TWMS J. App. Eng. Math. V.9, N.4, 2019, pp. 693-703

ON SUPER (a, d)-EAT VALUATION OF SUBDIVIDED CATERPILLAR

A. RAHEEM[1], M. JAVAID[2], M. A. UMAR[3], G. C. LAU[4], §

ABSTRACT. Let $G = (V(G), E(G))$ be a graph with $v = |V(G)|$ vertices and $e = |E(G)|$ edges. A bijective function $\lambda : V(G) \cup E(G) \leftrightarrow \{1, 2, \ldots, v + e\}$ is called an (a, d)-edge antimagic total (EAT) labeling(valuation) if the weight of all the edges $\{w(xy) : xy \in E(G)\}$ form an arithmetic sequence starting with first term a and having common difference d, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$. And, if $\lambda(V) = \{1, 2, \ldots, v\}$ then G is super (a, d)-edge antimagic total(EAT) graph. In this paper, we determine the super (a,d)-edge antimagic total (EAT) labeling of the subdivided caterpillar for different values of the parameter d.

Keywords: caterpillar, subdivided caterpillar, super(a, d)-EAT graph.

AMS Subject Classification: 05C78

1. Introduction and Preliminaries

Throughout in this paper, all graphs are simple, finite, and undirected. The graph G has the vertex-set $V(G)$ and edge-set $E(G)$. A general reference for graph-theoretic ideas can be consult[23]. A labeling (or valuation) of a graph is a mapping that carries graph elements to positive numbers. In this paper the domain will be the set of all vertices and edges and such a labeling is called a total labeling. Some labeling use the vertex-set only, or the edge-set only, and we shall call them vertex-labelings and edge-labelings respectively. A number of classification studies on edge antimagic total graphs has been intensively investigated. For further detail study on the antimagic labeling [13] a dynamic survey of graph labeling. The subject of edge-magic total labeling of graphs has its origin in the work of Kotzig and Rosa [16, 17], on what they called magic valuations of graphs. The notion of super edge-magic total labeling was introduced by Enomoto et al. [8] and they proposed following conjecture:

[1] Department of Mathematics, National University of Singapore, Singapore, 119076.
e-mail: rahimciit7@gmail.com; ORCID: https//orcid.org/0000-0001-8159-520X.
[2] Department of Mathematics, School of Science, University of Management and Technology, Pakistan.
e-mail: javaidmath@gmail.com; ORCID: https//orcid.org/0000-0001-7241-8172.
[3] Department of Mathematics, Govenment Degree Boys College (B), Sharqpur Shareef, Pakistan.
e-mail: owais054@gmail.com; ORCID: https//orcid.org/0000-0002-9063-1919.
[4] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Johor, Segamat, Malaysia.
e-mail: geeclau@yahoo.com; ORCID: https//orcid.org/0000-0002-9777-6571.
§ Manuscript received: August 23, 2017; accepted: May 3, 2018.

Conjecture: Every tree admits a super edge-magic total labeling.

In the support of this conjecture, many authors have considered super edge-magic total labeling for some particular classes of trees for example [3, 5, 10, 12, 14, 15, 20, 19]. However, this conjecture still remains open. Lee and Shah [18] have verified this conjecture for trees on at most 17 vertices with a computer help. Kotzig and Rosa [16] proved that every caterpillar is super edge-magic total. Sugeng et al.[22] proved some results related to super (a, d)-edge antimagic total labeling of stars and caterpillars for different values of the parameter d. Baca et al. [5] proved that disjoint union of caterpillars also admits super (a, d)-edge antimagic total labeling. Baca et al. [4] presented that if a tree with order greater or equal to 2 is super (a, d)-edge antimagic total then d must be less or equal to 3. In the present paper we find the super (a, d)-edge antimagic total labeling on subdivided caterpillar for $d = \{0, 1, 2\}$.

A graph G is called (a, d)-edge antimagic total ((a, d)-EAT) if there exist integers $a > 0, d \geq 0$ and a bijective mapping $\lambda : V(G) \cup E(G) \leftrightarrow \{1, 2, \ldots, v + e\}$ such that $W = \{w(xy) : xy \in E(G)\}$ forms an arithmetic sequence starting from a with common difference d, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$. W is called the set of edge-weights of the graph G. And, if $\lambda(V(G)) = \{1, 2, \ldots, v\}$ then G is super (a, d)-edge antimagic total graph.

In a caterpillar, if we subdivide the end edges then the resulting graph is called a subdivided caterpillar. It is denoted by $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n : n, l)$, where $\alpha_1 = (m_{1,1}, m_{1,2}, m_{1,3}, \ldots, m_{1,l}), \alpha_2 = (m_{2,1}, m_{2,2}, m_{2,3}, \ldots, m_{2,l}), \ldots, \alpha_n = (m_{n,1}, m_{n,2}, m_{n,3}, \ldots, m_{n,l})$.
The vertex-set and edge-set are defined as follow:

$$V(G) = \{c_i : 1 \leq i \leq n\} \cup \{a_{i,r}^{p_{i,r}} : 1 \leq i \leq n, 1 \leq p_{i,r} \leq m_{i,r}, 1 \leq r \leq l\}$$

and

$$E(G) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{a_{i,r}^{p_{i,r}} a_{i,r}^{p_{i,r+1}} : 1 \leq i \leq n, 1 \leq p_{i,r} \leq m_{i,r} - 1, 1 \leq r \leq l\}$$
$$\{a_{i,r}^{1} c_i : 1 \leq i \leq n, 1 \leq r \leq l\}$$

□

2. Main Results

Let us consider the following important Proposition that gives a necessary and sufficient condition for a graph to be super (a, d)-EAT labeling.

