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ON THE PARTIAL SUMS OF CONVEX HARMONIC UNIVALENT FUNCTIONS

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ABSTRACT. Partial sums of analytic univalent functions and partial sums of starlike have been investigated extensively by several researchers. In this paper, we investigate a partial sums of convex harmonic functions that are univalent and sense preserving in the open unit disk.

Keywords: Harmonic, Univalent, Convex, Partial sums.

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1. INTRODUCTION

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\Omega \subset \mathbb{C}$ if both u and v are real harmonic in Ω.
In any simply connected domain $\Omega \subset \mathbb{C}$, we may write $f = h + \overline{g}$, where h and g are analytic in Ω. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in Ω is that $|h^{'}(z)| > |g^{'}(z)|$ in Ω. (See [2]).

Denote by $\mathcal{S_H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{S_H}$, the analytic functions h and g can be expressed as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ |b_1| < 1. \tag{1}$$

A function f of the form (1) is harmonic convex of order $\alpha, 0 \leq \alpha < 1$, denoted by $K_H(\alpha)$, if it satisfies

$$\frac{\partial}{\partial\theta}\Big\{arg\big(\frac{\partial}{\partial\theta} f(re^{i\theta})\Big\} = Re\Big\{\frac{z\big(zh'(z)\big)' + \overline{z\big(zg'(z)\big)'}}{zh'(z) - \overline{zg'(z)}}\Big\} \geq \alpha,$$

where $0 \leq \theta \leq 2\pi, |z| = r < 1$.
As shown recently by Jahangiri [6] a sufficient condition for a function of the form (1) to be

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in $K_H(\alpha)$ is that

$$\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k|+\frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)\leq 2. \tag{2}$$

In 1985, Silvia[11] studied the partial sums of convex functions of order α. Later, Silverman [10], Abubaker and Darus[1], Dixit and Porwal[3], Frasin[4, 5], Raina and Bansal[8], Rosy et al.[9] and Porwal and Dixit[7] exhibited some results on partial sums for various classes of analytic functions. Here, we investigate a partial sums of convex harmonic functions.
Now, we let the sequences of partial sums of functions of the form (1) with $b_1=0$, have forms

$$f_m(z)=z+\sum_{k=2}^{m}a_kz^k+\sum_{k=2}^{\infty}\overline{b_kz^k},$$

$$f_n(z)=z+\sum_{k=2}^{\infty}a_kz^k+\sum_{k=2}^{n}\overline{b_kz^k},$$

$$f_{m,n}(z)=z+\sum_{k=2}^{m}a_kz^k+\sum_{k=2}^{n}\overline{b_kz^k}.$$

In the present paper, we determine sharp lower bounds for $Re\Big\{\frac{f}{f_m}\Big\}$, $Re\Big\{\frac{f_m}{f}\Big\}$, $Re\Big\{\frac{f'}{f'_m}\Big\}$, $Re\Big\{\frac{f'_m}{f'}\Big\}$,$Re\Big\{\frac{f}{f_n}\Big\}$, $Re\Big\{\frac{f_n}{f}\Big\}$, $Re\Big\{\frac{f}{f_{m,n}}\Big\}$, $Re\Big\{\frac{f_{m,n}}{f}\Big\}$, $Re\Big\{\frac{f'}{f'_{m,n}}\Big\}$ and $Re\Big\{\frac{f'_{m,n}}{f'}\Big\}$, where $f'(z)=\frac{\partial}{\partial\theta}f(re^{i\theta})=i\big(zh'(z)-\overline{zg'(z)}\big)$.

