_2. Also χ preserves adjacency. Hence χ induces a permutation π of {1, 2, ...,n} such that $\chi(v_i) = S_{x_{\pi(i)}}$ and $x_i = v_iv_{i+1} \to \chi(v_i)\chi(v_{i+1}) = S_{x_{\pi(i)}}S_{x_{\pi(i+1)}}, i = 1, 2, \ldots, n-1$.
Now
$s_i = T_{X_1}(v_i) \leq T_{X_2}(\chi(v_i)) = T_{X_2}(S_{x_{\pi(i)}}) = r_{\pi(i)}$,
$s_i^{'} = I_{X_1}(v_i) \geq I_{X_2}(\chi(v_i)) = I_{X_2}(S_{x_{\pi(i)}}) = r_{\pi(i)}^{'}$,
$s_i^{''} = F_{X_1}(v_i) \geq F_{X_2}(\chi(v_i)) = F_{X_2}(S_{x_{\pi(i)}}) = r_{\pi(i)}^{''}$
and

$$\begin{aligned}
r_i = T_{Y_1}(x_i) &\leq T_{Y_2}(\chi(v_i)\chi(v_{i+1})) = T_{Y_2}(S_{x_{\pi(i)}}S_{x_{\pi(i+1)}})\\
&= \min\{T_{Y_1}(x_{\pi(i)}), T_{Y_1}(x_{\pi(i+1)})\}\\
&= \min\{r_{\pi(i)}, r_{\pi(i+1)}\},
\end{aligned}$$

$$\begin{aligned}
r_i^{'} = I_{Y_1}(x_i) &\geq I_{Y_2}(\chi(v_i)\chi(v_{i+1})) = I_{Y_2}(S_{x_{\pi(i)}}S_{x_{\pi(i+1)}})\\
&= \max\{I_{Y_1}(x_{\pi(i)}), I_{Y_1}(x_{\pi(i+1)})\}\\
&= \max\{r_{\pi(i)}^{'}, r_{\pi(i+1)}^{'}\},
\end{aligned}$$

$$\begin{aligned}
r_i^{''} = F_{Y_1}(x_i) &\geq F_{Y_2}(\chi(v_i)\chi(v_{i+1})) = F_{Y_2}(S_{x_{\pi(i)}}S_{x_{\pi(i+1)}})\\
&= \max\{F_{Y_1}(x_{\pi(i)}), F_{Y_1}(x_{\pi(i+1)})\}\\
&= \max\{r_{\pi(i)}^{''}, r_{\pi(i+1)}^{''}\} \; i = 1, 2, \ldots, n.
\end{aligned}$$

That is,

$$s_i \leq r_{\pi(i)}, s_i^{'} \geq r_{\pi(i)}^{'}, s_i^{''} \geq r_{\pi(i)}^{''}, \tag{2}$$

and

$$r_i \leq \min\{r_{\pi(i)}, r_{\pi(i+1)}\}, r_i^{'} \geq \max\{r_{\pi(i)}^{'}, r_{\pi(i+1)}^{'}\}, r_i^{''} \geq \max\{r_{\pi(i)}^{''}, r_{\pi(i+1)}^{''}\}. \tag{3}$$

From (3), we get $r_i \leq r_{\pi(i)}$, $r_i^{'} \geq r_{\pi(i)}^{'}$, $r_i^{''} \geq r_{\pi(i)}^{''}$ and so $r_{\pi(i)} \leq r_{\pi(\pi(i))}$, $r_{\pi(i)}^{'} \geq r_{\pi(\pi(i))}^{'}$, $r_{\pi(i)}^{''} \geq r_{\pi(\pi(i))}^{''}$ $i = 1, 2, \ldots, n$. Continuing, we have $r_i \leq r_{\pi(i)} \leq \ldots \leq r_{\pi^j(i)} \leq r_i$, $r_i^{'} \geq r_{\pi(i)}^{'} \geq \ldots \geq r_{\pi^j(i)}^{'} \geq r_i^{'}$, $r_i^{''} \geq r_{\pi(i)}^{''} \geq \ldots \geq r_{\pi^j(i)}^{''} \geq r_i^{''}$ and so $r_i = r_{\pi(i)}$, $r_i^{'} = r_{\pi(i)}^{'}$, $r_i^{''} = r_{\pi(i)}^{''}$ for all $i = 1, 2, \ldots, n$, where π^{j+1} is the identity map. Again, from 3, we get $r_i \leq r_{\pi(i+1)} = r_{i+1}, r_i^{'} \geq r_{\pi(i+1)}^{'} = r_{i+1}^{'}, r_i^{''} \geq r_{\pi(i+1)}^{''} = r_{i+1}^{''}$ for all $i = 1, 2, \ldots, n$ where $r_{n+1} = r_1$.