Proposition 2.1. [4] If a (v, e)-graph G has a (s, d)-EAV labeling then

(i) G has a super $(s + v + 1, d + 1)$-EAT labeling,
(ii) G has a super $(s + v + e, d - 1)$-EAT labeling.

□

Theorem 2.1. The graph $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n : n, 5)$ is a super $(a, 0)$-EAT labeling with $a = 2v + s - 1$ and super $(a, 2)$-EAT labeling with $a = v + s + 1$, where $m \geq 3$ and $m \equiv 1(\text{mod } 2)$, $n \geq 2, l = 5, \alpha_1 = (m, m, m, m, 2m)$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_n = (2m, 2m-1, m-1, m, 2m)$, $s = (3m+2) + (4m-1)\lfloor \frac{n}{2} \rfloor + 4m(\lceil \frac{n}{2} \rceil - 1) + 2$ and $v = |V(G)|$.

Proof. Let us denote $v = |V(G)|$ and $e = |E(G)|$ then $v = 8mn - 2m - n + 2$ and $e = v - 1$. The vertex-set and edge-set of the graph G as following:

$$V(G) = \{c_i : 1 \le i \le n\} \cup \{a_{i,r}^{p_{i,r}} : 1 \le i \le n, 1 \le p_{ir} \le m_{ir}, 1 \le r \le 5\}$$

$$E(G) = \{c_i c_{i+1} : 1 \le i \le n-1\} \cup \{a_{ir}^{p_{ir}} a_{ir}^{p_{ir}+1} : 1 \le i \le n, 1 \le pir \le m_{ir}-1, 1 \le r \le 5\}$$
$$\{a_{ir}^1 c_i : 1 \le i \le n, 1 \le r \le 5\}$$

Now, we define the labeling $\lambda : V \to \{1, 2, \ldots, v\}$ as follows:
Throughout the labeling we will consider
$\alpha = 8m - 1$ and
$\eta = (3m+2) + (4m-1)\lfloor \frac{n}{2} \rfloor + 4m(\lceil \frac{n}{2} \rceil - 1)$

$$\lambda(c_i) = \begin{cases} \eta + m & \text{for } i = 1 \\ \eta + \frac{\alpha}{2}(i-3) + (9m-1) & \text{for } i \ge 3, \text{odd} \\ \frac{\alpha}{2}(i-2) + (5m+2) & \text{for } i = \text{even} \end{cases}$$

When $i = 1$ and $1 \le r \le 5$
for $p_{1,r} = 1, 3, 5, \ldots, m_{1,r}$

$$\lambda(u) = \begin{cases} \frac{p_{1,1}+1}{2} & \text{for } u = a_{11}^{p_{1,1}}, \\ (m+2) - \frac{p_{1,2}+1}{2} & \text{for } u = a_{1,2}^{p_{1,2}}, \\ (m+1) + \frac{p_{1,3}+1}{2} & \text{for } u = a_{1,3}^{p_{1,3}}, \\ (2m+3) - \frac{p_{1,4}+1}{2} & \text{for } u = a_{1,4}^{p_{1,4}}, \\ 3(m+1) - \frac{p_{1,5}+1}{2} & \text{for } u = a_{1,5}^{p_{1,5}}, \end{cases}$$

and for $p_{1r} = 2, 4, 6, \ldots, m_{1r} - 1$;

$$\lambda(u) = \begin{cases} \eta + \frac{p_{1,1}}{2} & \text{for } u = a_{1,1}^{p_{1,1}}, \\ \eta + m - \frac{p_{1,2}}{2} & \text{for } u = a_{1,2}^{p_{1,2}}, \\ \eta + m + \frac{p_{1,3}}{2} & \text{for } u = a_{1,3}^{p_{1,3}}, \\ \eta + 2m - \frac{p_{1,4}}{2} & \text{for } u = a_{1,4}^{p_{1,4}}, \\ \eta + (3m-1) - \frac{p_{1,5}}{2} & \text{for } u = a_{1,5}^{p_{1,5}}, \end{cases}$$

When $i =$ even and $1 \le r \le 5$:
For $p_{i,r} = 1, 3, 5, \ldots, m_{ir}$

$$\lambda(u) = \begin{cases} \eta + \alpha\left(\frac{i-2}{2}\right) + (3m-1) + \frac{p_{i,1}+1}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + 5m - \frac{p_{i,2}+1}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + (5m-1) + \frac{p_{i,3}+1}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + 6m - \frac{p_{i,4}+1}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + 7m - \frac{p_{i,5}+1}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \end{cases}$$

and for $p_{i,r} = 2, 4, 6, \ldots, m_{i,r} - 1$

$$\lambda(u) = \begin{cases} \alpha\left(\frac{i-2}{2}\right) + (3m+2) + \frac{p_{i,1}}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \alpha\left(\frac{i-2}{2}\right) + (5m+2) - \frac{p_{i,2}}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \alpha\left(\frac{i-2}{2}\right) + (5m+2) + \frac{p_{i,3}}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \alpha\left(\frac{i-2}{2}\right) + (6m+2) - \frac{p_{i,4}}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \alpha\left(\frac{i-2}{2}\right) + (7m+2) - \frac{p_{i,5}}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \end{cases}$$