2. Main Results

Theorem 2.1. *If $f(z)$ of the form* (1) *with $b_1=0$ satisfies condition* (2)*, then*

$$Re\Big\{\frac{f(z)}{f_m(z)}\Big\}\geq\frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)},\ (z\in\mathbb{D}) \tag{3}$$

The result (3) *is sharp with the function*

$$f(z)=z+\frac{1-\alpha}{(m+1)(m+1-\alpha)}z^{m+1}. \tag{4}$$

Proof. To obtain sharp lower bound given by (3), let us put

$$\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\frac{f(z)}{f_m(z)}-\frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}\Big]=$$

$$\frac{1+\sum\limits_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k+\sum\limits_{k=2}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}+\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k\Big]}{1+\sum\limits_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k+\sum\limits_{k=2}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}}$$

$$:=\frac{1+A(z)}{1+B(z)}.$$

Set $\dfrac{1+A(z)}{1+B(z)} = \dfrac{1+\omega(z)}{1-\omega(z)}$, so that $\omega(z) = \dfrac{A(z)-B(z)}{2+A(z)+B(z)}$. Then

$$\omega(z) = \frac{\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k\Big]}{2+2\Big(\sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big) + \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k\Big)}$$

Hence

$$|\omega(z)| \le \frac{\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} |a_k|\Big]}{2-2\Big(\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{\infty} |b_k|\Big) - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} |a_k|\Big)}$$

The last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{m} |a_k| + \sum_{k=2}^{\infty} |b_k| + \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} |a_k|\Big) \le 1 \tag{5}$$

It suffices to show that the L. H. S. of (5) is bounded above by $\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, which is equivalent to

$$\sum_{k=2}^{m}\Big(\frac{k(k-\alpha)}{1-\alpha}-1\Big)|a_k| + \sum_{k=2}^{\infty}\Big(\frac{k(k+\alpha)}{1-\alpha}-1\Big)|b_k| + \sum_{k=m+1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha} - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big)|a_k| \ge 0.$$

To see $f(z) = z + \dfrac{1-\alpha}{(m+1)(m+1-\alpha)}z^{m+1}$ gives the sharp result, we observe that for $z = re^{i\frac{\pi}{m}}$ we have

$$\frac{f(z)}{f_m(z)} = 1 + \frac{1-\alpha}{(m+1)(m+1-\alpha)}z^m \longrightarrow 1 - \frac{1-\alpha}{(m+1)(m+1-\alpha)} = \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}$$

when $r \to 1^-$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *If $f(z)$ of the form (1) with $b_1 = 0$ satisfies condition (2), then*

$$Re\Big\{\frac{f_m(z)}{f(z)}\Big\} \ge \frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)}, \ (z \in \mathbb{D}) \tag{6}$$

The result (6) is sharp with the function given by (4).

Proof. We may write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\Big[\frac{f_m(z)}{f(z)} - \frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)}\Big] =$$

$$\frac{1+\sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k} - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k\Big)}{1+\sum_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}}$$

where

$$|\omega(z)| \leq \frac{\frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}|a_k|\Big]}{2-2\Big(\sum\limits_{k=2}^{m}|a_k|+\sum\limits_{k=2}^{\infty}|b_k|\Big)-\frac{m(m+2-\alpha)}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty}|a_k|\Big)} \leq 1.$$

Equivalently

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|+\frac{m(m+2-\alpha)+(1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k|\Big) \leq 1. \tag{7}$$

since the L. H. S. of (7) is bounded above by $\sum\limits_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k|+\frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, the proof is complete. □

Theorem 2.3. *If $f(z)$ of the form* (1) *with $b_1 = 0$ satisfies condition* (2)*, then*

$$Re\Big\{\frac{f'(z)}{f'_m(z)}\Big\} \geq \frac{m}{m+1-\alpha}, \quad (z \in \mathbb{D}) \tag{8}$$

The result (8) *is sharp with the function given by* (4)*.*

Proof. We have

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+1-\alpha}{1-\alpha}\Big[\frac{f'(z)}{f'_m(z)}-\frac{m}{m+1-\alpha}\Big]$$

$$= \frac{1+\sum\limits_{k=2}^{m}kr^{k-1}e^{i(k-1)\theta}a_k-\sum\limits_{k=2}^{\infty}kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}+\frac{m+1-\alpha}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}kr^{k-1}e^{i(k-1)\theta}a_k\Big]}{1+\sum\limits_{k=2}^{m}kr^{k-1}e^{i(k-1)\theta}a_k-\sum\limits_{k=2}^{\infty}kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}}.$$