Hence from (1), (2) and (3),

$r_1 = \ldots = r_n = s_1 = \ldots = s_n$, $r_1^{'} = \ldots = r_n^{'} = s_1^{'} = \ldots = s_n^{'}$, $r_1^{''} = \ldots = r_n^{''} = s_1^{''} = \ldots = s_n^{''}$.

Hence X_1 and Y_1 are constant functions and we have also proved that (2) holds. The converse part is obvious.

Hence proved. □

Proposition 3.7. *Let $\mathcal{G} = (X_1, Y_1)$ and $\mathcal{G}^{'} = (X_1^{'}, Y_1^{'})$ be the SVNGs of connected graphs G and $G^{'}$, respectively. Let $L(\mathcal{G}) = (X_2, Y_2)$ and $L(\mathcal{G}^{'}) = (X_2^{'}, Y_2^{'})$ denote respectively the SVNLGs corresponding to $\mathcal{G}$ and $\mathcal{G}^{'}$. Assume that it is not the case that one of G and $G^{'}$ is complete graph K_3 and the other is bipartite complete graph $K_{1,3}$. If $L(\mathcal{G})$ and $L(\mathcal{G}^{'})$ are isomorphic, then $\mathcal{G}$ and $\mathcal{G}^{'}$ are line-isomorphic.*

Proof. Suppose that $L(\mathcal{G}) \cong L(\mathcal{G}^{'})$, then by Proposition 3.5, $L(G) \cong L(G^{'})$. Since $L(G)$ and $L(G^{'})$ are respectively the line graphs of G and $G^{'}$, so, $G \cong G^{'}$.

Let $\phi_1 : L(\mathcal{G}) \to L(\mathcal{G}^{'})$ and $\phi_2 : G \to G^{'}$ be the isomorphisms. Then

$T_{X_2}(S_{uv}) = T_{X_2^{'}}(\phi_1(S_{uv})) = T_{X_2^{'}}(S_{\phi_2(u)\phi_2(v)})$,

$I_{X_2}(S_{uv}) = I_{X_2^{'}}(\phi_1(S_{uv})) = I_{X_2^{'}}(S_{\phi_2(u)\phi_2(v)})$,

$F_{X_2}(S_{uv}) = F_{X_2^{'}}(\phi_1(S_{uv})) = F_{X_2^{'}}(S_{\phi_2(u)\phi_2(v)})$.

Therefore, $T_{Y_1}(uv) = T_{Y_1^{'}}(\phi_2(u)\phi_2(v))$, $I_{Y_1}(uv) = I_{Y_1^{'}}(\phi_2(u)\phi_2(v))$, $F_{Y_1}(uv) = F_{Y_1^{'}}(\phi_2(u)\phi_2(v))$.

Hence, $\mathcal{G}_1$ and $\mathcal{G}_2$ are line-isomorphic. Hence proved. □

4. Single-valued neutrosophic cliques

In this section, we propose the notion of SVNC consistent with single-valued neutrosophic cycles in SVNGs and present a complete characterization of the structure of the SVNC. To do this, we firstly introduce the concept of single-valued neutrosophic cycles.