When $i \geq 3$ and odd and $1 \leq r \leq 5$
For $p_{i,r} = 1, 3, 5, \dots, m_{i,r}$

$$\lambda(u) = \begin{cases} \alpha\left(\frac{i-3}{2}\right) + (7m+1) + \frac{p_{i,1}+1}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \alpha\left(\frac{i-3}{2}\right) + (9m+2) - \frac{p_{i,2}+1}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \alpha\left(\frac{i-3}{2}\right) + (9m+1) + \frac{p_{i,3}+1}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \alpha\left(\frac{i-3}{2}\right) + (10m+2) - \frac{p_{i,4}+1}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \alpha\left(\frac{i-3}{2}\right) + (11m+2) - \frac{p_{i,5}+1}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \end{cases}$$

and for $p_{i,r} = 2, 4, 6, \dots, m_{i,r} - 1$

$$\lambda(u) = \begin{cases} \eta + \alpha\left(\frac{i-3}{2}\right) + (7m-1) + \frac{p_{i,1}}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (9m-1) - \frac{p_{i,2}}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (9m-1) + \frac{p_{i,3}}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (10m-1) - \frac{p_{i,4}}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (11m-1) - \frac{p_{i,5}}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \end{cases}$$

The set of all edge-sums generated by the above scheme of labeling forms a consecutive integer sequence $s = (\eta+1)+1, (\eta+1)+2, \dots, (\eta+1)+e$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 0)$-EAT labeling and obtain the magic constant $a = 2v + s - 1 = \eta + 16mn - 4m - 2n + 5$. Similarly, by the Proposition 2.1, λ can be extended to a super $(a, 2)$-EAT labeling and obtain the magic constant $a = v + 1 + s = \eta + 8mn - 2m - n + 5$. □

Theorem 2.2. The graph $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n : n, 5)$ is a super $(a, 1)$-EAT labeling with $a = s + \frac{3}{2}v$ if v is even, where $m \geq 3$ and $m \equiv 1(\text{mod } 2)$, $n \geq 2, l = 5, \alpha_1 = (m, m, m, m, 2m)$ and $\alpha_2 = \alpha_3 = \dots = \alpha_n = (2m, 2m-1, m-1, m, 2m-1)$, $s = (3m+2) + (4m-1)\lfloor \frac{n}{2} \rfloor + 4m(\lceil \frac{n}{2} \rceil - 1) + 2$ and $v = |V(G)|$.

Proof. Let us suppose $v = |V(G)|$ and $e = |E(G)|$ then $v = 8mn - 2m - n + 2$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{c_i : 1 \leq i \leq n\} \cup \{a_{i,r}^{p_{i,r}} : 1 \leq i \leq n, 1 \leq p_{i,r} \leq m_{i,r}, 1 \leq r \leq 5\}$$

$$E(G) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{a_{ir}^{p_{ir}} a_{ir}^{p_{ir+1}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir} - 1, 1 \leq r \leq 5\}$$
$$\{a_{ir}^1 c_i : 1 \leq i \leq n, 1 \leq r \leq 5\}$$

Now we define the labeling $\lambda : V(G) \cup E(G) \to \{1, 2, \dots, v+e\}$ as in theorem 2.1.

It follows that the edge-weights of all edges of G constitute an arithmetic sequence $s = (\eta+1)+1, (\eta+1)+2, \dots, (\eta+1)+e$, with common difference 1. We denote it by $A = \{a_i : 1 \leq i \leq e\}$. Now for G we complete the edge labeling λ for super $(a, 1)$-edge antimagic total labeling with values in the arithmetic sequence $v+1, v+2, \dots, v+e$ with common difference 1. Let us denote it by $B = \{b_j : 1 \leq j \leq e\}$. Define $C = \{a_{2i-1} + b_{e-i+1} : 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2} - 1\}$. It is easy to see that C constitute an arithmetic sequence with $d = 1$ and $a = s + \frac{3}{2}v = \eta + 2 + \frac{3}{2}(8mn - 2m - n + 2)$. Since all vertices receive the smallest labels so λ is a super $(a, 1)$-edge antimagic total labeling. □

Theorem 2.3. The graph $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n : n, 5)$ is a super $(a, 0)$-EAT labeling with $a = 2v + s - 1$ and super $(a, 2)$-EAT labeling with $a = v + s + 1$, where $m \geq 3$

and $m \equiv 1(\text{mod } 2)$, $n \geq 2, l = 5, \alpha_1 = (m, m, m, m, 2m, 4m)$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_n = (2m, 2m-1, m-1, m, 2m, 4m)$, $s = (5m+2)+(8m-1)\lfloor \frac{n}{2} \rfloor + 8m(\lceil \frac{n}{2} \rceil - 1) + 2$ and $v = |V(G)|$.

Proof. Let us suppose $v = |V(G)|$ and $e = |E(G)|$ then $v = 16mn - 6m - n + 2$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{c_i : 1 \leq i \leq n\} \cup \{a_{ir}^{p_{ir}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir}, 1 \leq r \leq 5\}$$

$$E(G) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{a_{ir}^{p_{ir}} a_{ir}^{p_{ir+1}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir} - 1, 1 \leq r \leq 5\}$$
$$\{a_{ir}^{1} c_i : 1 \leq i \leq n, 1 \leq r \leq 5\}$$

Now, we define the labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows:
$\alpha = 16m - 1$ and
$\eta = (5m + 2) + (8m - 1)\lfloor \frac{n}{2} \rfloor + 8m(\lceil \frac{n}{2} \rceil - 1)$

$$\lambda(c_i) = \begin{cases} \eta + m & \text{for } i = 1 \\ \eta + \frac{\alpha}{2}(i-3) + (17m-1) & \text{for } i \geq 3, \text{odd} \\ \frac{\alpha}{2}(i-2) + (9m+2) & \text{for } i = \text{even} \end{cases}$$