Then

$$\omega(z) = \frac{\frac{m+1-\alpha}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}kr^{k-1}e^{i(k-1)\theta}a_k\Big]}{2+2\Big(\sum\limits_{k=2}^{m}kr^{k-1}e^{i(k-1)\theta}a_k-\sum\limits_{k=2}^{\infty}kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big)+\frac{m+1-\alpha}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty}kr^{k-1}e^{i(k-1)\theta}a_k\Big)}$$

In a similar fashion as in Theorem 2.1. the proof is complete. □

Theorem 2.4. *If $f(z)$ of the form* (1) *with $b_1 = 0$ satisfies condition* (2)*, then*

$$Re\Big\{\frac{f'_m(z)}{f'(z)}\Big\} \geq \frac{m+1-\alpha}{m+2(1-\alpha)}, \quad (z \in \mathbb{D}) \tag{9}$$

The result (9) *is sharp with the function given by* (4)*.*

Proof. Since

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+2(1-\alpha)}{1-\alpha}\Big[\frac{f'_m(z)}{f'(z)}-\frac{m+1-\alpha}{m+2(1-\alpha)}\Big],$$

proceeding exactly as in the proof of Theorem 2.3, we evidently have the required result. □

Theorem 2.5. *If $f(z)$ of the form* (1) *with $b_1 = 0$ satisfies condition* (2)*, then*

$$Re\Big\{\frac{f(z)}{f_n(z)}\Big\} \geq \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}, \quad (z \in \mathbb{D}) \tag{10}$$

The result (10) *is sharp with the function*

$$f(z) = z + \frac{1-\alpha}{(n+1)(n+1+\alpha)}\overline{z}^{n+1}. \tag{11}$$

Proof. Write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\frac{f(z)}{f_n(z)} - \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}\Big] =$$
$$\frac{1+\sum\limits_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k} + \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum\limits_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big]}{1+\sum\limits_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}}.$$

where

$$\omega(z) = \frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum\limits_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big]}{2+2\Big(\sum\limits_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big) + \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big(\sum\limits_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big)}.$$

Then

$$|\omega(z)| \le \frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum\limits_{k=n+1}^{\infty}|b_k|\Big]}{2-2\Big(\sum\limits_{k=2}^{\infty}|a_k| + \sum\limits_{k=2}^{n}|b_k|\Big) - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big(\sum\limits_{k=n+1}^{\infty}|b_k|\Big)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{\infty}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big(\sum_{k=n+1}^{\infty}|b_k|\Big) \le 1. \tag{12}$$

It suffices to show that the L. H. S. of (12) is bounded above by $\sum\limits_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, which is equivalent to

$$\sum_{k=2}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}-1\Big)|a_k| + \sum_{k=2}^{n}\Big(\frac{k(k+\alpha)}{1-\alpha}-1\Big)|b_k|$$
$$+\sum_{k=n+1}^{\infty}\Big(\frac{k(k+\alpha)}{1-\alpha} - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big)|b_k| \ge 0.$$

To see that $f(z) = z + \frac{1-\alpha}{(n+1)(n+1+\alpha)}\overline{z}^{n+1}$ gives the sharp result, we observe that for $z = re^{i\frac{\pi}{n+2}}$ one obtains

$$\frac{f(z)}{f_n(z)} = 1 + \frac{1-\alpha}{(n+1)(n+1+\alpha)}r^n e^{-\frac{i\pi}{n+2}(n+2)} \longrightarrow 1 - \frac{1-\alpha}{(n+1)(n+1+\alpha)} = \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}$$

when $r \to 1^-$. This completes the proof. □

Theorem 2.6. *If* $f(z)$ *of the form* (1) *with* $b_1 = 0$ *satisfies condition* (2), *then*

$$Re\Big\{\frac{f_n(z)}{f(z)}\Big\} \ge \frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2}, \ (z \in \mathbb{D}) \tag{13}$$

The result (13) *is sharp with the function given by* (11).