Definition 4.1. *Let $\mathcal{G} = (X, Y)$ be a SVNG. Then*

(i): *$\mathcal{G}$ is a cycle if and only if $G = (V, E)$ is a cycle.*

(ii): *$\mathcal{G}$ is called a single-valued neutrosophic cycle if and only if G is a cycle and there does not exist unique edge lm of G such that*

$$T_Y(lm) = \min\{T_Y(xy) \mid xy \in E\}, I_Y(lm) = \max\{I_Y(xy) \mid xy \in E\},$$
$$\text{and } F_Y(lm) = \max\{F_Y(xy) \mid xy \in E\}.$$

Definition 4.2. *Let $\mathcal{G} = (X, Y)$ be a SVNG of a graph $G = (V, E)$ and $H = (X^{'}, Y^{'})$ be a subgraph induced by $S \subseteq V$. Then H is a clique if $H^* = (S, T)$ is a clique and H is a SVNC if H is a clique and every cycle in H is a single-valued neutrosophic cycle.*

Example 4.1. *Consider a graph $G = (V, E)$, where $V = \{a_1, a_2, a_3\}$ and $E = \{a_1a_2, a_2a_3, a_1a_3\}$. Let $\mathcal{G}$ be a SVNG of G, given in Figure 3.*

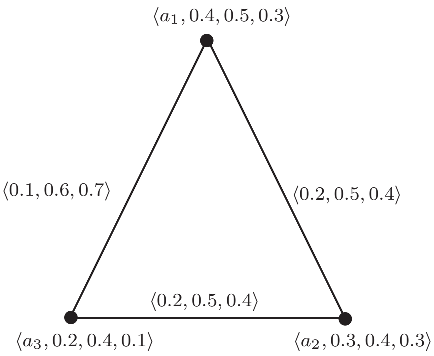

FIGURE 3. Not a single-valued neutrosophic clique.

Take $S = V$, then H is the same as $\mathcal{G}$. Routine computations show that H is a cycle but not a single-valued neutrosophic cycle. Hence H is a clique but not a SVNC.

Example 4.2. *Consider a graph $G = (V, E)$, where $V = \{a_1, a_2, a_3, a_4\}$ and $E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1a_3, a_2a_4\}$. Let $\mathcal{G}$ be a SVNG of G, given in Figure 4.*

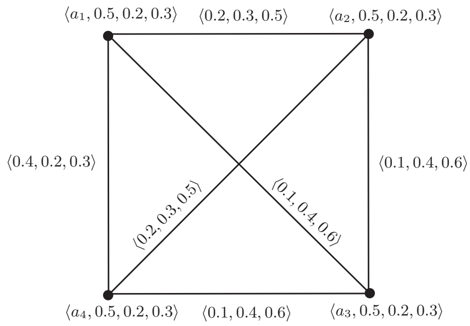

FIGURE 4. Single-valued neutrosophic clique.

Take $S = V$, then H is the same as $\mathcal{G}$. Routine computations show that every cycle in H is a single-valued neutrosophic cycle. Hence H is a clique and is also a SVNC.

Theorem 4.1. *Let $\mathcal{G} = (X, Y)$ be a SVNG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then H is a SVNC if and only if every cycle of length 3 in H is a single-valued neutrosophic cycle.*

Proof. Suppose that H is a SVNC. Then by above definition every cycle in H is a single-valued neutrosophic cycle and so every cycle of length 3 in H is also a single-valued neutrosophic cycle.

Conversely, assume that every cycle of length 3 is a single-valued neutrosophic cycle. To prove that H is a SVNC, we have to show that every cycle in H of length $n \geq 3$ is a single-valued neutrosophic cycle. The proof is by induction on the length of single-valued neutrosophic cycles in H. By assumption, every cycle of length 3 is a single-valued neutrosophic cycle. Induction hypothesis is that every cycle of length n is a single-valued neutrosophic cycle. Let $v_0, v_1, ..., v_n, v_{n+1}$ be any cycle C_{n+1} of length $n+1$ in H. Since H is a clique, H contains a cycle C_n of length n i.e $v_0, v_1, ..., v_n$ and is a single-valued neutrosophic cycle in H. Therefore $\exists$ at least two edges, say e_1 and e_2 in a single valued neutrosophic cycle C_n such that

$$\begin{aligned} T_Y(e_1) &= T_Y(e_2) = \min\{T_Y(e) \mid e \text{ is an edge in } C_n\}, \\ I_Y(e_1) &= I_Y(e_2) = \max\{I_Y(e) \mid e \text{ is an edge in } C_n\}, \\ F_Y(e_1) &= F_Y(e_2) = \max\{F_Y(e) \mid e \text{ is an edge in } C_n\}. \end{aligned}$$