When $i = 1$ and $1 \leq r \leq 6$
for $p_{1,r} = 1, 3, 5, \ldots, m_{1,r}$

$$\lambda(u) = \begin{cases} \frac{p_{1,1}+1}{2} & \text{for } u = a_{1,1}^{p_{1,1}}, \\ (m+2) - \frac{p_{1,2}+1}{2} & \text{for } u = a_{1,2}^{p_{1,2}}, \\ (m+1) + \frac{p_{1,3}+1}{2} & \text{for } u = a_{1,3}^{p_{1,3}}, \\ (2m+3) - \frac{p_{1,4}+1}{2} & \text{for } u = a_{1,4}^{p_{1,4}}, \\ 3(m+1) - \frac{p_{1,5}+1}{2} & \text{for } u = a_{1,5}^{p_{1,5}}, \\ (5m+3) - \frac{p_{1,6}+1}{2} & \text{for } u = a_{1,6}^{p_{1,6}}, \end{cases}$$

and for $p_{1,r} = 2, 4, 6, \ldots, m_{1,r} - 1$;

$$\lambda(u) = \begin{cases} \eta + \frac{p_{1,1}}{2} & \text{for } u = a_{1,1}^{p_{1,1}}, \\ \eta + m - \frac{p_{1,2}}{2} & \text{for } u = a_{1,2}^{p_{1,2}}, \\ \eta + m + \frac{p_{1,3}}{2} & \text{for } u = a_{1,3}^{p_{1,3}}, \\ \eta + 2m - \frac{p_{1,4}}{2} & \text{for } u = a_{1,4}^{p_{1,4}}, \\ \eta + (3m-1) - \frac{p_{1,5}}{2} & \text{for } u = a_{1,5}^{p_{1,5}}, \\ \eta + (5m-1) - \frac{p_{1,6}}{2} & \text{for } u = a_{1,6}^{p_{1,6}}, \end{cases}$$

When $i =$ even and $1 \leq r \leq 6$
for $p_{i,r} = 1, 3, 5, \ldots, m_{i,r}$:

$$\lambda(u) = \begin{cases} \eta + \alpha\left(\frac{i-2}{2}\right) + (5m-1) + \frac{p_{i,1}+1}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + 9m - \frac{p_{i,2}+1}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (9m-1) + \frac{p_{i,3}+1}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + 10m - \frac{p_{i,4}+1}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + 11m - \frac{p_{i,5}+1}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \\ \eta + \alpha\left(\frac{i-2}{2}\right) + 13m - \frac{p_{i,6}+1}{2} & \text{for } u = a_{1,6}^{p_{i,6}}, \end{cases}$$

and for $p_{i,r} = 2, 4, 6, \ldots, m_{i,r} - 1$

$$\lambda(u) = \begin{cases} \alpha\left(\frac{i-2}{2}\right) + (5m+2) + \frac{p_{i,1}}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \alpha\left(\frac{i-2}{2}\right) + (9m+2) - \frac{p_{i,2}}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \alpha\left(\frac{i-2}{2}\right) + (9m+2) + \frac{p_{i,3}}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \alpha\left(\frac{i-2}{2}\right) + (10m+2) - \frac{p_{i,4}}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \alpha\left(\frac{i-2}{2}\right) + (11m+2) - \frac{p_{i,5}}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \\ \alpha\left(\frac{i-2}{2}\right) + (13m+2) - \frac{p_{i,6}}{2} & \text{for } u = a_{1,6}^{p_{i,6}}, \end{cases}$$

When $i \geq 3$ odd $1 \leq r \leq 6$: and for $p_{i,r} = 1, 3, 5, \ldots, m_{i,r}$

$$\lambda(u) = \begin{cases} \alpha\left(\frac{i-3}{2}\right) + (13m+1) + \frac{p_{i,1}+1}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \alpha\left(\frac{i-3}{2}\right) + (17m+2) - \frac{p_{i,2}+1}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \alpha\left(\frac{i-3}{2}\right) + (17m+1) + \frac{p_{i,3}+1}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \alpha\left(\frac{i-3}{2}\right) + (18m+2) - \frac{p_{i,4}+1}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \alpha\left(\frac{i-3}{2}\right) + (19m+2) - \frac{p_{i,5}+1}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \\ \alpha\left(\frac{i-3}{2}\right) + (21m+2) - \frac{p_{i,6}+1}{2} & \text{for } u = a_{1,6}^{p_{i,6}}, \end{cases}$$

and for $p_{i,r} = 2, 4, 6, \ldots, m_{i,r} - 1$

$$\lambda(u) = \begin{cases} \eta + \alpha\left(\frac{i-3}{2}\right) + (13m-1) + \frac{p_{i,1}}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (17m-1) - \frac{p_{i,2}}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (17m-1) + \frac{p_{i,3}}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + 18m - \frac{p_{i,4}}{2} & \text{for } u = a_{1,4}^{p_{i,4}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (19m-1) - \frac{p_{i,5}}{2} & \text{for } u = a_{1,5}^{p_{i,5}}, \\ \eta + \alpha\left(\frac{i-3}{2}\right) + (21m-1) - \frac{p_{i,6}}{2} & \text{for } u = a_{1,6}^{p_{i,6}}, \end{cases}$$