Proof. It is easy to see that

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{n(n+2+\alpha)+2}{1-\alpha}\Big[\frac{f_n(z)}{f(z)} - \frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2}\Big] = \frac{1+\sum\limits_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k} - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum\limits_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big]}{1+\sum\limits_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}}.$$

Rest of the proof is omitted since it runs parallel to that from Theorem 2.2. □

Theorem 2.7. *If $f(z)$ of the form* (1) *with $b_1 = 0$ satisfies condition* (2)*, then*
(i) $Re\Big\{\frac{f(z)}{f_{m,n}(z)}\Big\} \geq \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}$, $(z \in \mathbb{D})$ *if* $n(n+2+\alpha)+2\alpha \geq m(m+2-\alpha)$ *or* $b_k = 0\ \forall k \geq 2$.
(ii) $Re\Big\{\frac{f(z)}{f_{m,n}(z)}\Big\} \geq \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}$, $(z \in \mathbb{D})$ *if* $n(n+2+\alpha)+2\alpha \leq m(m+2-\alpha)$ *or* $a_k = 0\ \forall k \geq 2$.

Proof. To prove (i), we may write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\frac{f(z)}{f_{m,n}(z)} - \frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}\Big] = \frac{P}{1+\sum\limits_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}},$$

where

$$P = 1+\sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k} + \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big].$$

So that

$$\omega(z) = \frac{\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big]}{Q},$$

where

$$Q = 2+2\Big(\sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big) + \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big).$$

Then

$$|\omega(z)| \leq \frac{\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big]}{2-2\Big(\sum\limits_{k=2}^{m}|a_k| + \sum\limits_{k=2}^{n}|b_k|\Big) - \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|+\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\Big)\leq 1. \tag{14}$$

Since the L. H. S. of (14) is bounded above by $\sum_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k|+\frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, it yields the following inequality

$$\sum_{k=2}^{m}\Big(\frac{k(k-\alpha)}{1-\alpha}-1\Big)|a_k|+\sum_{k=m+1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}-\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big)|a_k|$$
$$+\sum_{k=2}^{n}\Big(\frac{k(k+\alpha)}{1-\alpha}-1\Big)|b_k|+\sum_{k=n+1}^{\infty}\Big(\frac{k(k+\alpha)}{1-\alpha}-\frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big)|b_k|\geq 0.$$

To see $f(z)=z+\frac{1-\alpha}{(m+1)(m+1-\alpha)}z^{m+1}$ gives the sharp result, we observe that for $z=re^{i\frac{\pi}{m}}$ that

$$\frac{f(z)}{f_{m,n}(z)}=1+\frac{1-\alpha}{(m+1)(m+1-\alpha)}z^m\longrightarrow 1-\frac{1-\alpha}{(m+1)(m+1-\alpha)}=\frac{m(m+2-\alpha)}{(m+1)(m+1-\alpha)}$$

when $r\to 1^-$.
To prove (ii), note that

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\frac{f(z)}{f_{m,n}(z)}-\frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}\Big]$$
$$=\frac{P}{1+\sum_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{n}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}}.$$

where

$$P=1+\sum_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{n}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}$$
$$+\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=n+1}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big].$$

So that

$$\omega(z)=\frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=n+1}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big]}{Q},$$

where

$$Q=2+2\Big(\sum_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=2}^{n}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big)$$
$$+\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k+\sum_{k=n+1}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big).$$

Consequently

$$|\omega(z)| \le \frac{\frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big]}{2-2\Big(\sum\limits_{k=2}^{m}|a_k| + \sum\limits_{k=2}^{n}|b_k|\Big) - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big)}.$$