Also v_0, v_n, v_{n+1} is a single-valued neutrosophic cycle and hence $\exists$ at least two edges, say e_3 and e_4 in a single-valued neutrosophic cycle v_0, v_n, v_{n+1} such that

$$\begin{aligned} T_Y(e_3) &= T_Y(e_4) = \min\{T_Y(e) \mid e \text{ is an edge in } v_0, v_n, v_{n+1}\}, \\ I_Y(e_3) &= I_Y(e_4) = \max\{I_Y(e) \mid e \text{ is an edge in } v_0, v_n, v_{n+1}\}, \\ F_Y(e_3) &= F_Y(e_4) = \max\{F_Y(e) \mid e \text{ is an edge in } v_0, v_n, v_{n+1}\}. \end{aligned}$$

Then two cases arise, firstly, if one of the edges e_1 or e_2 is the same as one of the edges e_3 or e_4. In this case, take $e_1 = e_3$. Then e_2 and e_4 are the edges in C_{n+1} such that

$$\begin{aligned} T_Y(e_2) &= T_Y(e_4) = \min\{T_Y(e) \mid e \text{ is an edge in } C_{n+1}\}, \\ I_Y(e_2) &= I_Y(e_4) = \max\{I_Y(e) \mid e \text{ is an edge in } C_{n+1}\}, \\ F_Y(e_2) &= F_Y(e_4) = \max\{F_Y(e) \mid e \text{ is an edge in } C_{n+1}\} \end{aligned}$$

as required.
Secondly, all four edges e_1, e_2, e_3, e_4 are edges in C_{n+1} and either

$$\begin{aligned} T_Y(e_1) &= T_Y(e_2) = \min\{T_Y(e) \mid e \text{ is an edge in } C_{n+1}\}, \\ I_Y(e_1) &= I_Y(e_2) = \max\{I_Y(e) \mid e \text{ is an edge in } C_{n+1}\}, \\ F_Y(e_1) &= F_Y(e_2) = \max\{F_Y(e) \mid e \text{ is an edge in } C_{n+1}\} \end{aligned}$$

or

$$\begin{aligned} T_Y(e_3) &= T_Y(e_4) = \min\{T_Y(e) \mid e \text{ is an edge in } C_{n+1}\}, \\ I_Y(e_3) &= I_Y(e_4) = \max\{I_Y(e) \mid e \text{ is an edge in } C_{n+1}\}, \\ F_Y(e_3) &= F_Y(e_4) = \max\{F_Y(e) \mid e \text{ is an edge in } C_{n+1}\}. \end{aligned}$$

Hence in both cases, H is a SVNC. $\square$

Lemma 4.1. *Let $\mathcal{G} = (X, Y)$ be a SVNG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then every cycle of length 3 in H is a single-valued neutrosophic cycle if and only if for any three vertices u, v, w in H such that the edges $uv, vw \in E(H_t)$ implies $uw \in E(H_t)$ for all $t \in [0, 1]$.*

Lemma 4.2. *Let $\mathcal{G} = (X, Y)$ be a SVNG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then H is a disjoint union of cliques if and only if for any three vertices u, v, w in H such that the edges $uv, vw \in E(H_t)$ implies $uw \in E(H_t)$ for all $t \in [0, 1]$.*

As a consequence of Lemmas 4.1 and 4.2, we obtain

Theorem 4.2. *Let $\mathcal{G} = (X, Y)$ be a SVNG of a graph $G = (V, E)$ and $H = (X', Y')$ be a subgraph induced by $S \subseteq V$. Then H is a SVNC if and only if every cut set of H is a disjoint union of cliques.*

5. Conclusions

Single-valued neutrosophic models are more flexible and practical than fuzzy and intuitionistic fuzzy models. In this paper, we have introduced the concept of SVNLG of a SVNG and discussed some of their desirable properties. We have introduced the notion of SVNC and presented a complete characterization of the structure of the SVNC. SVNGs can be used in computer technology, networking, communication, economics, genetics, linguistics, sociology, etc, when the concept of indeterminacy is present. We are extending our research work to (i) Interval-valued neutrosophic line graphs, (ii) Intuitionistic neutrosophic line graphs, (iii) Vague neutrosophic line graphs, and (iv) Bipolar neutrosophic line graphs.

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