The set of all edge-sums generated by the above labeling scheme forms a consecutive integer sequence $s = (\eta+1)+1; (\eta+1)+2, \ldots, (\eta+1)+e$. Therefore, by Proposition 2.1, λ can be extended to a super $(a,0)$-EAT labeling and we obtain the magic constant $a = 2v + s - 1 = \eta + 32mn - 12m - 2n + 5$. Similarly, by Proposition 2.1, λ can be extended to a super $(a,2)$-EAT labeling and we obtain the magic constant $a = v+1+s = \eta + 16mn - 6m - n + 5$. □

Theorem 2.4. The graph $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n : n, 5)$ is a super $(a,1)$-EAT labeling with $a = s + \frac{3}{2}v$ if v is even, where $m \geq 3$ and $m \equiv 1(\text{mod } 2)$, $n \geq 2, l = 5, \alpha_1 = (m, m, m, m, 2m, 4m)$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_n = (2m, 2m-1, m-1, m, 2m, 4m)$, $s = (5m+2) + (8m-1)\lfloor \frac{n}{2} \rfloor + 8m(\lceil \frac{n}{2} \rceil - 1) + 2$ and $v = |V(G)|$.

Proof. Let us consider $v = |V(G)|$ and $e = |E(G)|$ then $v = 16mn - 6m - n + 2$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{c_i : 1 \leq i \leq n\} \cup \{a_{ir}^{p_{ir}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir}, 1 \leq r \leq 5\}$$

$$E(G) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{a_{ir}^{p_{ir}} a_{ir}^{p_{ir+1}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir} - 1, 1 \leq r \leq 5\}$$
$$\{a_{ir}^{1} c_i : 1 \leq i \leq n, 1 \leq r \leq 5\}$$

Now we define the labeling $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, v+e\}$ as in theorem 2.3.
It follows that the edge-weights of all edges of G constitute an arithmetic sequence $s = (\eta+1)+1, (\eta+1)+2, \ldots, (\eta+1)+e$, with common difference 1. We denote it by

$A = \{a_i : 1 \le i \le e\}$. Now for G we complete the edge labeling λ for super $(a, 1)$-edge antimagic total labeling with values in the arithmetic sequence $v + 1, v + 2, \ldots, v + e$ with common difference 1. Let us denote it by $B = \{b_j : 1 \le j \le e\}$. Define $C = \{a_{2i-1} + b_{e-i+1} : 1 \le i \le \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \le j \le \frac{e+1}{2} - 1\}$. It is easy to see that C constitute an arithmetic sequence with $d = 1$ and $a = s + \frac{3}{2}v = \eta + 2 + \frac{3}{2}(16mn - 6m - n + 2)$. Since all vertices receive the smallest labels so λ is a super $(a, 1)$-edge antimagic total labeling. □

Theorem 2.5. The graph $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n : n, l)$ is a super $(a, 0)$-EAT labeling with $a = 2v + s - 1$ and super $(a, 2)$-EAT labeling with $a = v + s + 1$, where $m \ge 3$ and $m \equiv 1(\text{mod } 2)$, $n \ge 2, l = 5, \alpha_1 = (m, m, m, m, m_5, \ldots, m_l)$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_n = (m_l, m_l - 1, m - 1, m, m_5, \ldots, m_l)$, $s = \left(\sum_{p=5}^{l}[m2^{p-5}] + 2m + 2\right) + \left(\sum_{p=5}^{l}[m2^{p-5}] + m - 1 + m2^{l-4}\right)\lfloor\frac{n}{2}\rfloor + \left(\sum_{p=5}^{l}[m2^{p-5}] + m + m2^{l-4}\right)\left(\lceil\frac{n}{2}\rceil - 1\right) + 2$, $m_p = m2^{p-5}$ for $5 \le p \le l$ and $v = |V(G)|$.

Proof. Let us consider $v = |V(G)|, e = |E(G)|$ then $v = (2mn + 2m - n + 2) + m(n - 1)2^{l-3} + n\sum_{p=5}^{l}[m2^{p-4}]$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{c_i : 1 \le i \le n\} \cup \{a_{ir}^{p_{ir}} : 1 \le i \le n, 1 \le p_{ir} \le m_{ir}, 1 \le r \le 5\}$$

$$E(G) = \{c_i c_{i+1} : 1 \le i \le n - 1\} \cup \{a_{ir}^{p_{ir}} a_{ir}^{p_{ir+1}} : 1 \le i \le n, 1 \le p_{ir} \le m_{ir} - 1, 1 \le r \le 5\}$$
$$\{a_{ir}^{1} c_i : 1 \le i \le n, 1 \le r \le 5\}$$

Now, we define the labeling $\lambda : V(G) \to \{1, 2, \ldots, v\}$ as follows:
Throughout the labeling we will consider

$a = \sum_{p=5}^{l}[m2^{p-5} + 2] + 2m + 2$,

$b = \sum_{p=5}^{l}[m2^{p-5}] + m - 1 + m2^{l-4}$,

$c = \sum_{p=5}^{l}[m2^{p-5}] + m + m2^{l-4}$,

$d = \sum_{p=5}^{l}[m2^{p-5}] + 2m - 1$,

$\alpha = \sum_{p=5}^{l}[m2^{p-4}] + m2^{l-3} + 5m - 1$,

$\eta = a + b\lfloor\frac{n}{2}\rfloor + c\left(\lceil n2\rceil - 1\right)$

$$\lambda(c_i) = \begin{cases} \eta + m & \text{for } i = 1 \\ \eta + \frac{\alpha}{2}(i-3) + (m2^{l-4} + c + d) & \text{for } i \ge 3, \text{odd} \\ \frac{\alpha}{2}(i-2) + (m-1)2^{l-4} + a & \text{for } i = \text{even} \end{cases}$$