This last expression is bounded above by 1, if and only if

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\Big) \le 1. \tag{15}$$

It suffices to show that the L. H. S. of (15) is bounded above by $\sum\limits_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, which is equivalent to

$$\sum_{k=2}^{m}\Big(\frac{k(k-\alpha)}{1-\alpha}-1\Big)|a_k| + \sum_{k=m+1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha} - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big)|a_k|$$
$$+\sum_{k=2}^{n}\Big(\frac{k(k+\alpha)}{1-\alpha}-1\Big)|b_k| + \sum_{k=n+1}^{\infty}\Big(\frac{k(k+\alpha)}{1-\alpha} - \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big)|b_k| \ge 0.$$

To see $f(z) = z + \frac{1-\alpha}{(n+1)(n+1+\alpha)}\overline{z}^{n+1}$ gives the sharp result, we observe that for $z = re^{i\frac{\pi}{n+2}}$ we get

$$\frac{f(z)}{f_{m,n}(z)} = 1 + \frac{1-\alpha}{(n+1)(n+1+\alpha)}r^n e^{-\frac{i\pi}{n+2}(n+2)} \longrightarrow 1 - \frac{1-\alpha}{(n+1)(n+1+\alpha)} = \frac{n(n+2+\alpha)}{(n+1)(n+1+\alpha)}$$

when $r \to 1^-$. The result follows. ∎

Theorem 2.8. *If $f(z)$ of the form (1) with $b_1 = 0$ satisfies condition (2), then*

(i) $Re\Big\{\frac{f_{m,n}(z)}{f(z)}\Big\} \ge \frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)}, (z \in \mathbb{D})$ *if* $n(n+2+\alpha)+2\alpha \ge m(m+2-\alpha)$ *or* $b_k = 0\ \forall k \ge 2$.

(ii) $Re\Big\{\frac{f_{m,n}(z)}{f(z)}\Big\} \ge \frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2}, (z \in \mathbb{D})$ *if* $n(n+2+\alpha)+2\alpha \le m(m+2-\alpha)$ *or* $a_k = 0\ \forall k \ge 2$.

Proof. To prove (i), we may write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\Big[\frac{f_{m,n}(z)}{f(z)} - \frac{(m+1)(m+1-\alpha)}{m(m+2-\alpha)+2(1-\alpha)}\Big]$$
$$= \frac{P}{1+\sum\limits_{k=2}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}},$$

where

$$P = 1 + \sum_{k=2}^{m}r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}$$
$$- \frac{(m+1)(m+1-\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty}r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=n+1}^{\infty}r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big].$$

Then

$$|\omega(z)| \le \frac{\frac{m(m+2-\alpha)+2(1-\alpha)}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big]}{2-2\Big(\sum\limits_{k=2}^{m}|a_k| + \sum\limits_{k=2}^{n}|b_k|\Big) - \frac{m(m+2-\alpha)}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big)} \le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{m(m+2-\alpha)+(1-\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\Big) \le 1. \tag{16}$$

Sufficiently, the L. H. S. of (16) is bounded above by $\sum\limits_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, the proof is complete.

To prove (ii), we consider that

$$\begin{aligned}\frac{1+\omega(z)}{1-\omega(z)} &= \frac{n(n+2+\alpha)+2}{1-\alpha}\Big[\frac{f_{m,n}(z)}{f(z)} - \frac{(n+1)(n+1+\alpha)}{n(n+2+\alpha)+2}\Big]\\ &= \frac{P}{1+\sum\limits_{k=2}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum\limits_{k=2}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}},\end{aligned}$$

where

$$\begin{aligned}P = 1 &+ \sum_{k=2}^{m} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=2}^{n} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\\ &- \frac{(n+1)(n+1+\alpha)}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} r^{k-1}e^{i(k-1)\theta}a_k + \sum_{k=n+1}^{\infty} r^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big].\end{aligned}$$

Then

$$|\omega(z)| \le \frac{\frac{n(n+2+\alpha)+2}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big]}{2-2\Big(\sum\limits_{k=2}^{m}|a_k| + \sum\limits_{k=2}^{n}|b_k|\Big) - \frac{n(n+2+\alpha)+2\alpha}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty}|a_k| + \sum\limits_{k=n+1}^{\infty}|b_k|\Big)} \le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| + \frac{n(n+2+\alpha)+(1+\alpha)}{1-\alpha}\Big(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\Big) \le 1. \tag{17}$$

Sufficiently, the L. H. S. of (17) is bounded above by $\sum\limits_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, which completes the proof. □

Theorem 2.9. *If $f(z)$ of the form* (1) *with $b_1 = 0$ satisfies condition* (2), *then*

$$Re\Big\{\frac{f'(z)}{f'_{m,n}(z)}\Big\} \ge \frac{m}{m+1-\alpha}, \ (z \in \mathbb{D}) \ if n > m \tag{18}$$

The result (18) *is sharp with the function given by* (4).

Proof. Consider

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{m+1-\alpha}{1-\alpha}\Big[\frac{f'(z)}{f'_{m,n}(z)} - \frac{m}{m+1-\alpha}\Big]$$

$$= \frac{P}{1+\sum\limits_{k=2}^{m} kr^{k-1}e^{i(k-1)\theta}a_k - \sum\limits_{k=2}^{n} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}},$$

where

$$P = 1 + \sum_{k=2}^{m} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=2}^{n} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}$$
$$+ \frac{m+1-\alpha}{1-\alpha}\Big[\sum_{k=m+1}^{\infty} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=n+1}^{\infty} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big].$$

Then

$$\omega(z) = \frac{\frac{m+1-\alpha}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty} kr^{k-1}e^{i(k-1)\theta}a_k - \sum\limits_{k=n+1}^{\infty} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big]}{Q},$$

where

$$Q = 2 + 2\Big(\sum_{k=2}^{m} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=2}^{n} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big)$$
$$+ \frac{m+1-\alpha}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} kr^{k-1}e^{i(k-1)\theta}a_k - \sum_{k=n+1}^{\infty} kr^{k-1}e^{-i(k+1)\theta}\overline{b_k}\Big).$$

Consequently, we get

$$|\omega(z)| \le \frac{\frac{m+1-\alpha}{1-\alpha}\Big[\sum\limits_{k=m+1}^{\infty} k|a_k| - \sum\limits_{k=n+1}^{\infty} k|b_k|\Big]}{2 - 2\Big(\sum\limits_{k=2}^{m} k|a_k| + \sum\limits_{k=2}^{n} k|b_k|\Big) - \frac{m+1-\alpha}{1-\alpha}\Big(\sum\limits_{k=m+1}^{\infty} k|a_k| + \sum\limits_{k=n+1}^{\infty} k|b_k|\Big)} \le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{m} k|a_k| + \sum_{k=2}^{n} k|b_k| + \frac{m+1-\alpha}{1-\alpha}\Big(\sum_{k=m+1}^{\infty} k|a_k| + \sum_{k=n+1}^{\infty} k|b_k|\Big) \le 1. \quad (19)$$

Since the L. H. S. of (19) is bounded above by $\sum\limits_{k=1}^{\infty}\Big(\frac{k(k-\alpha)}{1-\alpha}|a_k| + \frac{k(k+\alpha)}{1-\alpha}|b_k|\Big)$, the proof is complete. □

Theorem 2.10. *If f of the form* (1) *with $b_1 = 0$ satisfies condition* (2)*, then*

$$Re\Big\{\frac{f'_{m,n}(z)}{f'(z)}\Big\} \ge \frac{m+1-\alpha}{m+2(1-\alpha)}, \ (z \in \mathbb{D}) \quad (20)$$

The result (20) *is sharp with the function* $f(z) = z + \frac{1-\alpha}{(m+1)(m+1-\alpha)}z^{m+1}$.

Proof. Proceeding exactly as in the proof of Theorem 2.9, we evidently have the required result. □

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