When $i=1$:
for $p_{1r}=1,3,5,\dots,m_{1r}$, where $r=1,2,3,4$ and $5\le r\le l$, we define

$$\lambda(u)=\begin{cases}\frac{p_{1,1}+1}{2} & \text{for } u=a_{11}^{p_{11}},\\ (m+2)-\frac{p_{1,2}+1}{2} & \text{for } u=a_{1,2}^{p_{1,2}},\\ (m+1)+\frac{p_{1,3}+1}{2} & \text{for } u=a_{1,3}^{p_{1,3}},\\ (2m+3)-\frac{p_{1,4}+1}{2} & \text{for } u=a_{1,4}^{p_{1,4}},\end{cases}$$

$\lambda(a_{i,r}^{p_{1,r}})=(2m+3)+\sum_{k=5}^{r}[m2^{k-5}]-\frac{p_{1,r}+1}{2}$ respectively and for $p_{1,r}=2,4,6,\dots,m_{1,r}-1$, where where $r=1,2,3,4$ and $5\le r\le l$, we define

$$\lambda(u)=\begin{cases}\eta+\frac{p_{1,1}}{2} & \text{for } u=a_{1,1}^{p_{1,1}},\\ \eta+m-\frac{p_{1,2}}{2} & \text{for } u=a_{1,2}^{p_{1,2}},\\ \eta+m+\frac{p_{1,3}}{2} & \text{for } u=a_{1,3}^{p_{1,3}},\\ \eta+2m-\frac{p_{1,4}}{2} & \text{for } u=a_{1,4}^{p_{1,4}},\end{cases}$$

$\lambda(a_{i,r}^{p_{1,r}})=\eta+2m-1+\sum_{k=5}^{r}[m2^{k-5}]-\frac{p_{1,r}}{2}$ respectively.
When $i=$even
for $p_{i,r}=1,3,5,\dots,m_{i,r}$; where $r=1,2,3,4$ and $5\le r\le l$, we define

$$\lambda(u)=\begin{cases}\eta+\alpha\left(\frac{i-2}{2}\right)+d+\frac{p_{i,1}+1}{2} & \text{for } u=a_{1,1}^{p_{i,1}},\\ \eta+\alpha\left(\frac{i-2}{2}\right)+d+m2^{l-4}+1-\frac{p_{i,2}+1}{2} & \text{for } u=a_{1,2}^{p_{i,2}},\\ \eta+\alpha\left(\frac{i-2}{2}\right)+d+m2^{l-4}+\frac{p_{i,3}+1}{2} & \text{for } u=a_{1,3}^{p_{i,3}},\\ \eta+\alpha\left(\frac{i-2}{2}\right)+d+m+m2^{l-4}+4-\frac{p_{i,4}+1}{2} & \text{for } u=a_{1,4}^{p_{i,4}},\end{cases}$$

$\lambda(a_{i,r}^{p_{i,r}})=\eta+\alpha\left(\frac{i-2}{2}\right)+\sum_{k=5}^{r}[m2^{k-5}]+d+m+m2^{l-4}+1-\frac{p_{i,r}+1}{2}$ respectively.
and for $p_{i,r}=2,4,6,\dots,m_{i,r}-1$; where $r=1,2,3,4$ and $5\le r\le l$, we define

$$\lambda(u)=\begin{cases}a+\alpha\left(\frac{i-2}{2}\right)+\frac{p_{i,1}}{2} & \text{for } u=a_{11}^{p_{i,1}},\\ c+\alpha\left(\frac{i-2}{2}\right)+(m+2)-\frac{p_{i,2}}{2} & \text{for } u=a_{1,2}^{p_{i,2}},\\ c+\alpha\left(\frac{i-2}{2}\right)+(m+2)+\frac{p_{i,3}}{2} & \text{for } u=a_{1,3}^{p_{i,3}},\\ c+\alpha\left(\frac{i-2}{2}\right)+2(m+1)-\frac{p_{i,4}}{2} & \text{for } u=a_{1,4}^{p_{i,4}},\\ \alpha\left(\frac{i-2}{2}\right)+(11m+2)-\frac{p_{i,5}}{2} & \text{for } u=a_{1,5}^{p_{i,5}},\\ \alpha\left(\frac{i-2}{2}\right)+(13m+2)-\frac{p_{i,6}}{2} & \text{for } u=a_{1,6}^{p_{i,6}},\end{cases}$$

$\lambda(a_{i,r}^{p_{i,r}})=c+\alpha\left(\frac{i-2}{2}\right)+(2m+)+\sum_{k=5}^{r}[m2^{k-5}]-\frac{p_{i,r}}{2}$ respectively.
When $i\ge 3$ odd: and for $p_{i,r}=1,3,5,\dots,m_{i,r}$, where $r=1,2,3,4$ and $5\le r\le l$, we define

$$\lambda(u)=\begin{cases}(a+b)+\alpha\left(\frac{i-3}{2}\right)+(13m+1)+\frac{p_{i,1}+1}{2} & \text{for } u=a_{1,1}^{p_{i,1}},\\ (a+b)+\alpha\left(\frac{i-3}{2}\right)+m2^{l-4}+1-\frac{p_{i,2}+1}{2} & \text{for } u=a_{1,2}^{p_{i,2}},\\ (a+b)+\alpha\left(\frac{i-3}{2}\right)++m2^{l-4}+\frac{p_{i,3}+1}{2} & \text{for } u=a_{1,3}^{p_{i,3}},\\ (a+b)+\alpha\left(\frac{i-3}{2}\right)+m+1+m2^{l-4}-\frac{p_{i,4}+1}{2} & \text{for } u=a_{1,4}^{p_{i,4}},\end{cases}$$

$\lambda(a_{i,r}^{p_{i,r}}) = (a+b) + \alpha\left(\frac{i-3}{2}\right) + \sum_{k=5}^{r}[m2^{k-5}] + (m+4) + m2^{l-4} - \frac{p_{ir}+1}{2}$ respectively.
and for $p_{i,r} = 2, 4, 6, \ldots, m_{i,r} - 1$, where $r = 1, 2, 3, 4$ and $5 \leq r \leq l$, we define

$$\lambda(u) = \begin{cases} (c+d) + \eta + \alpha\left(\frac{i-3}{2}\right) + \frac{p_{i,1}}{2} & \text{for } u = a_{1,1}^{p_{i,1}}, \\ (c+d) + \eta + \alpha\left(\frac{i-3}{2}\right) + m2^{l-4} - \frac{p_{i,2}}{2} & \text{for } u = a_{1,2}^{p_{i,2}}, \\ (c+d) + \eta + \alpha\left(\frac{i-3}{2}\right) + m2^{l-4} + \frac{p_{i,3}}{2} & \text{for } u = a_{1,3}^{p_{i,3}}, \\ (c+d) + \eta + \alpha\left(\frac{i-3}{2}\right) + m2^{l-4} + m - \frac{p_{i,4}}{2} & textfor\ u = a_{1,4}^{p_{i,4}}, \end{cases}$$

$\lambda(a_{i,r}^{p_{i,r}}) = (c+d) + \eta + \alpha\left(\frac{i-3}{2}\right) + \sum_{k=5}^{r}[m2^{k-5}] + m + m2^{l-4} - \frac{p_{i,r}}{2}$ respectively.
The set of all edge-sums generated by the above labeling scheme forms a consecutive integer sequence $s = (\eta+1)+1, (\eta+1)+2, \ldots, (\eta+1)+e$. Therefore, by Proposition 2.1, λ can be extended to a super $(a, 0)$-EAT labeling and obtain the magic constant $a = 2v + s - 1 = \eta + 1 + 2(2mn + 2m - n + 2) + m(n-1)2^{l-2} + n\sum_{p=5}^{l}[m2^{p-3}]$. Similarly,by Proposition 2.1, λ can be extended to a super $(a, 2)$-EAT labeling and we obtain the magic constant $a = v + 1 + s = \eta + 3 + (2mn + 2m - n + 2) + m(n-1)2^{l-3} + n\sum_{p=5}^{l}[m2^{p-4}]$. □

Theorem 2.6. The graph $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n : n, l)$ is a super $(a, 1)$-EAT labeling with $a = s + \frac{3}{2}v$ if v is even, where $m \geq 3$ and $m \equiv 1 (\text{mod } 2)$, $n \geq 2, l = 5, \alpha_1 = (m, m, m, m, m_5, \ldots, m_l)$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_n = (m_l, m_l - 1, m - 1, m, m_5, \ldots, m_l)$, $s = \left(\sum_{p=5}^{l}[m2^{p-5}] + 2m + 2\right) + \left(\sum_{p=5}^{l}[m2^{p-5}] + m - 1 + m2^{l-4}\right)\lfloor\frac{n}{2}\rfloor + \left(\sum_{p=5}^{l}[m2^{p-5}] + m + m2^{l-4}\right)\left(\lceil\frac{n}{2}\rceil - 1\right) + 2$, $m_p = m2^{p-5}$ for $5 \leq p \leq l$ and $v = |V(G)|$.

Proof. Let us suppose $v = |V(G)|$ and $e = |E(G)|$ then $v = (2mn + 2m - n + 2) + m(n-1)2^{l-3} + n\sum_{p=5}^{l}[m2^{p-4}]$ and $e = v - 1$. We denote the vertex and edge sets of G as follows:

$$V(G) = \{c_i : 1 \leq i \leq n\} \cup \{a_{ir}^{p_{ir}} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir}, 1 \leq r \leq 5\}$$

$$E(G) = \{c_i c_{i+1} : 1 \leq i \leq n-1\} \cup \{a_{ir}^{p_{ir}} a_{ir}^{p_{ir}+1} : 1 \leq i \leq n, 1 \leq p_{ir} \leq m_{ir} - 1, 1 \leq r \leq 5\}$$
$$\{a_{ir}^1 c_i : 1 \leq i \leq n, 1 \leq r \leq 5\}$$

Now, we define the labeling $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, v+e\}$ as in theorem 2.5. It follows that the edge-weights of all edges of G constitute an arithmetic sequence $s = (\eta+1)+1, (\eta+1)+2, \ldots, (\eta+1)+e$, with common difference 1. We denote it by $A = |a_i : 1 \leq i \leq e|$. Now for G we complete the edge labeling λ for super $(a, 1)$-edge antimagic total labeling with values in the arithmetic sequence $v+1, v+2, \ldots, v+e$ with common difference 1. Let us denote it by $B = \{b_j : 1 \leq j \leq e\}$. Define $C = \{a_{2i-1} + b_{e-i+1} : 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1} : 1 \leq j \leq \frac{e+1}{2} - 1\}$. It is easy to see that C constitute an arithmetic sequence with $d = 1$ and $a = s + \frac{3}{2}v = \eta + 2 + \frac{3}{2}\left((2mn + 2m - n + 2) + m(n-1)2^{l-3} + n\sum_{p=5}^{l}[m2^{p-4}]\right)$.
Since all vertices receive the smallest labels so λ is a super $(a, 1)$-edge antimagic total labeling. □

3. Conclusion

In this paper, we have proved the super (a,d)-EAT labeling of the subdivided caterpillar $G \cong \zeta(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n : n, l)$,. However, the problem for super (a,d)-EAT labeling is still open for $\alpha_1 \neq \alpha_2 \neq, ..., \neq \alpha_n$ different values of magic constant.

References

[1] Ahmad, A., Baig, A. Q., and Imran, M., On super edge-magicness of graphs, Utilitas Math. to appear.

[2] Bača, M., Dafik, Miller, M., and Ryan, J., (2008), Edge-antimagic total labeling of disjoint unions of caterpillars, J. Comb. Math. Comb. Computing, 65, pp. 61–70.

[3] Bača, M., Y. Lin, Y., Miller, M., and Simanjuntak, R., (2001), New constructions of magic and antimagic graph labelings, Utilitas Math., 60, pp. 229–239.

[4] Bača, M., Lin, Y. and Muntaner-Batle, F.A., (2007), Super edge-antimagic labelings of the path-like trees, Utilitas Math.,73, pp. 117–128.

[5] Bača, M., and M. Miller, M., (2008), Super Edge-Antimagic Graphs, Brown Walker Press, Boca Raton, Florida USA.

[6] Bača, M.,, Lin, Y., Miller, M., and M. Z. Youssel, M. Z., Edge-antimagic graphs, *Discrete Math.*, to appear.

[7] Baskoro, E. T., and Cholily. Y., (2004), Expanding super edge-magic graphs, Proc. ITB Sains and Tek.,36:2, pp. 117–125.

[8] Enomoto, H., Llado, A. S., Nakamigawa, T., and Ringle, G., (1980), Super edge-magic graphs, SUT J. Math., 34, pp. 105–109.

[9] Figueroa-Centeno, R. M., Ichishima, R., and Muntaner-Batle, F. A., (2001), The place of super edge-magic labeling among other classes of labeling, Discrete Math., 231, pp. 153–168.

[10] Figueroa-Centeno, R. M., Ichishima, R., and Muntaner-Batle, F. A., (2002), On super edge-magic graphs, Ars Combin., 64, pp. 81–95.

[11] Figueroa-Centeno, R. M., Ichishima, R., and Mantaner-Batle, F. A., (2005), On edge-magic labeling of certain disjoint union graphs, Australas. J. Combin., 32, pp. 225–242.

[12] Fukuchi, Y., A recursive theorem for super edge-magic labeling of trees, (2000), SUT J. Math., 36, pp. 279–285.

[13] J. A. Gallian, (2009), A dynamic survey of graph labeling, Electronic J. Combin.

[14] Hussain, M., Baskoro, E. T., Slamin, (2009) On super edge-magic total labeling of banana trees, Utilitas Math., 79, pp. 243–251.

[15] Javaid, M., (2014), On super edge-antimagic total labeling of generalized extended w-trees, International Journl of Mathematics and soft Computing, 4, pp. 17–25.

[16] A. Kotzig, A., and Rosa, A., (1970), Magic valuations of finite graphs, Canad. Math. Bull., 13, pp. 451–461.

[17] Kotzig, A., and Rosa, A Magic valuation of complete graphs, (1972), Centre de Recherches Mathematiques, Universite de Montreal, CRM–175.

[18] Lee, S. M., and Shah, Q. X., All trees with at most 17 vertices are super edge-magic, (2002), 16th MCCCC Conference, Carbondale, University Southern Illinois.

[19] Raheem, A., Javaid, M., Baig, A. Q., (2016), Antimagic labeling of the union of subdivided stars, TWMS J. App. Eng. Math.,6(2), pp.244–250.

[20] Raheem, A., Javaid, M., Baig, A. Q., (2016), On antimagicness of generalized extene w-trees, science Internation Journal, 28(5),pp. 5057–5022.

[21] Raheem, A., (2018), On super (a, d)-edge antimagic total labeling of a subdivided stars, Ars Combin., 136, pp. 169–179.

[22] K. A. Sugeng. K. A., Miller, M., Baca, M., (a, d)-edge-antimagic total labeling of caterpillars, (2005), Lecture Notes Comput. Sci., 3330 pp. 169–180.

[23] West, D. B.,(1996), An Introduction to Graph Theory, Prentice-Hall.

Abdul Raheem is a Post Doctorate researcher fully supported by Punjab Higher Education Commission Lahore, Pakistan at Department of Mathematics, National University of Singapore, Singapore. He has received Ph.D. Mathematics (2012-2016) from COMSATS Institute Institute of Information Technology, Islamabad, Pakistan. He received his M.Phil Mathematics from Riphah International University Islamabad, Pakistan. Currently, he is working in different research area of graph theory such as computational graph theory, metric graph theory and chemical graph theory. He is referee of several international journals of mathematics and informatics.

Muhammad Javaid completed his Post Doctorate Mathematics (2015-17) from School of Mathematical Sciences, University of Science and Technology of China (USTC), Hefei, China and Ph.D. Mathematics (2009-14) from National University of Computer and Emerging Sciences, Lahore, Pakistan. He is currently working in the different areas of graph theory such as spectral theory of graphs, computational graph theory and chemical graph theory.

Muhammad Awais Umar completed his Ph.D. Mathematics (2017) from Abdus Salam School of Mathematical Sciences GC University Lahore, Pakistan. He is currently working in the different areas of graph theory such as computational graph theory and graph labeling.